MULTIALGEBRAIC STRUCTURES EXISTENCE FOR GRANULAR COMPUTING

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Abstract: In different fields a conception of granules is applied both as a group of elements defined by internal properties and as something inseparable whole reflecting external properties. Granular computing may be interpreted in terms of abstraction, generalization, clustering, levels of abstraction, levels of detail, and so on. We have proposed to use multialgebraic systems as a mathematical tool for synthesis and analysis of granules and granule structures. The theorem of necessary and sufficient conditions for multialgebraic systems existence has been proved.

Keywords: granular computing, multirelations, multioperations.

ACM Classification Keywords: I.2.4 Knowledge representation formalisms and methods: relation systems.

Introduction

Granular computing explores knowledge from different standpoints to reveal various types of structures and information embedded in the data [Zadeh, 1997, Bargiela, Pedrycz, 2002]. A paradigm of granular computing consists in grouping elements together (in a granule) by indistinguishability, similarity, proximity or functionality in arbitrary feature or signal spaces. Taking into account a semantic interpretation of why two objects are put into the same granule and how two objects are related with each other it provides one of a general methodology for intelligent data analysis on different levels of roughening or detailing [Pal et al., 2005, Yao, Yao, 2002].

Internal, external and contextual properties of granules, collective structure of a family of granules and hierarchical structure of granules represent a possible foundation for qualitative/quantitative characterization of levels of abstraction, detail, control, explanation, difficulty, organization and so on. Focusing on high conceptual level issues by ignoring much irrelevant details, granular computing are actively used in computational intelligence [Doherty et al., 2003], information granulation based on rough sets [Yao, 2001, Pal et al., 2005], data mining [Yao, 2006], interval analysis, cluster analysis, machine learning and many others [Yager, 2002, Lin, 2003, etc.] . The integration of multiple views on different types of granulation and granular structure may provide more useful data analysis tools [Lin, 2003, Yao, 2005]. One of a number of possible approaches is to use multialgebraic systems [Mashtalir, Shlyakhov, 2003] as mathematical apparatus for synthesis and analysis of granules and granule structures.

Thus, we need tools providing a granular linkage, i.e. formal operations and relations determined on granules. Furthermore, this linkage has to be induced either by information embedded in the data or by given close coupling with field of application. These questions are at present far from being solved. But the important point to note here is the search of necessary and sufficient conditions for existence of multialgebraic systems as enough general tool of granular computing.

Motivation of granular computing modeling by multialgebraic systems

If we choose any natural number $p \in \mathbb{N}$ then we can consider a ternary relation

$$\mathrm{E}(n_1, n_2, n_3) = \begin{cases} 1, & (n_1 + n_2)\ mod\ p = n_3\ mod\ p; \\ 0, & (n_1 + n_2)\ mod\ p \neq n_3\ mod\ p \end{cases} \tag{1}$$

where $a\ mod\ b$ defines a as modulo b residue, i.e. $a\ mod\ b \triangleq a - \lfloor a/b \rfloor \times b$ and $\lfloor \circ \rfloor$ is a floor function. It is easily seen, if we hold fixed $k \in \{1,2,3\}$ then we get an equivalence relation $\mathrm{P}_k^{\mathrm{E}}$, e.g. for $k = 1$

$$\mathrm{P}_1^{\mathrm{E}}(n_1,n_1')=1 \Leftrightarrow \mathrm{E}(n_1,n_2,n_3)\equiv \mathrm{E}(n_1',n_2,n_3)\,.$$

This equivalence partitions set of natural numbers into residue classes modulo p. Indeed, if remainders in division n_1 and n_1' by p are the same then for arbitrary n_2 and n_3 continued equality is

$$\begin{aligned}(n_1+n_2)\ mod\ p=(ps_1+r_1+ps_2+r_2)\ mod\ p=(r_1+r_2)\ mod\ p=\\ =(ps_1'+r_1+ps_2+r_2)\ mod\ p=(n_1'+n_2)\ mod\ p.\end{aligned} \tag{2}$$

Here it is implied that $n_1=ps_1+r_1, n_1'=ps_1'+r_1, n_2=ps_2+r_2$, and $s_1,s_1',s_2\in\mathbb{N}$. From (2) it follows that $\mathrm{E}(n_1,n_2,n_3)\equiv\mathrm{E}(n_1',n_2,n_3)$. The converse proposition is the valid one also. If $\mathrm{E}(n_1,n_2,n_3)\equiv\mathrm{E}(n_1',n_2,n_3)$ then remainders in division n_1 and n_1' by p have to be equal, if not when $n_1=ps_1+r_1$, $n_1'=ps_1+r_1'$ and $r_1\neq r_1'$ under $n_2=0$, $n_3=r_1$ we obtain, on the one hand,

$$(n_1+n_2)\ mod\ p=(ps_1+r_1)\ mod\ p=r_1\ mod\ p=n_3\ mod\ p\,,$$

i.e. $\mathrm{E}(n_1,0,r_1)=1$. On the other hand,

$$(n_1'+n_2)\ mod\ p=(ps_1'+r_1)\ mod\ p=r_1'\ mod\ p\neq n_3\ mod\ p\,.$$

Since $r_1',r_1\leq p$ и $r_1'\neq r_1$ then $\mathrm{E}(n_1',0,r_1)=0$, which contradicts the original assumption. Notice, from (1) we get $\mathrm{P}_1^{\mathrm{E}}=\mathrm{P}_2^{\mathrm{E}}=\mathrm{P}_3^{\mathrm{E}}$.

Let us sum up. The carrier of original relation is the set of natural numbers $\mathbb{N}$ but the induced equivalence demonstrates the significant carrier change: we have got a finite set $\Pi=\{0,1,\ldots,p-1\}$.

New relation, which will be named a multirelation and denoted by E^{M} in the sequel, is generated on new carrier. As before it is ternary relation but the domain is Π^3 instead of $\mathbb{N}^3$ and multirelation E^{M} acquires the new property that can be expressed as an operation

$$r_1\oplus r_2=r_3\Leftrightarrow \mathrm{E}_m(r_1,r_2,r_3)=1\Leftrightarrow \mathrm{E}(n_1,n_2,n_3)=1$$

where sign "$\oplus$" denotes p congruence addition and $n_i=ps_i+r_i,\ i=1,2,3,\ r_i\in\{0,1,\ldots,p-1\}$. Operations on equivalence classes here and subsequently will be denoted by F^{M}.

If they follow terminology of algebraic system a triplet $\langle \mathrm{A},\mathrm{R},\mathrm{Q}\rangle$ (here A is arbitrary set (carrier), R is a relation suite, Q is a set of operations) is called a model on conditions that $\mathrm{Q}=\varnothing$ and it is said to be an algebra if $\mathrm{R}=\varnothing$. Consequently, from the model $\langle\mathbb{N},\mathrm{E},\varnothing\rangle$ we pass on to the algebra $\langle\Pi,\varnothing,\oplus\rangle$ whose carrier is well-known algebraic structure viz a cyclic Abelian group of p-th order.

It is necessary to understand that original carrier can represent a set and carrier of induced relation on equivalence classes can be Cartesian product of different sets. Let us consider one more example

$$\mathrm{E}(n_1,n_2,n_3)=\begin{cases}1, & (n_1+n_2)\ mod\ p_1=n_3\ mod\ p_2;\\ 0, & (n_1+n_2)\ mod\ p_1\neq n_3\ mod\ p_2\end{cases} \tag{3}$$

where $p_1\neq p_2$. It should be noted that $\mathrm{P}_1^{\mathrm{E}}=\mathrm{P}_2^{\mathrm{E}}\neq\mathrm{P}_3^{\mathrm{E}}$, i.e. E_m is Cartesian product $\mathrm{A}\times\mathrm{B}$ where $\mathrm{A}=\{0,1,..,p_1-1\},\ \mathrm{B}=\{0,1,\ldots,p_2-1\}$. As may be seen from (3) the multirelation E^{M} as a ternary relation is defined on $\mathrm{A}^2\times\mathrm{B}$ and represents an operation from A^2 into B. There is no difficulty in understanding that under certain p_1 and p_2 not only an equivalence inequality appears but a level of partition detail and equivalence nesting are changed. For instance, if $p_1=4$, $p_2=2$ then $\mathrm{P}_1^{\mathrm{E}}=\mathrm{P}_2^{\mathrm{E}}\subseteq\mathrm{P}_3^{\mathrm{E}}$ as $\mathbb{N}$ is partitioned into 4 classes corresponding to residues of division $\mathrm{A}=\{0,1,2,3\}$ at the expense of $\mathrm{P}_1^{\mathrm{E}}=\mathrm{P}_2^{\mathrm{E}}$. Equivalence

$\mathrm{P}_3^{\mathrm{E}}$ partitions original set □ into 2 classes from even and odd numbers, i.e. $\mathrm{B}=\{0,1\}$. In this connection classes $\{0,2\}$ belong to the set of even numbers, classes $\{1,3\}$ form part of odd numbers set respectively.

In analyzed examples the original relation E induces the multirelation (more precisely the multioperation), i.e. an operation with ranges of definition as equivalence classes. It may seem that a similar situation is observed all the time, however this is by no means always the case. Consider the binary relation E (tab. 1) which is defined on the Cartesian product $\{1,2,3,4,5\}\times\{a_1,a_2,b_1,c_1,c_2,c_3,c_4,c_5,c_6\}$.

Table 1

		A		B	C					
		a_1	a_2	b_1	c_1	c_2	c_3	c_4	c_5	c_6
Π_{I}	1	0	0	1	1	1	1	1	1	1
	2	0	0	1	1	1	1	1	1	1
Π_{II}	3	1	1	1	0	0	0	0	0	0
	4	1	1	1	0	0	0	0	0	0
	5	1	1	1	0	0	0	0	0	0

It should be clear that the induced equivalences $\mathrm{P}_1^{\mathrm{E}}$ and $\mathrm{P}_2^{\mathrm{E}}$ dissect the first set $\mathrm{A}_1=\{1,2,3,4,5\}$ into 2 classes: $\Pi_{\mathrm{I}}=\{1,2\}$, $\Pi_{\mathrm{II}}=\{3,4,5\}$ and the second one $\mathrm{A}_2=\{a_1,a_2,b_1,c_1,c_2,c_3,c_4,c_5,c_6\}$ into 3 classes: $\mathrm{A}=\{a_1,a_2\}$, $\mathrm{B}=\{b_1\}$, $\mathrm{C}=\{c_1,c_2,c_3,c_4,c_5,c_6\}$. Thus, the multirelation E_m is defined on the Cartesian product of induced equivalence classes, i.e. $\{\Pi_{\mathrm{I}},\Pi_{\mathrm{II}}\}\times\{\mathrm{A},\mathrm{B},\mathrm{C}\}$ (tab. 2).

Table 2

	A	B	C
Π_{I}	0	1	1
Π_{II}	1	1	0

This multirelation can be represented as two explicit mappings associating $\{\Pi_{\mathrm{I}},\Pi_{\mathrm{II}}\}$ with $\{\mathrm{A},\mathrm{B},\mathrm{C}\}$. Denote induced mappings as F^{EM} and $(\mathrm{F}^{\mathrm{EM}})^{-1}$ in both directions then $\mathrm{F}^{\mathrm{EM}}(\Pi_{\mathrm{I}})=\{\mathrm{B},\mathrm{C}\}$, $\mathrm{F}^{\mathrm{EM}}(\Pi_{\mathrm{II}})=\{\mathrm{A},\mathrm{B}\}$ and $(\mathrm{F}^{\mathrm{EM}})^{-1}(\mathrm{A})=\{\Pi_{\mathrm{II}}\}$, $(\mathrm{F}^{\mathrm{EM}})^{-1}(\mathrm{B})=\{\Pi_{\mathrm{I}},\Pi_{\mathrm{II}}\}$, $(\mathrm{F}^{\mathrm{EM}})^{-1}(\mathrm{C})=\{\Pi_{\mathrm{I}}\}$. Single-valuedness is lacking in both cases therefore we are not able to indicate multioperation.

Thus, algebraic model can lead either to multimodels or to multialgebra and there arises an important question: when do two relations with different arities generate one carrier?

Necessary and sufficient conditions for multirelations carriers equality

Let $\mathrm{A}_1,\mathrm{A}_2,\ldots,\mathrm{A}_n$ be any given sets. Consider n-arity relation $\mathrm{E}(x_1,\ldots,x_n)$ on Cartesian product of arbitrary carriers $\mathrm{A}_1\times\ldots\times\mathrm{A}_n$. A trivial verification shows that

$$\mathrm{P}_k^{\mathrm{E}}(x_k,x'_k)=1\Leftrightarrow \mathrm{E}(x_1,\ldots,x_k,\ldots,x_n)\equiv \mathrm{E}(x_1,\ldots,x'_k,\ldots,x_n) \qquad (4)$$

constitutes an equivalence relation and partitions may be regarded on each A_k. The understanding of the appearance mechanism of $\mathrm{P}_k^{\mathrm{E}}$ awaits further investigation.

If $\mathrm{A}_k=\mathrm{A}_l$ then relations $\mathrm{P}_k^{\mathrm{E}}$ and $\mathrm{P}_l^{\mathrm{E}}$ can be compared. For instance,

$$[\mathrm{P}_k^{\mathrm{E}}(x_k,x'_k)=1\Rightarrow \mathrm{P}_l^{\mathrm{E}}(x_k,x'_k)=1]\Leftrightarrow \mathrm{P}_k^{\mathrm{E}}\subseteq \mathrm{P}_l^{\mathrm{E}},$$

i.e. $\mathrm{P}_k^{\mathrm{E}}$ fulfills more detail partition than $\mathrm{P}_l^{\mathrm{E}}$ and information can be analyzed with greater exactness. Using terminology of relation $\mathrm{E}(x_1,\ldots,x_n)$ we get in that case

$$\begin{aligned} &\mathrm{E}(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n) \equiv \mathrm{E}(x_1,\ldots,x_{k-1},x'_k,x_{k+1},\ldots,x_n) \Rightarrow \\ \Rightarrow\ &\mathrm{E}(x_1,\ldots,x_{l-1},x_k,x_{l+1},\ldots,x_n) \equiv \mathrm{E}(x_1,\ldots,x_{l-1},x'_k,x_{l+1},\ldots,x_n). \end{aligned}$$

Generally, on $\{1,2,\ldots,n\}$ the relation $\mathrm{E}(x_1,\ldots,x_n)$ produces the four-valued indicator function

$$f(k,l)=\begin{cases} -1, \ \mathrm{A}_k=\mathrm{A}_l, \mathrm{P}_k^{\mathrm{E}} \subseteq \mathrm{P}_l^{\mathrm{E}}; \\ \ \ 0, \ \mathrm{A}_k \neq \mathrm{A}_l, \mathrm{P}_k^{\mathrm{E}} \nparallel \mathrm{P}_l^{\mathrm{E}}; \\ \ \ 1, \ \mathrm{A}_k=\mathrm{A}_l, \mathrm{P}_k^{\mathrm{E}} = \mathrm{P}_l^{\mathrm{E}}; \\ \ \ 2, \ \mathrm{A}_k=\mathrm{A}_l, \mathrm{P}_l^{\mathrm{E}} \subseteq \mathrm{P}_k^{\mathrm{E}} \end{cases}$$

where symbol ” $\nparallel$ ” denotes relation incomparability. Let us introduce notations

$$\begin{aligned} \mathrm{X} &= \mathrm{E}(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n) \equiv \mathrm{E}(x_1,\ldots,x_{k-1},x'_k,x_{k+1},\ldots,x_n), \\ \mathrm{Y} &= \mathrm{E}(x_1,\ldots,x_{l-1},x_k,x_{l+1},\ldots,x_n) \equiv \mathrm{E}(x_1,\ldots,x_{l-1},x'_k,x_{l+1},\ldots,x_n) \end{aligned}$$

then it leds to the following sufficiently clear statement.

Proposition 1. For arbitrary n-arity relation $\mathrm{E}(x_1,\ldots,x_n)$ values of the indicator function $f(k,l)$ are specified by conditions

if $\mathrm{X} \Leftrightarrow \mathrm{Y}$ then $f(k,l)=1$,
if $\mathrm{X} \Rightarrow \mathrm{Y}$ then $f(k,l)=-1$,
if $\mathrm{X} \Leftarrow \mathrm{Y}$ then $f(k,l)=2$,
if $\mathrm{X} \nparallel \mathrm{Y}$ then $f(k,l)=0$.

Definition 1. Arbitrary n-arity relation $\mathrm{E}(x_1,\ldots,x_n)$ is said to be internally (*k*,*l*)-coherent if and only if $\mathrm{A}_k=\mathrm{A}_l$ and $\mathrm{P}_k^{\mathrm{E}}=\mathrm{P}_l^{\mathrm{E}}$.

It is reasonable to mention that equivalence relation

$$\mathrm{V}_{\mathrm{E}}(k,l)=1 \Leftrightarrow \mathrm{A}_k=\mathrm{A}_l, \mathrm{P}_k^{\mathrm{E}}=\mathrm{P}_l^{\mathrm{E}} \quad (f(k,l)=1)$$

is induced on $\{1,2,\ldots,n\}$. This relation can be expressed as matrix of internal coherence $\Phi(\mathrm{E})=(\mathrm{V}_{\mathrm{E}}(k,l))$.

Proposition 2. Under corresponding renumbering of n-arity relation $\mathrm{E}(x_1,\ldots,x_n)$ arguments, the matrix of internal coherence $\Phi(\mathrm{E})$ can be represented as block-diagonal matrix

$$\Phi(\mathrm{E})=\left(\begin{array}{cccccccc} 1 & \ldots & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \ddots & \vdots & & & & & \vdots \\ 1 & \ldots & 1 & & & & & \vdots \\ 0 & & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & 1 & \ldots & 1 \\ \vdots & & & & & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 1 & \ldots & 1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} r_1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} r_s \end{array} \tag{5}$$

where $r_1+r_2+\ldots+r_s=n$, $\begin{cases} \mathrm{A}_1=\ldots=\mathrm{A}_{r_1}=\mathrm{B}_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{A}_{n-r_s+1}=\ldots=\mathrm{A}_n=\mathrm{B}_n, \end{cases}$ $\begin{cases} \mathrm{P}_1^{\mathrm{E}}=\ldots=\mathrm{P}_{r_1}^{\mathrm{E}}=\mathrm{L}_1^{\mathrm{E}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{P}_{n-r_s+1}^{\mathrm{E}}=\ldots=\mathrm{P}_n^{\mathrm{E}}=\mathrm{L}_s^{\mathrm{E}}. \end{cases}$

Proposition 2 yields information that $\mathrm{E}(x_1,\ldots,x_n)$ has the carrier $\mathrm{B}_1^{r_1}\times\ldots\times\mathrm{B}_s^{r_s}$ and establishes s different equivalences $\mathrm{L}_i^{\mathrm{E}}$ on $\mathrm{B}_i\times\mathrm{B}_i$. From now on, $\mathrm{B}_i/\mathrm{L}_i^{\mathrm{E}}$ stands for cosets and $[\mathrm{B}_i/\mathrm{L}_i^{\mathrm{E}}]^{r_i}$ denotes the direct product of equal cosets, i.e. we can conclude that desired relation forms on $[B_1/L_1^E]^{r1}\times\ldots\times[B_s/L_s^E]^{rs}$ or actually on $B_i/L_i^E\times\ldots\times B_i/L_i^E$.

Definition 2. Arbitrary n-arity relation $E(x_1,\ldots,x_n)$ induces on $B_i/L_i^E\times\ldots\times B_i/L_i^E$ a relation E^M which will be referred to as a multirelation.

As it has already been stated above, it is important to understand that significance should be assigned to the simultaneous application of relations.

Definition 3. Two arbitrary relations n-arity $E_1(x_1,\ldots,x_n)$ on $A_1\times\ldots\times A_n$ and m-arity $E_2(x_1,\ldots,x_m)$ on $C_1\times\ldots\times C_m$ are externally (i,j)-coherent if and only if $A_i=A_j$ and $P_i^{E1}=P_j^{E2}$.

Obviously, on $\{1,2,\ldots,n\}\times\{1,2,\ldots,m\}$ an equivalence relation V_{E_1,E_2} and $(n\times m)$ matrix of external coherence $\Phi(E_1,E_2)$ are introduced similarly to the one relation case. More precisely, elements $l_{ij},\ i=\overline{1,n},\ i=\overline{1,m}$ of matrix $\Phi(E_1,E_2)$ are specified by expression

$$l_{ij}=\begin{cases}1, A_i=A_j, P_i^{E1}=P_j^{E2},\\ 0, \mathit{otherwise.}\end{cases}$$

Proposition 3. Two arbitrary relations $E_1(x_1,\ldots,x_n)$ on $A_1\times\ldots\times A_n$ and $E_2(x_1,\ldots,x_m)$ on $C_1\times\ldots\times C_m$ induce two multirelations E_1^M, E_2^M with the same carrier if and only if by rows (column) transpositions the matrix of external coherence $\Phi(E_1,E_2)$ is reduced to the block-diagonal form

$$\Phi^*(E_1,E_2)=\left(\begin{array}{cccccccc}
\overbrace{1\ \ldots\ 1}^{\beta_1} & \overbrace{0\ \ldots\ 0}^{\beta_2} & \ldots & \overbrace{\ldots\ldots}^{\beta_i} & \ldots & \overbrace{\ldots\ldots}^{\beta_s} & 0 & \\
\vdots\ \ddots\ \vdots & \vdots\ \ \vdots & & & & & \vdots & \Big\}\alpha_1\\
1\ \ldots\ 1 & 0\ \ldots\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & \\
0\ \ldots\ 0 & 1\ \ldots\ 1 & 0 & \ldots & \ldots & \ldots & 0 & \\
\vdots\ \ \ \vdots & \vdots\ \ddots\ \vdots & & & & & \vdots & \Big\}\alpha_2\\
\vdots\ \ \ 0 & 1\ \ldots\ 1 & 0 & \ldots & \ldots & \ldots & 0 & \\
\vdots\ \ \ \vdots & 0\ \ \ 0 & \ddots & & & & \vdots & \\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & 0 & 1\ \ldots\ 1 & 0 & \ldots & 0 & \\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & \vdots & \vdots\ \ddots\ \vdots & \vdots & & \vdots & \Big\}\alpha_i\\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & 0 & 1\ \ldots\ 1 & 0 & \ldots & 0 & \\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & \vdots & & \ddots & & & \\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & \vdots & & 0 & 1\ \ldots\ 1 & & \\
\vdots\ \ \ \vdots & \vdots\ \ \ \vdots & \vdots & & \vdots & \vdots\ \ddots\ \vdots & & \Big\}\alpha_s\\
0\ \ldots\ 0 & 0\ \ldots\ 0 & 0 & \ldots\ldots & 0 & 1\ \ldots\ 1 & &
\end{array}\right).$$

It should be emphasized that proposition 3 can be reformulated in terms of difunctional relations. Let us recall that a binary relation is difunctional if for all $i,i',j,j'\in\{1,2,\ldots,max(n,m)\}$ the implication

$$V_{E_1,E_2}(i,j')=1,\ V_{E_1,E_2}(i',j')=1, V_{E_1,E_2}(i',j)=1\ \Rightarrow\ V_{E_1,E_2}(i,j)=1.$$

holds.

Proposition 3*. Two arbitrary relations $E_1(x_1,\ldots,x_n)$ on $A_1\times\ldots\times A_n$ and $E_2(x_1,\ldots,x_m)$ on $C_1\times\ldots\times C_m$ induce two multirelations E_1^M, E_2^M with the same carrier if and only if V_{E_1,E_2} is a difunctional relation.

Now it is seems quite logical to assert that interpretation of multirelations may have two-valued nature. On the one hand, we have seen that a multirelation is induced by embedded properties of original information. On the other hand, data of arbitrary nature can be analyzed jointly with given equivalence relation associated with an object-oriented problems.

Necessary and sufficient conditions of multialgebraic systems existence

We have seen necessary and sufficient properties for multirelations carrier equality, however, we have not yet argued general existence conditions of multialgebraic systems. At this point it will be useful to introduce some terminology.

Let $\mathrm{A}_1,\ldots,\mathrm{A}_n,\ldots$ be arbitrary sets and let $\mathrm{P}_1,\ldots,\mathrm{P}_n,\ldots$ $(dom\,\mathrm{P}_i=\mathrm{A}_i)$ be corresponding equivalence relations then if $\mathrm{A}_i=\{x_1^i,\ldots,x_\alpha^i,\ldots\}$ we have

$$\begin{aligned}&\mathrm{A}(n)=\textstyle\prod_{i=1}^n \mathrm{A}_i,\\&\mathrm{P}(n)=\textstyle\prod_{i=1}^n \mathrm{P}_i,\\&\alpha(n)=\{\alpha_1,\ldots,\alpha_n\},\\&x_{\alpha(n)}=\{x_{\alpha_1}^1,\ldots,x_{\alpha_n}^n\}\in\mathrm{A}(n)\end{aligned}$$

and $\mathrm{P}(n)$ is the equivalence on $\mathrm{A}(n)$ in the sense that

$$\mathrm{P}(n)[x_{\alpha(n)},x_{\alpha'(n)}]=1\Leftrightarrow \mathrm{P}_i(x_{\alpha_i}^i,x_{\alpha_i}^i),\ i=\overline{1,n}\,.$$

Definition 4. An equivalence relation $\mathrm{P}(n)$ on $\mathrm{A}(n)$ will be referred to as partial factor with the notation $\mathrm{P}(n)=h-fac\,\mathrm{S}$ if and only if

$$\forall x_{\alpha(n)},x_{\alpha'(n)}\in\mathrm{A}(n):\mathrm{P}(n)[x_{\alpha(n)},x_{\alpha'(n)}]=1,\mathrm{S}(x_{\alpha(n)})=1\Rightarrow\mathrm{S}(x_{\alpha'(n)})=1\,.$$

It is obvious that the equivalence induces cosets $\mathrm{A}(n)/\mathrm{P}(n)$. If $[x_{\alpha(n)}]_{\mathrm{P}(n)}\in\mathrm{A}(n)/\mathrm{P}(n)$ is certain coset then pair $\mathrm{P}(n)$ and S defines n-arity multirelation

$$\mathrm{S}^{\mathrm{M}}([x_{\alpha(n)}]_{\mathrm{P}(n)})=1\Leftrightarrow\mathrm{S}(x_{\alpha(n)})=1\,. \qquad (6)$$

Under condition (6) a multirelation S^{M} is congruent dependent on $\mathrm{P}(n)$ and S what we denote by $\mathrm{S}^{\mathrm{M}}=con\,(\mathrm{P}(n),\mathrm{S})$ for brevity.

Remark 1. It is easy enough to understand that the definition of congruent dependence is correct if and only if $\mathrm{P}(n)=h-fac\,\mathrm{S}$.

Remark 2. It is a simple matter to show that $\mathrm{P}(n)=h-fac\,\mathrm{S}$ if and only if $\mathrm{P}(n)$ is induced by n-arity relation S satisfying (4).

Definition 5. Partial factor $\mathrm{P}(n)$ will be named factor (full factor) with notation $\mathrm{P}(n)=fac\,\mathrm{S}$ if and only if

$$\forall x_{\alpha(n-1)}\in\mathrm{A}(n-1),\forall x_{\alpha n}^n,x_{\alpha' n}^n\in\mathrm{A}(n):\ \mathrm{S}(x_{\alpha(n)})=\mathrm{S}(x_{\alpha'(n)})=1\Rightarrow\mathrm{P}_n[x_{\alpha n}^n,x_{\alpha' n}^n]=1$$

where $x_{\alpha(n)}=(x_{\alpha(n-1)},x_{\alpha n}^n)$, $x_{\alpha'(n)}=(x_{\alpha(n-1)},x_{\alpha' n}^n)$.

Consequently, we get a multioperation F^{M}

$$\mathrm{F}^{\mathrm{M}}([x_{\alpha(n)}]_{\mathrm{P}(n)})=[x_{\alpha n}^n]_{\mathrm{P}n}\Leftrightarrow\mathrm{S}(x_{\alpha(n)})=1$$

where $[x_{\alpha n}^n]_{\mathrm{P}_n}$ is the coset of the set A_n in regard to the equivalence P_n and the element $x_{\alpha n}^n$ belongs to this coset.

Remark 3. It is easy enough to see that $[x_{\alpha n}^n]_{\mathrm{P}_n}$ is unique coset. In this connection $\mathrm{F}^{\mathrm{M}}=con\,(\mathrm{P}(n),\mathrm{S})$ and $\mathrm{P}(n)=fac\,\mathrm{S}$ if and only if $\mathrm{P}(n)$ is induced by n-arity relation S satisfying (4).

The theorem of a multialgebraic system existence under given external equivalence and the same carrier had been proved [Mashtalir et al., 2003] and with mentioned notations it can be represented as follows.

Theorem 1. Suppose that A is arbitrary carrier, P is given equivalence on A^2 and $\Sigma_\mathrm{S} = \{\mathrm{P}, \mathrm{S}_1, \ldots, \mathrm{S}_\beta, \ldots\}$ is a family of n-arity relations then a model $\langle \mathrm{A}, \Sigma_\mathrm{S} \rangle$ generates multialgebraic system $\langle \mathrm{A/P}, \{\mathrm{F}_\xi^\mathrm{M}\}, \{\mathrm{S}_\eta^\mathrm{M}\} \rangle$ where $\mathrm{F}_\xi^\mathrm{M} = con\,(\mathrm{P}^n, \mathrm{S}_\xi)$, $\mathrm{S}_\eta^\mathrm{M} = con\,(\mathrm{P}^n, \mathrm{S}_\eta)$ and

$$\exists \Sigma_{1\mathrm{S}}, \Sigma_{2\mathrm{S}} \subset \Sigma_\mathrm{S} \ : \ \Sigma_{1\mathrm{S}} \cap \Sigma_{2\mathrm{S}} = \varnothing, \ \Sigma_{1\mathrm{S}} \cup \Sigma_{2\mathrm{S}} = \Sigma_\mathrm{S} \setminus \mathrm{P}, \quad \mathrm{S}_\xi \in \Sigma_{1\mathrm{S}}, \ \mathrm{S}_\eta \in \Sigma_{2\mathrm{S}}$$

if and only if

$$\forall \mathrm{S}_\beta \in \Sigma_\mathrm{S} \setminus \mathrm{P} \Rightarrow \mathrm{P}^n = \begin{cases} h-fac\,\mathrm{S}_\beta, & \mathrm{S}_\beta \in \Sigma_{1\mathrm{S}} \setminus \mathrm{P}, \\ fac\,\mathrm{S}_\beta, & \mathrm{S}_\beta \in \Sigma_{2\mathrm{S}} \setminus \mathrm{P}. \end{cases}$$

It should be emphasized that any n-arity relation E forms its equivalence $\mathrm{L}^\mathrm{E} = \prod_{i=1}^{s} \mathrm{L}_i^\mathrm{E}$ on the carrier $\mathrm{B}_1 \times \ldots \times \mathrm{B}_s$ (see the explication of expression (5)) which is determined by the matrix of internal coherence. Further, the carrier structure is direct product of matrix blocks. Hence, the consideration of relations and matrices of external coherence by pairs gives possibilities to establish conditions that due to proposition 3* all pairs $(\mathrm{S}_{\beta'}, \mathrm{S}_{\beta''})$ from this collection represent difunctional relations $\mathrm{V}_{\mathrm{S}_{\beta'}, \mathrm{S}_{\beta''}}$. Granting remarks 1–3, we can restate theorem 1 and give more strong assertion of necessary and sufficient conditions for multialgebraic systems existence.

Theorem 2. Let $\{\mathrm{S}_1, \ldots, \mathrm{S}_\beta, \ldots\}$ be a family of arbitrary arity relations whose carriers may be different then multialgebraic system is induced if and only if

i) $\Sigma_\mathrm{S} = \{\mathrm{L}^\mathrm{E}, \mathrm{S}_1, \ldots, \mathrm{S}_\beta, \ldots\}$, L^E is an equivalence induced by $\mathrm{S}_1, \ldots, \mathrm{S}_\beta, \ldots$,

ii) $\exists \Sigma_{1\mathrm{S}}, \Sigma_{2\mathrm{S}} \subset \Sigma_\mathrm{S} : \Sigma_{1\mathrm{S}} \cap \Sigma_{2\mathrm{S}} = \varnothing, \Sigma_{1\mathrm{S}} \cup \Sigma_{2\mathrm{S}} = \Sigma_\mathrm{S} \setminus \mathrm{L}^\mathrm{E}$,

iii) $\forall \mathrm{S}_\beta \in \Sigma_\mathrm{S} \setminus \mathrm{L}^\mathrm{E} \Rightarrow \underbrace{\mathrm{L}^\mathrm{E} \times \ldots \times \mathrm{L}^\mathrm{E}}_{n} = \begin{cases} h-fac\,\mathrm{S}_\beta, & \mathrm{S}_\beta \in \Sigma_{1\mathrm{S}} \setminus \mathrm{L}^\mathrm{E}, \\ fac\,\mathrm{S}_\beta, & \mathrm{S}_\beta \in \Sigma_{2\mathrm{S}} \setminus \mathrm{L}^\mathrm{E}, \end{cases}$

iv) $\forall \mathrm{S}_{\beta'}, \mathrm{S}_{\beta''} \in \Sigma_\mathrm{S} \setminus \mathrm{L}^\mathrm{E}$ and $\mathrm{V}_{\mathrm{S}_{\beta'}, \mathrm{S}_{\beta''}}$ is difunctional relation.

Thus, a factorization of information in any feature space conceptually is one of the basic methods providing an interpretation of data. On the one hand, identification can be required up to given or explored equivalence relations set. With another, construction of equivalence classes often represents an essence and a purpose of data processing. We have introduced and proved conditions describing interdependence of different levels information representations.

Conclusion

Different types of granulation represent different aspects of data and provide different types of knowledge embedded in data. An intelligent data analysis based on granular computing deals with theories, methodologies, techniques and tools that provide consideration what is relevant and permit to ignore irrelevant details. Granular computing involves two-way communications upward and downward in a hierarchy of different abstraction levels that represent different granulated views of problems understanding. It is reasonable to assume that granules relations satisfying various axiomatics and operations with operands corresponding to granules offer advantages for formalization of transformations and interpretations in multilevel processing of arbitrary nature data.

There exist two distinct varieties of relations concerning data to be analyzed. First of all, we should emphasized internal (embedded) interrelationships of original data. Thus, latent information that induces relationships between granules has to be explored. In the second place, a relation concerned with applications can be introduced on the original data. In both cases joint analysis has to be carried into effect.

Usually there are possibilities of empirical verification of the properties only at the lower level of abstractions, i.e. with the use of original data. Partitions and coverings can be normally valid models of granulation, and properties of relations along with operations in conformity with equivalence or tolerance classes generate a basic interest. In other words, the problem consists in an examination of original data to know properties of granule families. In our opinion multialgebraic systems can be sufficiently adequate tools in order to formalize elements of detailing or roughening such as granule, granulated view, granularity and hierarchy in the framework at least formal mathematical structures. We have established necessary and sufficient conditions of producing relations (with the same carrier) on granules induced by relations associated with original data. Furthermore, we have found conditions of multialgebraic systems existence. As development of these results it should be indicated the investigation of specific algebraic structures on original data such as semigroup, group, ring, different vector spaces etc. There arise several problems (it seems that peculiar but, vice versa, very important). Among them it should be noted comparisons of granule families for which there are no two ways about an introduction of an admissible metric on granule structures, e.g. on set partitions [Bobrowsky et al., 2006, Mashtalir et al., 2006] since it is often necessary to have dealings with a whole family of partitions and we have to be able to compare these partitions.

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