IDENTIFICATION AND OPTIMAL CONTROL OF SYSTEM DESCRIBED BY QUASILINEAR PARABOLIC EQUATIONS

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Abstract: This paper treats the problem of control of a quasilinear parabolic equation with controls in the coefficients, in the boundary conditions and in the right side of the equation. The difference approximations problem is constructed. The minimization of the modified cost function is found by applying the penalty method combined with PQI method. The problem of identification of the unknown coefficients for a heat exchange process is solved numerically. Numerical results are reported.

Keywords: Optimal control, parabolic equations, Finite difference method, Stability theory, PQI method.

1. Introduction

An interesting and well investigated problem is the identification of coefficients in partial differential equations. In constract to this, the identification of nonlinear phenomena is less developed. This refers also to the nonlinear boundary conditions for the heat equation. The technical background consists in the identification of the heat exchange coefficient in our case. There are many papers dealing with the identification problem. Methods of solving of these problems are listed in the following:

1) Gradient or conjugate gradient methods, for example, [Farag, 2006].
2) The lagrangian method, for example, [Arada, 2003].
3) The regularization methods, for example, [Lur'e, 1995].

This paper treats the problem of control of a quasilinear parabolic equation with controls in the coefficients, in the boundary conditions and in the right side of the equation. The difference approximations problem is constructed. The minimization of the modified cost function is found by applying the penalty method combined with PQI method [El-Gendi,1995]. The problem of identification of the unknown coefficients for a heat exchange process is solved numerically. Numerical results are reported.

2. Control Problem of Heat Equation

Let D be a bounded domain of the N-dimensional Euclidean space E_N , l , T be given positive numbers, let $\Omega = \{(x,t) : x \in D \ , t \ \in \}$. Let $V = \{v : v = v_1, v_2, \ldots, v_N \in E_N, \ \|v\|_{E_N} \le R\}$ where R > 0 are given numbers. We consider the heat exchange process described by the equation

$$(1) \qquad \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(u,v)\frac{\partial u}{\partial x}\right) = f(x,t,u,v) \ , \ (x,t) \in \Omega$$

with initial and boundary conditions

$$(2) \qquad u(x,t)\big|_{t=0} = \phi(x) \ , \ x \in D$$

(3) $$\lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=0}=Y_0(t),\ \lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=l}=Y_1(t)\ ,\ t\in[0,T],$$

where $\phi(x),\ Y_0(t)\ ,Y_1(t)\in L_2(0,T)$. Besides, the functions $\lambda(u,v),\ f(u,v)$ are continuous on $(u,v)\in[r_1,r_2]\times E_N$, have continuous derivatives in u and $\forall\ (u,v)\in[r_1,r_2]\times E_N$ the derivatives $\frac{\partial\lambda(u,v)}{\partial u},\ \frac{\partial f(u,v)}{\partial u}$ are bounded and $[r_1,r_2$ are given numbers.

On The set V ,under the conditions (1)-(3) and additional restrictions

(4) $$\xi_0\ \le\ \lambda(u,v)\ \le\mu_0\quad,\qquad r_1\ \le\ u(x,t)\ \le\ r_2$$

is required to minimized the functional

(5) $$J_\alpha(v)\ =\ \beta_0\int_0^T\ [u(0,t)-y_0(t)]^2 + \beta_1\int_0^T\ [u(l,t)-y_1(t)]^2+\ \alpha\ \|v\ -\ \omega\ \|^2_{E_N}$$

where $y_0(t)\ ,\ y_1(t)\in L_2(0,T)$ are given numbers, $\alpha\ge0,\ \beta_i\ge0, i=\overline{0,2}\ ,\ \beta_0+\beta_1\ne0$ are given numbers, $\omega=\{\omega_1\ ,\omega_2\ ,\ldots,\omega_N\ \}\in E_N$ is also given.

<u>Definition 2.1:</u> The solution of problem (1)-(4) is the function $u(x,t)\in V_2^{1,0}(\Omega)$ and satisfies the integral

(6) $$\int_0^l\int_0^T\ [u\frac{\partial\eta}{\partial t}-\ \lambda(u,v)\ \frac{\partial u}{\partial x}\frac{\partial\eta}{\partial x}+\eta\ \ f(x,t,u,v)\ ]\ dx\ dt = -\int_0^l\phi(x)\ \eta(x,0)\ dx\ -\int_0^T Y_0(x)\ \eta(0,t)\ dt\ -\int_0^T Y_1(x)\ \eta(l,t)\ dt$$

$\forall\ \eta=\eta(x,t)\in W_2^{1,1}(\Omega)\ and\ \ \eta(x,T)=0.$

In [Farag, 2002], the existence and uniqueness of the solution of problem (1)-(5) are studied. Optimal control problems of the coefficients of differential equations do not always have solution [Tikhonov, 1974].

3. Modified Optimal Control Problem

It is well known that the penalty function methods are very effective techniques for solving constrained optimization problems via unconstrained problems. In recent years, these methods have been widely used to solve constrained optimal control problems. Applications of the exterior, interior and mixed penalty function methods in constrained optimal control problems can then be found. The inequality constrained problem (1) through (5) is converted to a problem without inequality constraints by adding a penalty function [Ibeijuba, 1983] to the objective (5), yielding the following $\Phi_{\alpha,s}(v,A_s)$ function:

(7) $$\Phi_{\alpha,s}(v, A_s) = J_\alpha(v) + P_s(v)$$

$$P_s(v) = A_s \int_0^l \int_0^T [\ F(u,v) + Q(u) + B(u)]\ dx\ dt$$

$$F(u,v) = [\ \max\ \{\xi_0 - \lambda(u,v), 0\}]^2 + [\ \max\ \{\lambda(u,v) - \mu_0, 0\}]^2$$

$$Q(u) = [\ \max\ \{r_0 - u(x,t;v), 0\}]^2,\ \ B(u) = [\ \max\ \{u(x,t;v) - r_2, 0\}]^2$$

where $A_s,\ s = 1,2,\ldots$ are positive numbers, $\lim\limits_{s\to\infty} A_s = +\infty$.

4. The Discrete Control Problem

Difference methods of solution of optimal control problems for partial differential equations are investigated comparatively small [Potapov,1978]and [Tagiev, 1982].

In this section, we shall find the discrete control problem for (7), (1)-(3) and two theorems prove the estimates of stability for the difference approximations problem (22)-(25) and an estimate on v . From [Ladyzenskaya, 1973], we give the net norms for the net functions

$[Z] = \{((Z_1)_i^j, \ldots, ((Z_m)_i^j\},\ i = \overline{0,N},\ j = \overline{0,M}$ with m components: $\|Z\|_{L_2(\overline{\omega}_{h\tau})} = [h\tau \sum\limits_{i=0}^{N-1} \sum\limits_{j=1}^{M} (Z_i^j)^2\ ]^{\frac{1}{2}}$, $\max\limits_j \|Z\|_{L_2(\overline{\omega}_h)} = [h \sum\limits_{i=0}^{N-1} (Z_i^j)^2\ ]^{\frac{1}{2}}$, for the net functions $[Z] = \{((Z_1)_i, \ldots, ((Z_m)_i\}, i = \overline{0,N}$ the norm is $\|Z\|_{L_2(\overline{\omega}_h)} = [h \sum\limits_{i=0}^{N-1} (Z_i)^2\ ]^{\frac{1}{2}}$ and also for the net functions $[Z] = \{((Z_1)^j, \ldots, ((Z_m)^j\},\ j = \overline{0,M}$ the norm is $\|Z\|_{L_2(\overline{\omega}_\tau)} = [\tau \sum\limits_{j=1}^{M} (Z_j)^2\ ]^{\frac{1}{2}}$.

For discretization the optimal control problem (1)-(3),(7) in $\overline{\Omega}$ we introduce the net $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$ where $\overline{\omega}_h = \{x_i = i\,h,\ i = \overline{0,N},\ N\ h = l\},\ \overline{\omega}_\tau = \{t_j = j\,\tau,\ i = \overline{0,M},\ M\tau = T\}$

Here and further for arbitrary net functions $u = u_i^j = u(x,t) = u(x_i, t_j), x = x_i \in \overline{\omega}_h, t = t_j \in \overline{\omega}_\tau$ adopt denotations [Ladyzenskaya, 1973]:

$$\hat{u} = u_i^{j+1} = u(x_i, t_{j+1}), u^* = u_i^{j-1} = u(x_i, t_{j-1}), u^- = u_{i-1}^j = u(x_{i-1}, t_j)$$

$$u^+ = u_{i+1}^j = u(x_{i+1}, t_j), u_x = \frac{u^+ - u}{h}, u_{\bar{x}} = \frac{u - u^-}{h}, u_t = \frac{\hat{u} - u}{\tau}, u_{\bar{x}} = \frac{u - u^*}{\tau}$$

The given functions in (6) approximate as follows:

$$\lambda_i^j = \frac{1}{h\tau} \int_{t_{j-1}}^{t_j} \int_{x_i}^{x_{i+1}} \lambda(u(x,t),v)\ dx\, dt,\ i = \overline{0,N-1},\ j = \overline{1,M}$$

$$f_i^j = \frac{1}{h\tau} \int_{t_{j-1}}^{t_j} \int_{x_i}^{x_{i+1}} f(u(x,t),v)\ dx\, dt,\ i = \overline{0,N-1},\ j = \overline{1,M}$$

$$\phi_i = \frac{1}{h}\int_{x_j}^{x_{i+1}} \phi(x)\ dx\ ,\ i = \overline{0, N-1},$$

$$(Y_0)^j = \frac{1}{\tau}\int_{t_{j-\frac{\tau}{2}}}^{t_{j+\frac{\tau}{2}}} Y_0(t)\ dt\ , (Y_1)^j = \frac{1}{\tau}\int_{t_{j-\frac{\tau}{2}}}^{t_{j+\frac{\tau}{2}}} Y_1(t)\ dt\ ,\ j = \overline{1, M}$$

The discrete analogy of the integral identity (6) writes in the form

$$(8)\quad h\tau \sum_{i=0}^{N-1}\sum_{j=1}^{M} u_i^j\,(\eta_i^j)_t - \sum_{i=0}^{N-1}\sum_{j=1}^{M}[\,-\lambda_i^j (u_i^j)_x (\eta_i^j)_x + f_i^j \eta_i^j\,]$$
$$= -h\sum_{i=0}^{N-1}\phi_i\ \eta_i^0 - \tau\sum_{j=1}^{M}\eta_0^j\,(Y_0)^j - \tau\sum_{j=1}^{M}\eta_N^j\,(Y_1)^j$$

for any network function η_i^j, $\eta_i^M = 0$.

From [Samarski, 1992] and equality to zero the coefficients of η_i^j, we obtain the difference approximations problem for (1)-(3):

$$(9)\quad \begin{cases} (u_i^j)_{\bar t} - (\lambda_i^j (u_i^j)_{\bar x})_{\bar x} = f_i^j,\ i = \overline{0, N-1},\ j = \overline{1, M} \\ u_i^0 = \phi_i,\ i = \overline{0, N-1} \\ -\lambda_0^j (u_0^j)_x - (Y_0)^j,\ j = \overline{1, M} \\ -\lambda_{N-1}^j (u_{N-1}^j)_x - (Y_1)^j = -h\ f_0^j - h\,(u_0^j)_{\bar t},\ j = \overline{1, M} \end{cases}$$

But the functional (7) is approximated by the following way:

$$(10)\quad DF(v) = \beta_0\ \tau\sum_{j=1}^{M}\ [u_0^j - (y_0)^j]^2 + \beta_0\ \tau\sum_{j=1}^{M}\ [u_N^j - (y_1)^j]^2$$
$$+ h\tau\sum_{i=0}^{N-1}\sum_{j=1}^{M}\ [F(u_i^j, v) + Q(u_i^j) + B(u_i^j)]$$

A similar theorems 1,2 in [Farag,2004], the estimates of stability for the difference approximations problem (9) and an estimate on V are given as follows:

Theorem 4.1

Suppose that that the all functions in the system (1)-(4) satisfy the above enumerated conditions. Besides, $\lambda(u,v),\ f(u,v)$ satisfy the Lipschits conditionon V with constant L, $\forall\ (x,t) \in \Omega$, and for any $v \in V$. Then the estimates of stability for the difference approximations problem (9) are

$$(11)\quad \begin{cases} \|u\|^2_{L_2(\bar\Omega_{h\tau})} \le C_2\ [\|\phi\|^2_{L_2(\bar\Omega_h)} + \|Y_0\|^2_{L_2(\bar\Omega_\tau)} + \|Y_1\|^2_{L_2(\bar\Omega_\tau)}] \\ \|u_x\|^2_{L_2(\bar\Omega_{h\tau})} \le C_2\ [\|\phi\|^2_{L_2(\bar\Omega_h)} + \|Y_0\|^2_{L_2(\bar\Omega_\tau)} + \|Y_1\|^2_{L_2(\bar\Omega_\tau)}] \\ \max_j \|u^j\|^2_{L_2(\bar\Omega_h)} \le C_2\ [\|\phi\|^2_{L_2(\bar\Omega_h)} + \|Y_0\|^2_{L_2(\bar\Omega_\tau)} + \|Y_1\|^2_{L_2(\bar\Omega_\tau)}] \end{cases}$$

where C_2 is a constant depending only the constants in (1)-(4) and L.

The proof of theorem 4.1 is then directly followed by theorem 1 in [Farag, 2004].

Theorem 4.2

Suppose that that the all functions in the system (1)-(4) satsify the above enumerated conditions. Besides, $\lambda(u,v),\ f(u,v)$ satisfy the Lipschits conditionon V with constant L , $\forall\ (x,t) \in \Omega$ and for any $v \in V$. Then the stability estimation of the solution of difference approximations problem (9) on $v \in V$ are

$$(12)\ h\sum_{i=1}^{N-1}(\delta u_i^j)^2 + h\tau\sum_{i=0}^{N-1}\sum_{j=0}^{M}(\delta u_i^j)^2 + h\tau\sum_{i=0}^{N-1}\sum_{j=0}^{M}(\delta u_i^j)_x^2 \le C_3\,\|\delta v\|_{E_N}^2$$

where C_3 is a constant depending only the constants in (1)-(4) and L .

The proof of theorem 4.2 is then directly followed by theorem 2 in [Farag, 2004].

5. Numerical Results

5.1 Numerical Approach

The outlined of the algorithm for solving OCP problem are as follows:

1- Given $it = 0$, $\varepsilon_1 > 0$, $\varepsilon_1 > 0$, $A_{it} > 0$, $v^{it} \in V$.

2- At each iteration it do

2.1 - Solve (1)-(3), then find $u(.,v^{it})$.

2.2 - Minimize the functional (10) using PQI method.

2.3 - Find optimal control v_*^{it+1} .

End do.

3- If $\left\| DF(v^{it+1}) - DF(v^{it}) \right\| < \varepsilon_1$, then Stop, else, go to Step 4.

4- Set $v^{it+1} = v^{it}$, $A_{it+1} = \varepsilon_2\ A_{it}$, $it = it + 1$ and go to Step 2.

5.2 Numerical Example

The numerical results were carried out for the following example of exact solution:

$$u(x,t) = x + t\,,\ \lambda(u,v) = \frac{1}{1+u^2}\ ,\ x \in [0,0.8]\ , t \in [\ 0,\ 0.001]$$

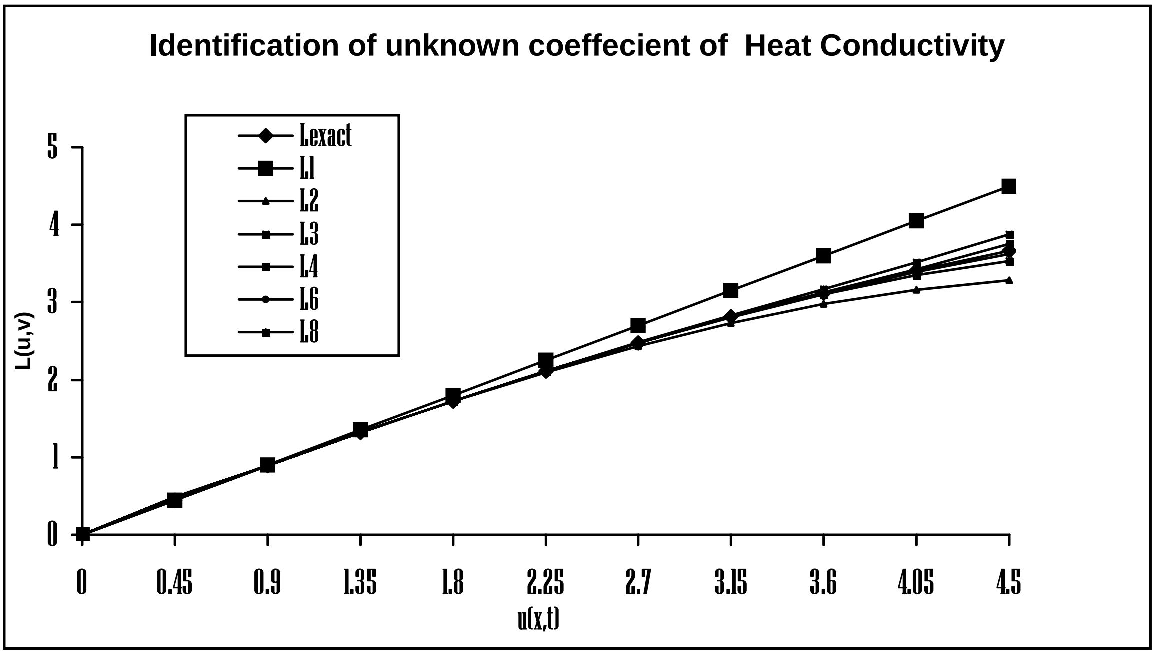
The iteration number it for the function to be minimized $DF(v)$, the exact, approximate values of $\lambda(u,v)$ with v^* tabulated in table 1 and the absolute error $\chi = \left| \dfrac{\lambda_{Exact} - \lambda_{Approx}}{\lambda_{Exact}} \right|$ also.

Table 1

it	λ_{Exact}	λ_{Approx}	$\chi = \left\lvert \dfrac{\lambda_{Exact} - \lambda_{Approx}}{\lambda_{Exact}} \right\rvert$
1	.7352941E+00	.1224296E+00	.8334957E+00
2	.7352941E+00	.4115356E+00	.4403116E+00
3	.7352941E+00	.4743363E+00	.3549026E+00

4	.7352941E+00	.6568843E+00	.1066374E+00
5	.7352941E+00	.7134509E+00	.2970675E-01
6	.7352941E+00	.7150049E+00	.2759337E-01

In Table 2, we report the number NEF of function evaluations needed to attain the solution with accuracy on the modified function $DF(v)$ of the order 10^{-6}. The above algorithm takes six iterations for decreasing $DF(v)$ to the value 0.8393609E-04. Knowing the computed optimal control values v^* obtained by using the previous numerical algorithm, we can calculate the approximate values of the unknown coefficient $\lambda(u,v)$ can be represented in a series as $\lambda(u,v)=\sum_{i=1}^{k} u^k v_k$. In the following Figure the curves denoted by L_1 , L_2 , ... *and* L_{Exact} are the approximate and exact values of $\lambda(u,v)$. Obviously by increasing k the coefficient $\lambda(u,v)$ will agree with precise ones.



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