STABILITY OF FUZZY DIFFERENTIAL EQUATIONS

Alexey Bychkov, Mikhail Merkuryev

Abstract: While modeling process of thinking and other types of subjective perception, mathematical methods are extremely important. This paper proposes a mathematical apparatus for such a formalization based on Pytyev's theory of possibilities. For describing and investigation of fuzzy undefined processes a new class of differential equations is constructed. An important task of fuzzy differential equation investigation is obtaining of sufficient conditions of stability

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Introduction

The paper is devoted to classical problems of description of event space and construction of abstract axiomatic of theory of possibilities. The questions of constructing theory of possibilities were investigated by authors of [Zadeh, 1978, Dubois, 1988, Pytyev, 2000]. But it appeared that within these models it wasn't anticipated that for description of event, besides possibility measure, necessity measure is also needed.

Event, in possibility-theoretic formulation, is a combination of states of reality that we consider as a whole. After it happens, we can say for sure which elements it consists of. That's why an event is a combination of states of reality that doesn't depend on our perception. Going on to mathematical specification, we'll consider crisp events, i.e., events whose characteristic functions look like $\chi_A(x) = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A} \end{cases}$.

The most essential difference between theory of probabilities and theory of possibilities is that treatment of physical sense of events described within theory of possibilities fundamentally differs from rate interpretation in terms of theory of probabilities.

Elements of theory of possibilities

Let's consider some definition from [Pytyev, 2000].

Definition 1. Let's call subjective scale $L = ([0,1], \le, +, \circ)$ segment $[0,1]$ with classical order $\le$, operation of sum "+" and operation of multiplication "$\circ$".

Definition 2. Let's denote as sum of two elements $a, b \in L$ maximum of these elements, i.e., $a + b \overset{\Delta}{=} \max(a,b)$.

Definition 3. Let's denote as product of two elements $a, b \in L$ minimum of these elements, i.e., $a \circ b \overset{\Delta}{=} \min(a,b)$.

It's easy to make sure that operations introduced in such a way satisfy all classical properties, i.e., they are commutative, associative and mutually distributive:

- $a+b=b+a$, $a\circ b=b\circ a$;
- $(a+b)+c=a+(b+c)$, $(a\circ b)\circ c=a\circ(b\circ c)$;
- $a\circ(b+c)=\min(a,\max(b,c))=\max(\min(a,b),\min(a,c))=(a\circ b)+(a\circ c)$;
- $a+(b\circ c)=\max(a,\min(b,c))=\min(\max(a,b),\max(a,c))=(a+b)\circ(a+c)$.

Let's define neutral elements $\tilde{0}$ and $\tilde{1}$ as $\tilde{0}\overset{\Delta}{=}0$ and $\tilde{1}\overset{\Delta}{=}1$. For these elements the following equalities hold: $0\circ a=\min(0,a)=0=\tilde{0}$, $1\circ a=\min(1,a)=a$, $\tilde{0}+a=\max(0,a)=a$, $\tilde{1}+a=\max(a,1)=\tilde{1}$, where $a\in[0,1]$.

Order on L is matched with operations of addition and multiplication, i.e.

$$a\le b\Rightarrow\begin{cases}a\circ b\le b\circ c,\\ a+c\le b+c,\end{cases}\quad a,b,c\in L;\ \tilde{0}<\tilde{1}.$$

Let's consider an abstract space X . The physical sense of elements of this space is inessential for us on this stage of constructing a model. Let's call an arbitrary subset $A\subseteq X$ an event.

The question of qualitative description of events is naturally raised. Let's introduce an evaluation of an event — possibility.

Let's denote as $\mathbf{A}$ an algebra of sets on X .

Definition 4 [Pytyev, 2000]. Let's call function $P:A$ possibility, if:

1. $P(A)\ge 0$ for $\forall A\in\mathbf{A}$;

2. $P(A)$ is countable-additive, i.e., for $\forall\{A_i\}\in\mathrm{A}$: $\bigcup_{i=1}^{\infty}A_i\in\mathrm{A}\Rightarrow P(\bigcup_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}P(A_i)=\sup_{i=\overline{1,\infty}}P(A_i)$.

Note that unlike classical theory of measure, condition of countable additivity doesn't put limits on sets $\{A_i\}_{i=1}^{\infty}\in\mathbf{A}$ — in theory of possibilities condition $A_i\cap A_j=\varnothing,\ i\ne j$ isn't required.

Lemma 1 (monotonousness of measure). If $A,B\in\mathbf{A}$, $A\subseteq B$, then $P(A)\le P(B)$.

Lemma 2 (continuity in regard to monotonously increasing sequence). Let $\{A_i\}_{i=1}^{\infty}\in\mathbf{A}$, $A_i\subset A_{i+1}$, and $\bigcup_{i=1}^{\infty}A_i\in\mathbf{A}$. Then $P(\lim_{n\to\infty}A_n)=\lim_{n\to\infty}P(A_n)$.

Lemma 3 (lower semicontinuity in regard to monotonously decreasing sequence). Let $\{A_i\}_{i=1}^{\infty}\in\mathbf{A}$, $A_{i+1}\subset A_i$, and $\bigcap_{i=1}^{\infty}A_i\in\mathbf{A}$. Then $P(\lim_{n\to\infty}A_n)\le\lim_{n\to\infty}\inf_{m>n}P(A_m)$.

Lemma 3 implies interesting consequences that aren't properties of measure in classical functional analysis. It worth saying that Lemma 3 itself characterizes exactly possibility and subjective scale.

Property of semicontinuity of possibility means that for arbitrary sequence $\{A_i\}_{i=1}^{\infty}\in\mathbf{A}$ such as $\lim_{n\to\infty}A_n=A$ we cannot say for sure that $\lim_{i\to\infty}P(A_i)=P(A)$. In practice for a decreasing sequence we cannot define the possibility of its limit set, given possibilities of its elements. In particular, we cannot define value of $P(\varnothing)$ by continuity, because $P(A)$ isn't continuous in $A=\varnothing$. Indeed, let's consider a sequence $\{A_i\}_{i=1}^{\infty}\in\mathbf{A}$ such as

$\lim_{n\to\infty} A_n = \varnothing$, $A_{n+1} \subset A_n$ (absolutely decreasing sequence). Then $P(\varnothing) = P(\lim_{n\to\infty} A_n) \le \lim_{n\to\infty} \inf P(A_n)$. Therefore, let's define $P(\varnothing)$ as an arbitrary number form segment $[0, \inf_{A\in\mathbf{A}} P(A)]$. With these assumptions $P(A \bigcup \varnothing) = P(A)$, $P(A \bigcap \varnothing) = \varnothing \ \ \forall A \in \mathbf{A}$. Henceforward we'll define $P(\varnothing) = 0$, unless stated otherwise.

Let X be an arbitrary space, $\mathbf{A}$ is an algebra of sets defined on X, and $P(\cdot)$ is a possibility on $\mathbf{A}$. A question of extension the possibility measure onto wider class of sets arises naturally. Let's use a classic approach from functional analysis. For this let's introduce a concept of outer measure.

Definition 5. Let's denote the set of subsets of X as $\beta(X)$. Function $P^*(\cdot): \beta(\mathrm{X}) \to L$, defined as

$$P^*(B) = \inf_{\{E_j\}\in \mathrm{A}} \sup_j P(E_j) \ , \tag{1}$$

where $\{E_j\} \in \mathbf{A}$ are such as $B \subset \bigcup_{j=1}^{\infty} E_j$, we shall call an outer possibility measure.

This definition is sensible, because always such sets $\{E_j\} \in \mathbf{A}$, that for $\forall A \subset \mathrm{X}$, $A \subset \bigcup_{j=1}^{\infty} E_j$, always exist. Indeed, having $E_1 = X$, $E_2 = E_3 = ... = \varnothing$, we obtain that any set $A \subseteq X$ is covered by sets $\bigcup_{j=1}^{\infty} E_j = X$.

Lemma 4. For arbitrary set $A \in \mathbf{A}$ outer possibility measure is equal to possibility measure, i.e., $P^*(A) = P(A)$.

Proof. Let's choose a sequence $\{E_j\} \in \mathbf{A}$ such as $E_1 = A$, $E_2 = E_3 = ... = \varnothing$. We obtain that $A \subset \bigcup_{j=1}^{\infty} E_j$, and, accordingly, $\inf_{\{E_j\}} \sup_j P(E_j) \le P(A)$. So, $P^*(A) \le P(A)$ holds.

By definition of greatest lower bound for arbitrary $\varepsilon > 0$ there exists a sequence of sets $\exists \{E_j\} \in \mathrm{A}$ such as $\sup_j P(E_j) = P(\bigcup_{j=1}^{\infty} E_j) < P^*(A) + \varepsilon$.

Because $A = A \bigcap (\bigcup_{j=1}^{\infty} E_j) = \bigcup_{j=1}^{\infty} (A \bigcap E_j)$, for arbitrary $A \in \mathbf{A}$ the following holds:

$$P(A) = P(\bigcup_{j=1}^{\infty} (A \bigcap E_j)) = \sum_{\mathrm{j}=1}^{\infty} P(A \bigcap E_j) = \sup_{j=\overline{1,\infty}} P(A \bigcap E_j) \le \sup_{j=\overline{1,\infty}} P(E_j).$$

It implies that $P(A) < P^*(A) + \varepsilon$. But when $\varepsilon \to 0$, we'll have unstrict inequality, i.e. $P(A) \le P^*(A)$. We obtain that $P(A) = P^*(A)$.

Lemma 5. Outer possibility measure is a non-negative function of a set, i.e., $P^*(A) \ge 0$ for arbitrary $A \subset X$.

The proof follows from non-negativity of possibility measure.

Lemma 6. Outer possibility measure $P^*(\cdot)$ is monotonous, i.e., for $\forall A, B \subset X$ such as $A \subset B$, $P^*(A) \le P^*(B)$ holds.

Proof. $P^*(B) = \inf_{\{E_j\}\in \mathrm{A}} \sup_j P(E_j) \ \ \inf_{\{E_j\}} \sup_j P(E_j) \le P(A)$

Let $\{E_j\}_{j=1}^{\infty} \in \mathbf{A}$ be a sequence of sets that covers B, i.e., $A \subset B \subset \bigcup_{j=1}^{\infty} E_j$. But this sequence also covers A, i.e., $A \subset \bigcup_{j=1}^{\infty} E_j$. That's why $\inf_{\substack{\{E_j\} \\ A \subseteq \bigcup_j Ej}} \sup_j P(E_j) \le \inf_{\substack{\{E_j\} \\ B \subseteq \bigcup_j Ej}} \sup_j P(E_j)$, or $P^*(A) \le P^*(B)$.

Theorem 1 (extension of possibility measure). Outer measure P^* defined on all subsets of X, is a possibility measure.

Extending the measure, we obtain just upper bound, that formally satisfies definition of measure.

Properties of measures clearly state that this kind of measure is not enough for adequate description of models. So, the lower bound of measure is also needed.

Like measure of possibility, measure of "sureness" is also bounded above. Naturally, the new measure can be introduced on the same scale $L = ([0,1], \le, +, \circ)$ as possibility. When we try to describe an event with two values, these values must be connected somehow.

Definition 6. Necessity measure is a function $N : \mathbf{A} \to L$ that satisfies the following requirements:

- $N(A) \ge 0$ for $\forall A \in \mathrm{A}$;
- $N(A)$ is countable-multiplicative, i.e., for arbitrary sequence of sets $\{A_i\} \in \mathbf{A}$ such as $\bigcap_{i=1}^{\infty} A_i \in \mathbf{A}$, $N(\bigcap_{i=1}^{\infty} A_i) = \prod_{i=1}^{\infty} A_i = \underline{\inf}_{i=1,\infty} N(A_i)$ holds.

Necessity satisfies all conditions and has all the features possibility doesn't have. That's why necessity and possibility should be considered together, because their advantages perfectly match. That allows describing correctly, besides events themselves, all countable operations on them.

So, we shall consider $(X, \mathbf{A}, P, N)$ model with two measures, which we'll call (*PN*)-model, treating at adequate for description of questions put in the beginning of the paper.

Having introduced possibility and necessity measures, we obtained an apparatus that allows to find values of these measures and describe an experiment adequately.

Fuzzy differential equations

Definition 7. Given a possibility space $(X, \mathbf{A}, P, N)$ and measurable space $(Y, \mathbf{B})$, let's call *fuzzy perceptive variable* any $(\mathbf{A}, \mathbf{B})$-measurable function $\xi : X \to Y$.

Definition 8. Fuzzy perceptive variables ξ and η are called independent, if $P\{\xi = u, \eta = v\} = \min(P\{\xi = u\}, P\{\eta = v\})$.

Definition 9. Fuzzy perceptive variable ξ scalar or vector is called *normal*, if its distribution looks like $P\{\xi = u\} = \varphi\left\{\left\|\Xi^{-1/2}(u - u_0)\right\|^2\right\}$, where $\varphi(x)$ is a decreasing function that specified for $x \ge 0$ such as $\varphi(x) \xrightarrow[x \to \infty]{} 0$, $\varphi(0) = 1$.

Definition 10. Fuzzy perceptive process is a function $\xi(x,t) : X \times R \to Y$.

Definition 11. Normal fuzzy perceptive process is a process of fuzzy roaming, if it is:

- A process with independent increments, i.e., for any four moments of time $t_1 < t_2 \le t_3 < t_4$ fuzzy perceptive variables $\xi(t_2) - \xi(t_1)$ and $\xi(t_4) - \xi(t_3)$ are independent.
- Its transient possibility is $P\{\xi(t) = x \mid \xi(t_0) = x_0\} = \varphi\left(\frac{\left\|\Xi^{-1/2}(x - x_0)\right\|^2}{t - t_0}\right)$.
- $\xi(0) = 0$.

Definition 12. Given a piecewise-constant function $f: \ f(t) = \{y_k: \ t_k \le t < t_{k+1}\}$, $k = 0...N-1$, $t_0 = 0$, $t_N = T$, and a scalar process of fuzzy roaming $w(t)$, define

$$\int_0^T f(t)dw(t) = \sum_{k=0}^{N-1} y_k (w(t_{k+1}) - w(t_k)).$$

This fuzzy perceptive variable is called integral of a piecewise-constant function by a process of fuzzy roaming.

Theorem 2. A piecewise-continuous function $f(t)$, and two sequences of piecewise-constant functions $f_n(t)$ and $\bar{f}_n(t)$ that converge to $f(t)$ in average, are given. If $Q = \mathrm{P}\lim_{n\to\infty}\int_0^T f_n(t)dw(t)$ exists, limit $\overline{Q} = \mathrm{P}\lim_{n\to\infty}\int_0^T \bar{f}_n(t)dw(t)$ also exists, and $\overline{Q} = Q$.

Definition 13. A sequence of fuzzy perceptive variables $\xi_n(x)$ converges to $\xi(x)$ in possibility ($\mathrm{P}\lim_{n\to\infty}\xi_n = \xi$), if $P\{|\xi_n - \xi| > c\} \xrightarrow[n\to\infty]{} 0$.

Definition 14. Given a piecewise-continuous function $f(t)$ and a sequence $f_n(t)$, converging to it in average, let's denote

$$\int_0^T f(t)dw(t) = P\lim_{n\to\infty}\int_0^T f_n(t)dw(t).$$

This limit is called integral of piecewise-continuous function by a process of fuzzy roaming.

Let's consider a process of fuzzy roaming $w(t,x)$ and a fuzzy perceptive variable $y_0(x)$. Let's construct the integral equation

$$y(t,x) = y_0(x) + \int_{t_0}^t a(y(s),s)ds + \int_{t_0}^t b(y(s),s)dw(s,x), \qquad (2)$$

where the last integral is an integral of fuzzy roaming.

Theorem 3. Let crisp functions $a(y,t)$ and $b(y,t)$ be continuous by t and satisfy Lipschitz condition on $D = I \times [t_0, t_0 + \Delta t]$, i.e.,

$$|a(y,t) - a(z,t)| \le L|y - z|, \ \ |b(y,t) - b(z,t)| \le L|y - z| \text{ for any } x, y \in R.$$

If for arbitrary fuzzy y_0 and any fixed $x_0 \in X$, for which $P(\{x_0\}) \ge \alpha > 0$, holds $[y_0(x_0) - \Delta y, \ y_0(x_0) + \Delta y] \subseteq I$, then for this x_0 the problem (2) has a unique solution on segment $t \in [t_0; t_0 + h]$, where

$$h=\min\left[k\frac{1}{2L}\cdot\min\left\{1,\frac{1}{\sqrt[4]{\varphi^{-1}(\alpha)}}\right\},\frac{\Delta y}{\max\limits_t|a(x)|+\sqrt{\varphi^{-1}(\alpha)}\cdot\max\limits_t|b(x)|},\Delta t\right],\ k\in(0,1).$$

Stability of fuzzy differential equations

Like in theory of probabilities, in theory of possibilities several types of stability are possible.

Let's consider a system of fuzzy differential equations:

$$y(t)=y(t_0)+\int_{t_0}^{t}a(y)ds+\int_{t_0}^{t}b(y,x)dw(s,x),\qquad (3)$$

where $a(y)=0$, and $b(y,x)$ has $y=0$ as the only modal value.

Definition 15. If for any $x\in X$, for which $P(\{x\})=1$, $y(t,x)=\bar{y}(t)$, then function $\bar{y}(t)$ is called a modal path of the differential equation.

System (3) has a modal path $y=0$.

Definition 16. Modal path $y(t)$ is called weakly stable, if for any $\varepsilon>0$ $\delta(\varepsilon)>0$, $\alpha(\varepsilon)<1$ exist such as when $|y_0-\bar{y}(0)|<\delta$ and $P(\{x\})>\alpha$, $|y(y_0,t,x)-\bar{y}(t)|<\varepsilon$ holds.

Let's call *trajectory derivative of function* $V(y)$ *with regard to system* the following expression:

$$\dot{V}|_{ab}(y,x)=\sum_{i=1}^{n}\frac{\partial V(y)}{\partial y_i}\cdot(a_i(y)+b_i(y,x)\cdot\xi_i(x)),$$

where $\xi(x)$ is a normal fuzzy perceptive variable that corresponds to process of fuzzy roaming $w(\cdot)$ and independent from b.

Let's call crisp analogue of (3) equation

$$y(t)=y(t_0)+\int_{t_0}^{t}a(y)ds.$$

Its trajectory derivative is

$$\dot{V}|_{a}(y)=\sum_{i=1}^{n}\frac{\partial V(y)}{\partial y_i}\cdot a_i(y).$$

Definition 17. α-cut of a fuzzy perceptive variable $\xi(x)$ is the expression $[\xi(x)]_\alpha=\{y:P(y)\ge\alpha\}$ for $\alpha\in(0,1]$.

Lemma 7. For any $x\in X$, when V is continuous and g and h are piecewise-continuous, the following asymptotic inequality holds:

$$\left[V(y(t_2))-V(y(t_1))\right]_\alpha\in\left[\int_{t_1}^{t_2}\inf[\dot{V}|_{ab}(y(t),\cdot)]_\alpha dt+o(t_2-t_1),\int_{t_1}^{t_2}\sup[\dot{V}|_{ab}(y(t),\cdot)]_\alpha dt+o(t_2-t_1)\right].$$

Theorem 4. If for system (3) a Lyapunov-like function $V(y)$ exists such as the following condition holds:

- for all $0<c\le r$ such $\alpha(c)\in(0,1)$ exists that on $V(y)=c$ curve $[\dot{V}|_{ab}(y,\cdot)]_{\alpha(c)}<0$, then system (3) has a weakly stable modal path $y = 0$.

In many cases investigation of weak stability of (3) is possible via investigation of its crisp analogue.

Definition 18. System (3) is called regular if for any $\varepsilon > 0$ there exists $\alpha(\varepsilon) \in [0,1)$ such as $H([b(\cdot)]_\alpha, \{0\}) < \varepsilon$ ($H(A,B)$ is Hausdorff distance between sets).

Theorem 5. Assume that for a regular system (3) such a condition holds: in a neighbourhood of zero a Lyapunov-like function $V(y)$ exists such as for any solution $y(y_0, t)$ of system (3) $\dot{V}|_a(y) \le \psi(V(y))$, where $\psi(z)$ is a decreasing function, for which $\psi(0) = 0$. Then system (3) has a weakly stable modal path $y = 0$.

Conclusion

We have created a new type of differential equation, and coined a new definition of fuzzy stability that differs from all other types of stability under uncertainties. For this type of stability a necessary condition is proven.

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Authors' Information

Alexey Bychkov – Taras Shevchenko National University of Kyjv, Cybernetics department, ul.Volodimirska, 64, Kyjv, Ukraine, e-mail: bychkovtk@gmail.com

Mikhail Merkuryev – Taras Shevchenko National University of Kyjv, Cybernetics department, ul.Volodimirska, 64, Kyjv, Ukraine, e-mail: bychkovtk@gmail.com