P SYSTEMS GÖDELIZATION

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Abstract: This paper presents a method for assigning natural numbers to Transition P systems based on a Gödelization process. The paper states step by step the way for obtaining Gödel numbers for each one of the fundamental elements of Transition P systems –multisets of objects, evolution rules, priorities relation, membrane structure- until defining the Gödel number of a given Transition P system.

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Introduction

A P system can be defined as a membrane structure in which multiset of objects and evolution rules have been placed in regions defined by membranes. One of the most important characteristic of these computational systems it is the following: objects evolve by the application of evolution rules in a non-deterministic and massively parallel manner all over the system. One way to control the application of evolution rules in membranes is to define a priority relationship among rules in membranes. This priority relation defines a partial order relation and a hierarchy in application of evolution rules inside membranes. Associated to rules, there is one more feature, the capability of dissolving membranes. P systems having these two features (priority relationship and dissolving capability) have been demonstrated that are Turing complete [Păun 2002].

P systems compute transiting from one configuration to the next one by application of evolution rules. A configuration of a P system is defined by the membrane structure of the system and the set of multiset allocated inside membranes of the structure. Hence, a computation is defined as the set of configuration, starting from the initial configuration, the systems transits. It is said that the system performs a successful computation when a configuration in which no one rule can by applied in the system is reaches. This configuration is named "halting configuration". The result of a successful computation is the number of objects present in a determined elemental membrane or the number of objects the system outputs to the environment.

In [Turing 1936] demonstrated that "every computable function corresponds to a Turing machine, and that every Turing machine could be mapped into a unique natural number. As a consequence, the computable functions are enumerable. Moreover, the numbers corresponding to computable functions are known as "Gödel numbers"". Hence, it could be interesting to establish a correspondence between elements of a P system and natural numbers. This gödelization process will permit to obtain some computational benefits. First of all, it is possible to obtain a uniform representation for every element of the P system; it could be possible to reduce the analysis of the system to elementary natural number operations (for example, evolution rules application as divisions [Suzuki 2000]). Secondly this encoding process can produce a new way of packing information and simulating P systems in digital devices with the appropriate algorithms.

This paper proposes a way for obtaining the Gödel number for every element of a P system and some operation for manipulating them with their associated Gödel numbers, and finally the Gödel number of the P system is defined.

Multisets Gödelization

A multiset is defined as a mapping from a non-empty and finite set, *U* over the natural number set. More formally,

$$\begin{aligned} M &: U \to N \\ &a \to M\ a \end{aligned} \tag{1}$$

where *M a* is the number of copies for the *a* element in the multiset *M*. Representing by $\mathcal{M}(U)$ the set of every multiset of objects over the set *U*, it can be defined the following operations:

Multisets operations:

Let M, M_1 y $M_2 \in \mathcal{M}(U)$

- Multisets inclusion:
  - $M_1 \subset M_2 \Leftrightarrow \forall a \in U, M_1\ a < M_2\ a$
- Multisets addition $M_1 + M_2$:
  - $\forall a \in U,\ (M_1 + M_2)\ a = M_1\ a + M_2\ a$
- Multiset subtraction $M_1 - M_2$:
  - If $M_2 \subset M_1,\ \forall a \in U,\ (M_1 - M_2)\ a = M_1\ a - M_2\ a$

It can be easily demonstrated that $(\mathcal{M}(U),+)$ is a commutative monoid with identity element.

Gödel number associated to a Multiset

In this section the Gödel number for every multiset $M \in \mathcal{M}(U)$ is defined.

Let $P_m = \{p_1, \dots, p_m\}$ be the set of the first *m* natural prime numbers starting in *2*; and let $U=\{a_1, \dots, a_m\}$ be a non-empty and finite set of objects with $card(U)=m$, the following one to one map is defined:

$$\begin{aligned} b &: U \to P_m \\ &a_i \to p_i \end{aligned} \tag{2}$$

Moreover, given *b* and P_m there is a function $\mathcal{G}_m$ satisfying:

$$\forall M \in \mathcal{M}(U), \exists n \in \mathcal{N} \rceil \mathcal{G}_m(M)=n \wedge \mathcal{G}_m^{-1}(n) = M$$

defined by:

$$\mathcal{G}_m : \mathcal{M}(U) \to \mathcal{N}$$

$$\begin{aligned} &M \to n = p_1^{M\ a_1} \times \ldots \times p_m^{M\ a_m} \\ &\Phi \to 1 \end{aligned} \tag{3}$$

where $p_i = b(a_i)\ \forall a_i \in U$ as in equation *(2)*.

Definition: $\forall M \in \mathcal{M}(U)$ the *Gödel number* associated to *M* is $\mathcal{G}_m(M)$.

Operations with multiset using Gödel numbers

From the definition of Gödel number associated to a multiset it can be easily demonstrated the following proposition:

Proposition 1: Let M_1 y $M_2 \in \mathcal{M}(U)$ and $\mathcal{G}_m(M_1) = n_1$ y $\mathcal{G}_m(M_2) = n_2$, their associated Gödel numbers, then:

- Multisets inclusion:
  - $M_1 \subset M_2 \Leftrightarrow n_1$ *divides to* n_2

- Multisets addition:
  - o $\mathcal{G}_m (M_1 + M_2) = n_1 \times n_2$
- Multisets subtraction: If $M_1 \subset M_2$, *then*
  - o $\mathcal{G}_m (M_1 - M_2) = n_1 \div n_2$

From now on, in order to simplify the text, we will represent multiset of objects with small letters. Hence, $u \in \mathcal{M}(U)$ represents one multiset of object in the set *U*.

Evolution rules Gödelization

In this section the Gödel number for one evolution rule is defined. After that, operations over evolution rules are translated to their associated Gödel numbers. Finally, priority relationship over evolution rules is encoded in Gödel numbers and it is incorporated to the Gödel number associated to the evolution rule.

In order to define evolution rules are needed the following ingredients: a finite set of labels *L* for numbering membranes in a membrane structure and a non empty and finite set of objects *U* to define multisets of objects. Let us to represent the set of evolution rules with labels in *L* and objects in *U* by $\mathcal{R}(U,L)$.

An evolution rule $r \in \mathcal{R}(U,L)$ is a tuple $r = (u,v,\delta)$ where:

- $u \in \mathcal{M}(U)$
- $v \in \mathcal{M}(U \times \mathcal{D})$ with $\mathcal{D} = \{out, here\} \cup \{in_j \mid j \in L\}$
- $\delta \in \{0,1\}$

Moreover, an evolution rule can be also represented by a set of *(n+2)* multisets of objects plus one natural number δ representing the rule dissolving capability. Hence,

$$r = (u_a, u_0, u_1, u_2, \ldots, u_n, \delta) \qquad (4)$$

where:

- $u_a \in \mathcal{M}(U)$, *the rule antecedent,*
- $u_0 \in \mathcal{M}(U)$, *the multiset to be sent to the environment,*
- $u_i \in \mathcal{M}(U)$, $\forall i \in \{1, \ldots, n\}$, *the multiset to be sent to region i,*
- $\delta \in \{0,1\}$, *the dissolving capability of the rule.*

Using this representation for evolution rules, it is established a gödelization for evolution rules.

Gödel number associated to an evolution rule

First of all, in order to define the Gödel number for an evolution rule $r \in \mathcal{R}(U,L)$ it is necessary to consider a set of *n+3* natural prime numbers $P = \{p_a, p_o, p_1, \ldots, p_n, p_\delta\}$, and a map *g* defined as follows:

$$\begin{aligned} g : L \cup \{a, o, \delta\} &\to P \\ a &\to p_a \\ o &\to p_o \\ \delta &\to p_\delta \\ i &\to p_i, \forall i \in \{1,2,\ldots,n\} \end{aligned} \qquad (5)$$

Then, from equations *(5)* and *(6)* the Gödel number associated to one evolution rules is defined by:

$$\mathcal{G}_m : \mathcal{R}(U,L) \to \mathcal{N}$$

$$r = (u_a, u_o, u_1, u_2, \dots, u_n, \delta) \to p_a^{G_m(u_a)}.p_o^{G_m(u_o)}.\prod_{i=1}^{n} p_i^{G_m(u_i)}.p_\delta^{\delta} \tag{6}$$

Rules operation using Gödel numbers

Let $r_1, r_2 \in \mathcal{R}(U,L)$ be two evolution rules, being $r_1 = (u_1, v_1, \delta_1)$ and $r_2 = (u_2, v_2, \delta_2)$ and let $s \in \mathcal{N}$ be a natural number

It is defined $r_1 + r_2 \in \mathcal{R}(U,L)$ by:

$$r_1 + r_2 = (u_1 + u_2, v_1 + v_2, \delta_1 \vee \delta_2) \tag{7}$$

and $sr_1 \in \mathcal{R}(U,L)$ by:

$$sr_1 = (su_1, sv_1, \delta_1) \tag{8}$$

From *(6)* and *(7)* it can be easily demonstrated that:

$$G_m : R(U,L) \to N$$

$$r_1 + r_2 \to p_a^{G_m(u_{1a}+u_{2a})}.p_o^{G_m(u_{1o}+u_{2o})}.\prod_{i=1}^{n} p_i^{G_m(u_{1i}+u_{2i})}.p_\delta^{\delta_1 \vee \delta_2} = \tag{9}$$

$$p_a^{G_m(u_{1a}).G_m(u_{2a})}.p_o^{G_m(u_{1o}).G_m(u_{2o})}.\prod_{i=1}^{n} p_i^{G_m(u_{1i}).G_m(u_{2i})}.p_\delta^{\delta_1 \vee \delta_2}$$

and from *(6)* and *(8)*

$$G_m : R(U,L) \to N$$

$$sr_1 \to p_a^{(G_m(u_{1a}))^s}.p_o^{(G_m(u_{1o}))^s}\prod_{i=1} p_i^{(G_m(u_{1i}))^s}\, p_\delta^{\delta} \tag{10}$$

Rules priority using Gödel numbers

Let $\mathcal{R}_j = \{r_1, r_2, \dots, r_i\}$ be the set of evolution rules from region *j* of a P system; and let $P_i = \{p_1, p_2, \dots, p_i\}$ the set of the first *i* prime natural numbers starting in *2*.

Then, it is defined the map:

$$pri : R_j \to N$$
$$r_i \to p_i \tag{11}$$

Let ρ_j be the priority relationships associated to $\mathcal{R}_j$, the set of evolution rules in region *j*, and let $ct(\rho_j)$ the transitive and non reflexive closure of ρ_j.

Let $r_k, r_{s1}, r_{s2}, \dots, r_{st} \in \mathcal{R}_j$ with $(r_k, r_{s1}), (r_k, r_{s2}), \dots, (r_k, r_{st}) \in ct(\rho_j)$ where rule r_k has a higher priority over rules $r_{s1}, r_{s2}, \dots, r_{st}$, that is $r_k > r_{s1}, r_k > r_{s2}, \dots, r_k > r_{st}$ and $r_k \neq r_{s1}, r_k \neq r_{s2}, \dots, r_k \neq r_{st}$.

It is defined:

$$gprior : R_j \to N$$
$$r_k \to pri(r_k).\prod_{i=1}^{t} pri(r_{si}) \tag{12}$$

Let $r \in \mathcal{R}_j(U,L)$ be an evolution rule in region *j* with label in *L* and objects in *U*. In particular, let $r = (u_a, u_o, u_1, u_2, \dots, u_n, \delta)$ that rule.

The Gödel number associated to the rule *r* with priority is defined by:

$$\mathcal{G}_m : \mathcal{R}_j(U,L) \to \mathcal{N}$$

$$r = (u_a, u_o, u_1, u_2, \ldots, u_n, \delta) \rightarrow p_p^{gprior(r)} p_a^{G_m(u_a)} . p_o^{G_m(u_o)} . \prod_{i=1}^{n} p_i^{G_m(u_i)} . p_\delta^{\delta} \quad (13)$$

where p_a, p_o, p_i, p_δ are natural prime numbers defined in the one to one mapping *g* equation *(13)*, and p_p is a different one natural prime number.

Membrane Structure Gödelization

The main element of a P system is the membrane structure. The membrane structure can be represented as a directed and non ordered tree, having the skin membrane as root of the tree. Nodes are membranes and edges represent the relationship to be *'directly included in'*.

Gödel number associated to a membrane tree

Let $P_m = \{p_1, p_2, \ldots, p_m\}$ be the set of the m first natural number starting in *2* and let μ be a membrane structure with labels in $L=\{1,2, \ldots, m\}$.

The Gödel number associated to a membrane is defined by the following map:

$$\begin{aligned} &G_m : \mu \rightarrow P_m \\ &\qquad m_k \rightarrow p_k, \quad \forall k \in L \end{aligned} \quad (14)$$

Now, it is possible to define the Gödel number associated to a membrane tree T_μ with labels in *L* as follows:

$$G_m(T_\mu) = \prod_{k=1}^{m} G_m(m_k), \quad \forall k \in L \quad (15)$$

P systems Gödelization

From the previous Gödel numbers associated to each one of the different components of Transition P systems, it is possible to associate to each Transition P system a Gödel number. Let *Π* be a Transition P system of degree *m*.

$$\Pi = (V, \mu, \varpi_1, \ldots, \varpi_m, (R_1, \rho_1), \ldots (R_m, \rho_{m1}), i_0) \quad (16)$$

where:

- *V* is a finite and not empty set of objects.
- μ is a membrane structure labelled in a one to one manner from *1* to *m*.
- ϖ_i, $1 \le i \le m$, multisets of objects over *V* associated to regions *1, …, m*.
- R_i, $1 \le i \le m$, finite set of evolution rules over V associated to regions *1, …, m.*
- ρ_i, $1 \le i \le m$, priority relationships defined over the set of evolution rules R_i.

Gödel number associated to a P system

Now it can be defined the Gödel number associated to the Transition P system *Π* as follows: let $p_1, p_2, \ldots, p_{2m+3}$ be the first *2m+3* natural prime numbers starting from 2.

$$G_m(\Pi) = p_1^{Card(V)} p_2^{G_m(T_\mu)} p_3^{G_m(\varpi_1)} \ldots p_{m+2}^{G_m(\varpi_m)} p_{m+3}^{\sum_{r_1^i \in R_1} G_m(r^i{}_1, \rho_1)} \ldots p_{2m+2}^{\sum_{r_m^i \in R_1} G_m(r^i{}_m, \rho_m)} p_{2m+3}^{i_0} \quad (17)$$

where:

- *Card(V)*, is the cardinal of *V*.

- $G_m(A_\mu)$ is the number defined in *(14)* and *(15)* for the membrane structure μ.
- $G_m(\varpi_i)$, $1 \le i \le m$, are the defined numbers in (2) and (3) for the multiset of objects ϖ_i, $1 < i < m$.
- $G_m(ri, \rho_i)$, $1 \le i \le m$, is the defined number in *(12)* and *(13)* for evolution rules r_i associated to regions R_i with their corresponding priorities ρ_i, $1 \le i \le m$.

Conclusions

Gödel numbers are fundamental in history of computation. Turing in [Turing 1936] showed that every Turing machine could be mapped into a unique natural number and that every computable function corresponds to Turing machine. Hence, the computable functions are enumerable. Here we present the way for mapping Transition P systems into natural number using a Gödelization process. The whole process is described in this paper step by step, in order to define the appropriate Gödel numbers to each one of the different fundamental elements in which Transition P systems are decomposed. Moreover, some operations over multisets of objects and evolution rules have been defined in terms of Gödel numbers associated to them. The different applications of this method to the study of membrane systems are unexplored and it must be subject of study in different aspects related to hardware/software implementations using Gödel numbers, or how to use this encoding process in order to pack information related to P systems and how affect it in the development of algorithms for implementing P systems in digital devices.

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