THE METRICS AND MEASURE OF REFUTABILITY ON FORMULAS IN THE THEORY T

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Abstract: The paper discusses statements of experts about objects represented as the formulas in language L=L(T) some theory T and offers techniques for introducing metrics on such statements and measure of refutability. The research will find a use to problems of the best matching of the statements, of construction the decision functions of pattern recognition and development of expert's systems. The offered refutability functions satisfy all requirements (for informativeness) formulated in [1, 2].

Keywords: pattern recognition, distance between statements, theory of models, metrics, know base.

ACM Classification Keywords: I.2.6. Artificial Intelligence - knowledge acquisition.

Introduction

The theory and methods of constructing decision function of pattern recognition on the basis of an analysis of empirical information represented as tabulated data have been well advanced by now. In addition to this there is an increasing interest in the construction of decision function on the basis of an analysis of experts information provided in the form of logical "knowledge" of several experts which are represented by the predicate formulas [1-2, 5-6] (these "knowledge" can be partly or completely contradictory) in some L=L(T). It involves the problem of matching the statements of the experts about hierarchical objects and also the problems of introducing a distance on these statements and of defining their refutability. Obviously the statements ("knowledge") can differ on quantity of refutability contained in them. The refutability reflects importance of the information informed by the expert.

This work is a natural continuation of works [1-2] and the familiarity with them is supposed. The paper discusses logical statements of experts about hierarchical objects recorded as the logical predicate formulas. By making use of the methods of mathematical logic and of the model theory we offer the techniques for introducing metrics on these statements and measure of their refutability . We study the properties of entered metrics and connected to them measures. The work was supported by the Russia Foundation for Basic Research № 07-01-00331 and by program "Logika" of Novosibirsk State University.

The distances on the formulas and its properties

Let L = L(T) be a first order language consisting of final number of predicate symbol which are selected for record and study the connections between variables in particular application area. For each variable x_i there is the unary predicate P_{x_i} determining only on the range of values of variable x_i on the set A_n. Let A_n be nonempty set of power n ($\leq n$). A_n is the join of all values of considering variables which are included in the predicates. In many applied problems the value of variables and their number are final and so the model with final set are considered.

Definition 0.[5-6]. The interpretation γ is the map putting in conformity to each symbol $P_i^{n_i}$ of signature Ω of language L ($\Omega = \langle P_1^{n_1}, P_2^{n_2}, \ldots \rangle$, where $P_i^{n_i}$ - n_i-ary predicate) particular n_i-ary relation $P_i^{A_n}$ determined on the set A_n. It allows to speak about model $\langle A_n; \Omega^{A_n} \rangle$ of signature Ω. Let's consider models only of final signature.

Let there is a final number s of the experts and the area of possible values of variables. The models (in sense of model theory) are set by experts. Each expert j sets the interpretation of each predicate symbol $P_i^{n_i}$ of the language L by the appropriate relation $\gamma\left(P_i^{n_i}\right) = P_i^{\mathrm{M}_j} \subseteq A_n^{n_i}$ in model M_j. Then we have "knowledge" of the experts recorded as the formula which are set by formula's subset in each model M_j [4-5] under the interpretation of the expert j.

Let $Mod_n(L)$ be a set of all models T of the language L (T) determined on the set A_n by the experts.

We shall consider σ- algebra of subset F on the set $\mathrm{A} = A_n^{<\omega} = \bigcup\limits_{k<\omega} A_n^k$ ($A_n^k = A_n \times \ldots \times A_n$). Only those subset S_j from F will be interested us for which the formula ψ of language L, appropriated to "knowledge" of the experts, will be discovered such that under the interpretation of "knowledge" of expert j ($j = 1, \ldots, s$) $S_j = \psi(\mathrm{M}_j) = \{\bar{a} \mid \mathrm{M}_j \models \psi(\bar{a})\}$ (that is formula's subset which is appropriated to the formula ψ in model M_j [5-6]).

Let U be a set of all predicate symbols used by the experts; B- is the closure of the set U under logical operations $\neg$, $\wedge$, $\vee$, $\rightarrow$ and quantors $\forall$ and $\exists$ on variables. Obviously, the set of formulas interesting for us is contained in B.

Definition 1. [7]. By the probabilistic measure μ on the set B we mean a function $\mu : B \rightarrow$ [0,1] satisfying conditions for ϕ and $\psi \in B$:

1) if $\models \phi \equiv \psi$, then $\mu(\phi) = \mu(\psi)$; 2) if $\models \phi$, then $\mu(\phi) = 1$;

3) if $\models \neg\phi$, then $\mu(\phi) = 0$; 4) $\mu(\neg\phi) = 1 - \mu(\phi)$;

5) if $\models \neg(\phi \wedge \psi)$, then $\mu(\phi \vee \psi) = \mu(\phi) + \mu(\psi)$.

Let a probabilistic measure μ is set on sets from F.

Instead of "knowledge" of the expert recorded as predicate $P^{\mathbf{M}_i}$ in model $\mathbf{M}_i$ further we shall consider it approximation – predicate $\widetilde{P}^{\mathbf{M}_i}$.

Under approximation $\widetilde{P}^{\mathbf{M}_i}$ of predicate $P^{\mathbf{M}_i}$ is understood closer definition of the domain of truth of this predicate in model $\mathbf{M}_i$ by one of ways:

1) to leave the relation without changes;

2) to eliminate these elements from $P^{\mathbf{M}_i}$ in which truth the expert i is not absolutely sure ;

3) to add in relation new elements and eliminate some old, for example, with allowance for the "knowledge" of other experts;

4) to execute items 2) and 3) simultaneously.

Let's enter a distance on a set of "knowledge" of the experts with the help of models which are set be them. The models differ by the interpretations.

Let's define a distance between the formula's subset (predicates) in each model for T $\mathrm{M}_i \in Mod_n(L)$ as a measure of their symmetrical difference.

Definition 2. We call $\rho_{\mathbf{M}_i}\left(P_k^{\mathbf{M}_i}, P_j^{\mathbf{M}_i}\right) = \mu\left(\tilde{P}_k^{\mathbf{M}_i} \Delta \tilde{P}_j^{\mathbf{M}_i}\right)$ the distance between predicates $P_k^{\mathbf{M}_i}$ and $P_j^{\mathbf{M}_i}$ determined in model $\mathbf{M}_i$.

Remark. This definition is correct if the predicates of equal arity and with an identical set of variables. If the considered predicates have different arity or different set of variables and if the expert consider insignificant the absent in one of the formulas variable x_i wt suppose that it receives anyone from possible values. Otherwise (if it is significant) we determine this variable by adding to the necessary formula the predicate P_{x_i} selecting values of this variable. It's clear that the entered concept is easily spreaded on formula's subset. Further we shall study a distance between the formulas of the same variables (equal arity).

The distance between the formulas determined on the set of models $Mod_n(L)$ we shall define as mean on a set of distances in models

Definition 3. We call $\rho_1\left(P_k(\overline{x}), P_j(\overline{x})\right) = \dfrac{\sum_{\mathrm{M}_i \in Mod_n(L)} \rho_{\mathrm{M}_i}\left(P_k^{\mathrm{M}_i}, P_j^{\mathrm{M}_i}\right)}{\left|Mod_n(L)\right|}$ the distance between the formulas $P_k(\overline{x})$ and $P_j(\overline{x})$ determined on the set $Mod_n(L)$.

Now we shall consider a way of definition of a distance between the sentences (closed formulas).

We denote by $Mod(\phi)$ the set of models in $Mod_n(L)$ on which the sentence ϕ is true, that is $Mod(\phi) = \{\mathbf{M}_i \in Mod_n(L) \mid \mathbf{M}_i \models \phi\}$.

Clear, there are such models for which the sentence is true and such models for which it is false (if it is not a tautology). It is natural to measure the difference of information contained in the sentences by the number of the models on which sentences have different truth values.

Definition 4. We call $\rho_2(\phi, \psi) = \dfrac{\left|Mod\left((\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)\right)\right|}{\left|Mod_n(L)\right|}$ the distance between the sentences ϕ and ψ.

Let's consider one more way of definition of a distance between the formulas. Let's add the first order language L by constant from the set $M = Mod_n(L)$. For this set M we shall consider any tuples $\overline{a}$ which lengths are equal to arity of the formulas $l(\overline{a})$. At the substitution of tuples in the formulas in the supposition that the formulas have identical arity (as it to achieve was indicated above) the formulas become the sentences.

Definition 5. We call $\rho_3(\phi(\overline{x}), \psi(\overline{x})) = \min_{\overline{a} \in M^{l(\overline{a})}} \rho_2(\phi(\overline{a}), \psi(\overline{a}))$ the distance between the formulas $\phi(\overline{x})$ and $\psi(\overline{x})$.

The following theorem is proved and from theorem follows that the offered distances are the metrics really. Some additional properties of entered distances are proved in the theorem.

Theorem 1. For any formulas ("knowledge" of the experts) ϕ, ψ, χ and for any function ρ_i on T the following assertions are valid:

1. $0 \le \rho_i(\phi, \psi) \le 1.$ 2. $\rho_i(\phi, \psi) = \rho_i(\psi, \phi)$ (symmetry).

3. If $\rho_i(\phi, \psi) = \rho_i(\phi_1, \psi_1)$ and $\rho_i(\phi_1, \psi_1) = \rho_i(\phi_2, \psi_2)$ then $\rho_i(\phi, \psi) = \rho_i(\phi_2, \psi_2)$ (transitivity equality).

4. $\rho_i(\phi, \psi) \le \rho_i(\phi, \chi) + \rho_i(\chi, \psi)$ (nonequality of a triangle).

5. $\phi \equiv \psi \Leftrightarrow \rho_i(\phi, \psi) = 0$ ($\phi \equiv \psi$ here and further denotes equivalence of the formulas concerning to all models of the experts, that is for anyone expert i (assigning model $\mathbf{M}_i$) $\phi^{\mathbf{M}_i} = \psi^{\mathbf{M}_i}$ correctly).

6. $\phi \equiv \neg\psi \Rightarrow \rho_i(\phi, \psi) = 1.$ 7. $\rho_i(\phi, \psi) = 1 - \rho_i(\phi, \neg\psi) = \rho_i(\neg\phi, \neg\psi)$.

8. $\rho_i(\phi, \psi) = \rho_i(\phi \wedge \psi, \phi \vee \psi)$. 9. $\rho_i(\phi, \neg\phi) = \rho_i(\phi, \psi) + \rho_i(\psi, \neg\phi)$.

The proof of the theorem follows from definitions, properties of a probabilistic measure and logical evaluations.

The measure of refutability and their properties

From the point of view of importance of the information messaged by the expert it is natural to assume that the more above refutability (informativeness) of the formula, the smaller the number of models on which it is executed (the smaller measure of the set on which the formula is true). Therefore we shall enter the refutability as follows.

Definition 6. We call $I_i(P(\overline{x})) = \rho_i(P(\overline{x}), 1)$ the measure of refutability of formula $P(\overline{x})$, where 1 is the identical true predicate, that is $\overline{x} = \overline{x}$.

For entered distances obtained :

$$I_i(P)=\begin{cases} \dfrac{\sum\limits_{\mathbf{M}_i \in Mod_n(L)} \mu\left(\neg P^{\mathbf{M}_i}\right)}{\left|Mod_n(L)\right|}, & if\ \ \rho_1 \\ \dfrac{\left|Mod\left(\neg P\right)\right|}{\left|Mod_n(L)\right|}, & if\ \ \rho_2 \\ \dfrac{\min\limits_{\bar{a} \in M^{l(\bar{a})}} \left|Mod\left(\neg P(\bar{a})\right)\right|}{\left|Mod_n(L)\right|}, & if\ \ \rho_3 \end{cases}$$

The following theorem is proved:

Theorem 2. For any formulas ("knowledge" of the experts) ϕ , ψ and anyone ρ_i on T the following assertions are valid:

1. $0 \le I_i(\phi) \le 1.$ 2. $I_i(1) = 0.$

3. $I_i(0) = 1$. 4. $I_i(\phi) = 1 - I_i(\neg\phi).$

5. $I_i(\phi) \le I_i(\phi \wedge \psi).$ 6. $I_i(\phi) \ge I_i(\phi \vee \psi).$

7. $I_i(\phi \wedge \psi) = \rho_i(\phi,\psi) + I_i(\phi \vee \psi).$ 8. If $\phi \equiv \psi$, then $I_i(\phi) = I_i(\psi).$

9. If $\rho_i(\phi,\psi) = 0$, then $I_i(\phi \wedge \psi) = I_i(\phi \vee \psi) = I_i(\phi).$

10. $I_i(\phi \wedge \psi) = \dfrac{I_i(\phi) + I_i(\psi) + \rho_i(\phi,\psi)}{2}.$

11. $I_i(\phi \vee \psi) = \dfrac{I_i(\phi) + I_i(\psi) - \rho_i(\phi,\psi)}{2}.$

The entered definitions, formulated above the properties of the metric and logical evaluations are used for proof of this theorem (analog [2, 4]).

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