LAGRANGEAN APPROXIMATION FOR COMBINATORIAL INVERSE PROBLEMS

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***Abstract**: Various combinatorial problems are effectively modelled in terms of (0,1) matrices. Origins are coming from n-cube geometry, hypergraph theory, inverse tomography problems, or directly from different models of application problems. Basically these problems are NP-complete. The paper considers a set of such problems and introduces approximation algorithms for their solutions applying Lagragean relaxation and related set of techniques.*

***Keywords**: Approximation algorithms, Lagragean relaxation*

***ACM Classification Keywords:** G.2.1 Discrete mathematics: Combinatorics*

***Conference**: The paper is selected from Sixth International Conference on Information Research and Applications – i.Tech 2008, Varna, Bulgaria, June-July 2008*

1. Introduction

A set of diverse combinatorial problems are defined and investigated in terms of discrete structures; in the simplest case these are $(0,1)$-matrices. Considered optimization problems come from specific subject areas – astronomy, medical diagnostics, seismology, etc. and are effectively modeled in terms of n-cube geometry, hypergraph degree sequences, image restoration, and other mathematical means. The common to these problems is that they look for inverses of some direct simple tasks. Most of these and related problems are NP-hard therefore approximate and heuristic algorithms are of interest. The area is studied intensively and we will brief in references [2-4].

Approximation algorithms introduced in this paper are based on Lagrangean relaxation and on variable splitting technique. These are well known widely implemented techniques of getting approximations. But in each particular case it is yet a question if the Lagrangean approach is effective. The problem under consideration is to be transformed into a form of Integer Linear Optimization with several groups of constraints. For each group of constraints it is necessary to have developed algorithms of their solutions. Finally it is yet a question whether the integration into Lagrangean framework will approach the optimal solution. To learn these possibilities a software environment is created for experimentations, which in addition provides solutions of Problems 1-3 considered below. As a demonstration example the Problem 2 is considered taking into account that differences between these problems are not critical.

Section 2 contains the necessary initial information and problem definitions. $(0,1)$-matrix interpretations are given in Section 3. In Section 4 Lagrangean relaxation method is applied to solve these problems. The splitting problems are given in Section 5 where Algorithms to solve the fragmental problems are constructed.

2. Problem descripion

We start with listing of minimal set of source problems.

<u>P1. *n*-dimensional unit cube subsets with given partition (projection) sizes [4].</u>

Vertices of n-dimensional unit cube is given by $E^n = \{(x_1, \cdots, x_n) / x_i \in \{0,1\}, i = \overline{1,n}\}$. Consider partition of E^n into the two subcubes $E^{n-1}_{x_i=1}$ and $E^{n-1}_{x_i=0}$ in accord to values of an arbitrary variable x_i. Similarly, each vertex subset $M \subset E^n$ can be partitioned into the $M_{x_i=1}$ and $M_{x_i=0}$.

Let M be an m-vertex subset of E^n. The vector $S=(s_1,\cdots,s_n)$ is called **associated (characteristic) vector of partitions** for the set M if $s_i=\left|M_{x_i=1}\right|$ for $i, 1\le i\le n$. The existence problem in this regard is to find out the existence of an m-vertex subset of E^n with given associated vector of partitions and a Boolean function by the given associated vector of activities. I

<u>P2. Uniform hypergraphs with given Subsumed graphs' Degree Sequences</u>

$k\ge 2$ is an integer and $V(G)$ is the vertex set with $\left|V(G)\right|\ge k$. Edges $E(G)$ of G are defined as members of $\binom{V(G)}{k}$ which is the set of all k-subsets of $V(G)$. If G is k-hypergraph, $k\ge 3$ and $u\in V(G)$, then a $(k-1)$-hypergraph G_u is defines as follows. The vertex set $V(G_u)=V(G)-u$, and for each edge $B\in E(G)$ with $u\in V(G)$, $B-u$ is included as an edge of $E(G_u)$. We say that G *subsumes* the collection of hypergraphs $\left\{G_u / u\in V(G)\right\}$.

If G is a hypergraph and $u\in V(G)$, then $\deg(u)$ (degree of u), the number of edges containing u, is the number of edges of the subsumed graph G_u. The *degree sequence* of G is the multiset $DegSeq(G)=\left\{\deg(u)/u\in V(G)\right\}$.

Problem (Subsumed graphs' Degree Sequences). [3]

Given $DegSeq(g_i), i=1,\cdots,n$ of n graphs $g_1,\cdots,g_n$, is there an n vertex hypergraph G such that the subsumed graphs $G_1,\cdots,G_n$ satisfy $DegSeq(G_i)=DegSeq(g_i)$ for $i=1,\cdots,n$?

<u>P3. Reconstruction of weighted $(0,1)$-matrices [4].</u>

The general image reconstruction problem is defined as follows: an image of ($m\times n$) pixels of p different colors, has to be reconstructed. We are given the number $r(i,c)$ of pixels of each color c in each row i and also the number $s(j,c)$ of pixels of each color c in each column j; is it possible to reconstruct an image, for all i,j,c?

3. (0,1) matrix model of problems P1. P2. P3

Consider a $(0,1)$-matrix A of size $m\times n$. Let $R=(r_1,\cdots,r_m)$ and $S=(s_1,\cdots,s_n)$ denote the row and column sums of A respectively, and let $U(R,S)$ be the set of all $(0,1)$-matrices with row sums R and column sums S. A necessary and sufficient condition for the existence of a $(0,1)$ matrix of the class $U(R,S)$ was found by Gale and Ryser [R,1966].

We reformulate the basic problems P1, P2 and P3 in terms of $(0,1)$-matrices. Common to all problems are the given integer vectors $R=(r_1,\cdots,r_m)$ and $S=(s_1,\cdots,s_n)$.

Problem 1. Existence of a $(0,1)$ matrix with different rows in the class $U(R,S)$

Given vectors $R=(r_1,\cdots,r_m)$ and $S=(s_1,\cdots,s_n)$. Does there exist a matrix $X=\{x_{i,j}\}$ in the class $U(R,S)$ with different rows?

Problem 2. Existence of a $(0,1)$ matrix in the class $U(R,S)$, with given intersections of pairs of rows

Given $R=(r_1,\cdots,r_m)$, $S=(s_1,\cdots,s_n)$ and $R'=(r'_1,\cdots,r'_{C_m^2})$ vectors. Enumerate pairs of rows and let $p(i',i'')$ indicates the number of the pair (i',i'') for $1\le i'<i''\le m$. Then the problem is in existence of

a matrix $X=\{x_{i,j}\}$ in the class $U(R,S)$ with the following property: rows i' and i'' intersect (by 1's) in $r'_{p(i',i'')}$ places.

Problem 3. Existence of a $(0,1)$ matrix in the class $U(R,S)$ with given intersections of adjacent pairs of rows

Given vectors $R=(r_1,\cdots,r_m)$, $S=(s_1,\cdots,s_n)$ and $R'=(r'_1,\cdots,r'_{m-1})$. Does there exist a matrix $X=\{x_{i,j}\}$ in $U(R,S)$ with the given intersections of adjacent pairs of rows - rows i and $i+1$ intersect (by 1's) in exactly r'_i places ($i=1,\cdots,m-1$)?

Note . Consider rows i' and i'' and let $r_{i'}\le r_{i''}$. If rows are different, then their intersection size is less than $r_{i''}$. Assuming that $r_1\le\cdots\le r_m$, the requirement of different rows in Problem 1 can be replaced by the property: intersection size for all pairs of rows, (i',i''), $1\le i'<i''\le m$ is less than $r_{i''}$.

While the Problem 2 is NP-complete, the complexity of Problems 1 and 3 are not known: Problem 1 is a well known open problem [4]. Complexity issue of the Problem 3 is not addressed yet.

4. Integer linear programming formulations and Lagrangean relaxation formulas

Let X be a $(0,1)$-matrix of size $m\times n$. Enumerate pairs of rows and let $p(i',i'')$ indicates the number of the pair (i',i''), for $1\le i'<i''\le m$. For each pair of rows, (i',i''), we define n binary variables $y_{p(i',i''),j}$, such that. $(y_{p(i',i''),j}=1)\Leftrightarrow(x_{i',j}=1)\,\&\,(x_{i'',j}=1)$.

Obviously it can be provided by the following set of algebraic conditions: $\begin{cases} y_{p(i',i''),j}\le x_{i',j}\\ y_{p(i',i''),j}\le x_{i'',j}\\ y_{p(i',i''),j}\ge x_{i',j}+x_{i'',j}-1\end{cases}$

Now Problems 1-3 above can be formulated in terms of integer linear programming. We focus only on Problem 2 giving the details for that case. Problems 1 and 3 can be reformulated as integer programming, then relaxed and solved, - by a similar way.

Recall that we assume $r_1\le\cdots\le r_m$.

Problem IP2 Given integer vectors $R=(r_1,\cdots,r_m)$, $S=(s_1,\cdots,s_n)$ and $R'=(r'_1,\cdots,r'_{C_m^2})$. The problem is in existence of an $m\times n$ binary matrix $X=\{x_{i,j}\}$ and a $(C_m^2)\times n$ binary matrix $Y=\{y_{i,j}\}$ such that

$$(\text{IP2})\begin{cases}(1)\sum_{i=1}^{m}x_{i,j}=s_j,\ j=1,\cdots,n\\ (2)\sum_{j=1}^{n}x_{i,j}=r_i,\ i=1,\cdots,m\\ (3)\begin{cases}y_{p(i',i''),j}\le x_{i',j}\\ y_{p(i',i''),j}\le x_{i'',j}\\ y_{p(i',i''),j}\ge x_{i',j}+x_{i'',j}-1\end{cases}\quad 1\le i'<i''\le m,\ j=1,\cdots,n\\ (4)\sum_{j=1}^{n}y_{p(i',i''),j}=r'_{p(i',i'')}\quad 1\le i'<i''\le m\\ (5)x_{i,j}\in\{0,1\},\ y_{i,j}\in\{0,1\}\end{cases}$$

3. Lagrangean relaxation and variable splitting

In a way to solve this problem we apply the Lagrangean relaxation and variable splitting technique.

Lagrangean relaxation for integer linear programming

Consider the following optimization problem

(P) $Max_x\{fx / Ax \le b, Cx \le d, x \in Z\}$ in which some constraints are complicating (suppose $Ax \le b$), in the sense that one would be able to solve the same integer programming problem has these constraints not been present: $Max_x\{fx / Cx \le d, x \in Z\}$. One can take advantage of this situation by constructing a so-called Lagrangean relaxation in the following way. Let $\lambda \ge 0$ be a vector of multipliers and let (LR_λ) be the problem

(LR_λ) $Max_x\{fx + \lambda(b - Ax) / Cx \le d, x \in Z$. ($LR_\lambda$) is the Lagrangean relaxation of (P). Let $v(LR_\lambda)$ is the value of optimal solution of (LR_λ). The problem (LD) $Min_{\lambda \ge 0} v(LR_\lambda)$ is called the Lagrangean dual of (P) relative to the $Ax \le b$. The optimal value of LD is a smallest upper bound on the optimal value of (P). For Problem 2 we will use variable splitting technique - we split our problem into separate vertical and horizontal subproblems, then the horizontal subproblem is further separated into subproblems for each pair of rows. Thus we consider Lagrangean relaxation of (IP2). We duplicate variables $x_{i,j}$, getting 2 independent sets of variables $x_{i,j}^h$ and $x_{i,j}^v$, and then dualize the copy (duplication) constraint using Lagrangean multipliers $\lambda_{i,j}$.

$$
\text{(IP2LR)}\quad
\begin{cases}
\max\{\sum \lambda_{i,j}(x_{i,j}^h - x_{i,j}^v)\} \\
(1)\sum_{i=1}^{m} x_{i,j}^v = s_j,\ j = 1,\cdots,n \\
(2)\sum_{j=1}^{n} x_{i,j}^h = r_i,\ i = 1,\cdots,m \\
(3)\begin{cases}
y_{p(i',i''),j} \le x_{i',j}^h \\
y_{p(i',i''),j} \le x_{i'',j}^h \qquad 1 \le i' < i'' \le m,\ j = 1,\cdots,n \\
y_{p(i',i''),j} \ge x_{i',j}^h + x_{i'',j}^h - 1
\end{cases} \\
(4)\sum_{j=1}^{n} y_{p(i',i''),j} = r'_{p(i',i'')} \qquad 1 \le i' < i'' \le m \\
(5) x_{i,j}^h, x_{i,j}^v \in \{0,1\},\ y_{i,j} \in \{0,1\}
\end{cases}
$$

Split the problem into sub problems – horizontal and vertical

$$
\text{(IP2-v)}\quad
\begin{cases}
\max\{\sum \beta_{i,j} x_{i,j}^v\} \\
(1)\sum_{i=1}^{m} x_{i,j}^v = s_j,\ j = 1,\cdots,n \\
(2) x_{i,j}^v \in \{0,1\}
\end{cases}
$$

$$\text{(IP2-h)}\begin{cases}\max\{\sum \alpha_{i,j}x^h_{i,j}\}\\ (1)\sum_{j=1}^{n} x^h_{i,j}=r_i, \quad i=1,\cdots,m\\ (2)\begin{cases} y_{p(i',i''),j}\le x^h_{i',j}\\ y_{p(i',i''),j}\le x^h_{i'',j}\\ y_{p(i',i''),j}\ge x^h_{i',j}+x^h_{i'',j}-1\end{cases} \quad 1\le i' < i'' \le m,\ j=1,\cdots,n\\ (3)\sum_{j=1}^{n} y_{p(i',i''),j}=r'_{p(i',i'')} \quad 1\le i'<i''\le m\\ (4)x^h_{i,j}\in\{0,1\}, y_{i,j}\in\{0,1\}\end{cases}$$

Using similar reasons IP2-h is split into subproblems for each pair of rows:

$$\text{(IP2-h1)}\begin{cases}\max\{\sum_{j=1}^{n}(\alpha'_j x'_j+\alpha''_j x''_j)\}\\ (1)\sum_{j=1}^{n}x'_j=r', \quad \sum_{j=1}^{n}x''_j=r''\\ (2)\begin{cases} y_j\le x'_j\\ y_j\le x''_j\\ y_j\ge x'_j+x''_j-1\end{cases} \quad j=1,\cdots,n\\ (3)\sum_{j=1}^{n}y_j=r^*\\ (4)x'_j,x''_j\in\{0,1\}, y_j\in\{0,1\},\ j=1,\cdots,n\end{cases}$$

Further we apply an iterative procedure to find the optimisation coefficients $\lambda_{i,j}$. On each iteration we consider C_m^2+1 separate subproblems (C_m^2 horizontal and 1 vertical). Each horizontal subproblem is formulated as a parameterised set system problem.

4. Algorithms for solving subproblems for pairs of rows

(IP2-h1) is equivalent to the following problem.

Problem of Weighted Threads. Given 2 sets of weighted elements $X'=\{x'_1,\cdots,x'_n\}$ and $X''=\{x''_1,\cdots,x''_n\}$. $\alpha'_i\ge 0$ is the weight of $x'_i\in X'$, and $\alpha''_i\ge 0$ is the weight of $x''_i\in X''$. Given also positive integers $r',r'',r^*, r'\le r'', r^*<r''$. The problem is in finding subsets $\tilde{X}'\subseteq X'$ and $\tilde{X}''\subseteq X''$, such that: $\left|\tilde{X}'\right|=r'$ and $\left|\tilde{X}''\right|=r''$, and

1. $\sum\limits_{x_i' \in \tilde{X}'} \alpha_i' + \sum\limits_{x_i'' \in \tilde{X}''} \alpha_i'' \to \max$

2. $\left| \left\{ (x_i', x_i'') / x_i' \in \tilde{X}', x_i'' \in \tilde{X}'' \right\} \right| = r^*$

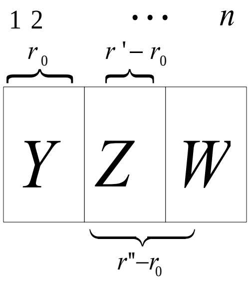
In its short description a three stage selection algorithm is constructed.

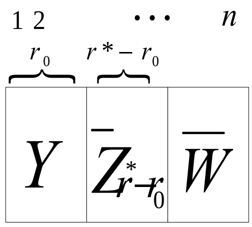
1. Arranging elements in X' and X'' by decreasing order of their weights $\overrightarrow{X'} = \{x_{i_1}', \cdots, x_{i_n}'\}$, $x_{i_1}' \geq \cdots \geq x_{i_n}'$, and $\overrightarrow{X''} = \{x_{j_1}'', \cdots, x_{j_n}''\}$, $x_{j_1}'' \geq \cdots \geq x_{j_n}''$ and taking the first iteration for $\tilde{X}'$ and $\tilde{X}''$ as $X_{r'}' = \{x_{i_1}', \cdots, x_{i_{r'}}'\}$ and $X_{r''}'' = \{x_{j_1}'', \cdots, x_{j_{r''}}''\}$.

2. If $|Y| = r^*$ then $X_{r'}'$ and $X_{r''}''$ are the required subsets. Otherwise consider cases:

a) $|Y| = r_0 < r^*$ and b) $|Y| = r_0 > r^*$. It is enough to consider the first case:

Shift the elements of Y to the left.





Arrange elements (pairs) in $Z \cup W$ by decreasing order of sum of elements (weights) of the pair and denote by $\overline{Z}_{r^*-r_0}$ first $r^* - r_0$ elements, and by $\overline{W}$ - the reminder:

Construct the sets $\tilde{X}'$ and $\tilde{X}''$ by the elements of $Y \cup \overline{Z}_{r^*-r_0}$ (first element of each pair goes to $\tilde{X}'$, second goes to $\tilde{X}''$). Remaining $r' - r^*$ elements of $\tilde{X}'$ and $r'' - r^*$ elements of $\tilde{X}''$ are formed as follows: Arrange elements by decreasing order in $\overline{W}'$ and $\overline{W}''$, where $\overline{W}'$ and $\overline{W}''$ consist of respectively the first (belonging to X') and second (belonging to X'') elements of pairs of $\overline{W}$. Consider first $r' - r^*$ subset in each set and denote them by $\overline{W}'_{r'-r^*}$ and $\overline{W}''_{r'-r^*}$. Subsets of remaining elements we denote by $\overline{W}'_{rem}$ and $\overline{W}''_{rem}$.

1. If there are no elements with the same index in $\overline{W}'_{r'-r^*}$ and $\overline{W}''_{r'-r^*}$, then these elements go to the $\tilde{X}'$ and $\tilde{X}''$ respectively.

2. Otherwise we replace the last element in subset $\overline{W}'_{r'-r^*}$ by the first element of $\overline{W}'_{rem}$ or replace the last element in $\overline{W}''_{r'-r^*}$ by the first element of $\overline{W}''_{rem}$ depending on the sum of corresponding weights.

Remaining $r''-r'$ elements for $\tilde{X}''$ we take from $\overline{W}''$.

Problem of Weighted Threads is just one example of fragmental problems that arise in splitting of optimisation of (0,1) matrices. A series of similar problems arise when different conditions are applied as a consequence of application area modelling. These problems are relatively simple and a large set of them and their solutions are collected in a software library serving the experimentation software system created in this regard.

5. Conclusion

Lagragean relaxation and the related set of techniques is one of the ways of constructing approximation algorithms for hard and unsolved combinatorial problems. A compact class of optimisation problems are effectively modelled in terms of (0,1) matrices. During the Lagrangean relaxation a number of relatively simple optimisation problems arise as a result of splitting the problems into fragmental subproblems. The **Weighted Threads Problem** of this class is solved and the whole chain of approximation is formalised for an example demonstration problem. A software system created on this base provides experimentation environment for treatment of combinatorial NP problems.

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