GENERATING MORE BOUNDARY ELEMENTS OF SUBSET PROJECTIONS

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***Abstract**: Composition problem is considered for partition constrained vertex subsets of n dimensional unit cube E^n. Generating numerical characteristics of E^n subsets partitions is considered by means of the same characteristics in $n-1$ dimensional unit cube, and construction of corresponding subsets is given for a special particular case. Using pairs of lower layer characteristic vectors for E^{n-1} more characteristic vectors for E^n are composed which are boundary from one side, and which take part in practical recognition of validness of a given candidate vector of partitions.*

***Keywords**: monotone Boolean functions, (0,1)-matrices*

***ACM Classification Keywords**: G.2.1 Discrete mathematics: Combinatorics*

***Conference**: The paper is selected from Seventh International Conference on Information Research and Applications – i.Tech 2009, Varna, Bulgaria, June-July 2009*

Introduction

Set systems are considered represented as subsets of vertices of n dimensional unit cube E^n. Given a subset of E^n, its numerical characterization is composed by partitions and their sizes, which are the coordinates of the corresponding associated vector of partitions. General numerical characterization of vertex subsets of E^n through their partitions is considered in [SA, 2006], [S, 2009] where a complete and simple structural description of the set of all integer-valued vectors, which serve as associated vectors of partitions is given, The description is through the set of its boundary elements. Elements of boundary set with minimum and maximum weight are known by [SA 2001], [SA, 2006]. Current research focuses on generating more boundary elements via the known ones for smaller dimensions, and constructs a set of corresponding subsets of vertices.

Basic Structural Description

Let E^n be the set of vertices of n-dimensional unit cube, $E^n = \{(x_1, \cdots, x_n) / x_i \in \{0,1\}, i = 1, \cdots, n\}$. For an arbitrary variable x_i, consider partition of the cube into two subcubes according to the value of $x_i, 1 \le i \le n$. Denote these subcubes by $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ correspondingly. Similar to this, each subset of vertices $M \subseteq E^n$ can be partitioned into $M_{x_1=1}$ and $M_{x_1=0}$. For a given m, $0 \le m \le 2^n$ consider an m-element subset M of E^n. Vector $S = (s_1, \cdots, s_n)$ is called associated vector of partitions of M if $s_i = |M_{x_1=1}|$ for all $1 \le i \le n$. Let $\psi_m(n)$ denote the set of all associated vectors of partitions of m-subsets of E^n. Let Ξ^n_{m+1} denote the n dimensional $(m+1)$-valued grid, i.e., the set of all integer-valued vectors $S = (s_1, s_2, \cdots, s_n)$ with $0 \le s_i \le m$, $i = 1, \cdots, n$. Obviously $\psi_m(n) \subseteq \Xi^n_{m+1}$.

Vector $S \in \psi_m(n)$ is called an upper (lower) boundary vector for $\psi_m(n)$ if no vector $R \in \Xi_{m+1}^n$ with $R > S$ ($S > R$) belongs to $\psi_m(n)$. By $\widehat{\psi}_m(n)$ and $\breve{\psi}_m(n)$, respectively, denote the sets of all upper and lower boundary vectors of $\psi_m(n)$. The sets $\widehat{\psi}_m(n)$ and $\breve{\psi}_m(n)$ contain equal numbers of elements [S, 2009]. Let r denote this number of elements and let $\widehat{\psi}_m(n) = \{\widehat{S}^1, \cdots, \widehat{S}^r\}$ and $\breve{\psi}_m(n) = \{\breve{S}^1, \cdots, \breve{S}^r\}$ so that ($\widehat{S}^j$, $\breve{S}^j$) is a pair of complementary vertices: $\widehat{s}_i^j = m - \breve{s}_i^j$, for $1 \le i \le n$. For each vector $\widehat{S}^j$ all coordinates are greater than or equal to $m/2$ and for each vector $\breve{S}_j$ all coordinates are less than or equal to $m/2$. For a pair $\widehat{S}_j$, $\breve{S}_j$, from $\widehat{\psi}_m(n)$ and $\breve{\psi}_m(n)$ respectively, let $I(\widehat{S}_j)$ (the notion $I(\breve{S}_j)$ also may be used) denotes the vector set of the sub-cube of Ξ_{m+1}^n spanned by these vectors: $I(\widehat{S}_j) = \{Q \in \Xi_{m+1}^n / \widehat{S}_j \le Q \le \breve{S}_j\}$. proves that the collection of all sub-cubes $\{I(\widehat{S}_j) / \widehat{S}_j \in \widehat{\psi}_m(n)\}$ composes the basic set $\psi_m(n)$:

Theorem 1. $\psi_m(n) = \bigcup_{j=1}^{r} I(\widehat{S}_j)$

Boundary Cases

Basic object of analysis in this section is the set of all monotone Boolean functions in E^n, with exactly m ones. Let $M_m^1(n)$ be the set of all associated vectors of partitions of m-subsets corresponding to the ones of these monotone Boolean functions. Similarly $M_m^0(n)$ is the set of all associated vectors of partitions of m-subsets corresponding to the zeros of such monotone Boolean functions. It is easy to check that $\widehat{\psi}_m(n) \subseteq M_m^1(n)$ and that $\breve{\psi}_m(n) \subseteq M_m^0(n)$. Thus for description of $\psi_m(n)$ suffice to find those monotone Boolean functions which correspond to $\widehat{\psi}_m(n)$ (and/or $\breve{\psi}_m(n)$). Suppose the weights (sum of all coordinates) of vectors of $\widehat{\psi}_m(n)$ belong to some interval $[L_{\min}, L_{\max}]$. A specific set of monotone Boolean functions is constructed in [AS, 2001], for which the corresponding associated vectors of partitions belong to $\widehat{\psi}_m(n)$ and have the weight $L_{\min}$. The value of $L_{\min}$ and the coordinates of associated vectors are also analyzed in detail.

Let $D^{i_1,\cdots,i_n}$ be the set of vertices of E^n arranged in decreasing order of numeric value of the binary vectors $< x_{i_1}, \cdots, x_{i_n} >$. It is easy to check that the first 2^{n-1} elements of $D^{i_1,\cdots,i_n}$ form the set of vertices of $E_{x_{i_1}=1}^{n-1}$ and the reminder 2^{n-1} elements form $E_{x_{i_1}=0}^{n-1}$ being arranged in the same decreasing order of numeric value of $< x_{i_2}, \cdots, x_{i_n} >$. Denote these sets by $D_{x_{i_1}=1}^{i_2\cdots,i_n}$ and $D_{x_{i_1}=0}^{i_2\cdots,i_n}$ respectively. Similarly the first 2^{n-2} elements of $D_{x_{i_1}=1}^{i_2\cdots,i_n}$ form $E_{x_{i_1}=1,x_{i_2}=1}^{n-2}$ and the rest 2^{n-2} elements form $E_{x_{i_1}=1,x_{i_2}=0}^{n-2}$ - arranged in decreasing order of numeric value of the vectors $< x_{i_3}, \cdots, x_{i_n} >$, etc. It follows that each initial part (subset) of $D^{i_1,\cdots,i_n}$ (respectively $D_{x_{i_1}=1}^{i_2\cdots,i_n}$, $D_{x_{i_1}=0}^{i_2\cdots,i_n}$, etc.) of arbitrary size serves as the set of ones of some monotone Boolean function. Let $D^{i_1,\cdots,i_n}(m)$ denote the m-length initial segment of $D^{i_1,\cdots,i_n}$. Denote by D^n the overall set of all enumerations $D^{i_1,\cdots,i_n}$, and let $D^n(m)$ be the m-length initial segments of enumerations of $D^n(m)$.

The class of monotone Boolean functions, which have sets of ones belonging to $D^n(m)$ we denote by $f^{D^n(m)}$ and let $S^{D^n(m)}$ be the corresponding set of associated vectors of partitions.

Theorem 2. $S \in \widehat{\psi}_m(n)$ has weight $L_{\min}$ if and only if $S \in S^{D^n(m)}$.

The proof of theorem which is not the target to bring here is by induction on n.

Note. Let us consider vector $S = (s_1, \cdots, s_n)$ of $S^{D^n(m)}$, which is the associated vector of partitions for some $D^{i_1,\cdots,i_n}(m)$. It is easy to check that $s_{i_1} \geq s_{i_2} \geq \cdots \geq s_{i_n}$. Further, vector S obeys the following very useful property: any vector $(s_{i_i}, \cdots, s_{i_j} + 1, *, \cdots, *)$ for arbitrary $j, 1 \leq j \leq n$, does not belong to $\psi_m(n)$.

Generating New Boundary Elements

Thus, we consider the problem of description of all monotone Boolean functions, which correspond to $\widehat{\psi}_m(n)$.

Let we have the solution of this problem for $n-1$: given m_1, $0 \leq m_1 \leq 2^{n-1}$, and we may describe the monotone Boolean functions corresponding to $\widehat{\psi}_{m_1}(n-1)$.

Consider the way of composing the target construction through composing the ones which we have in dimensions n-1: consider partition of E^n according to some variable x_i, and consider the pairs of monotone Boolean functions in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ respectively, which correspond to the upper boundary vectors and find such pairs, that in E^n (their union in E^n) correspond to $\widehat{\psi}_m(n)$.

Consider partition of E^n according to the value of x_1 and consider monotone Boolean functions in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ with m_1 and m_2 ones respectively, where m_1, m_2 are arbitrary partition of m with the only requirement that $m_1 \geq m_2$. We intend to find feasible pairs of monotone functions from $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ to get at first monotone functions in E^n, and then - monotone functions which correspond to $\widehat{\psi}_m(n)$.

Consider the following particular case.

In $E^{n-1}_{x_1=1}$, as well as in $E^{n-1}_{x_1=0}$ consider monotone functions for which corresponding associated vectors belong to sets $\widehat{\psi}_{m_1}(n-1)$ and $\widehat{\psi}_{m_2}(n-1)$ respectively, and have the minimum possible weights. By theorem 2 these are functions of the form $f^{D^{n-1}(m_1)}$ and $f^{D^{n-1}(m_2)}$ respectively. Consider the case of the same order of variables, let it be $D^{2,\cdots n}(m_1)$ and $D^{2,\cdots n}(m_2)$. Denote by $S^{'} = (s_2^{'}, \cdots, s_n^{'})$ and $S^{''} = (s_2^{''}, \cdots, s_n^{''})$ - the corresponding associated vectors. Obviously this pair is feasible for getting a monotone function in E^n. Now we prove that it is feasible for getting an upper boundary vector as well, that is the vector $S = (m_1, s_2^{'} + s_2^{''}, \cdots, s_n^{'} + s_n^{''}) = (s_1, s_2, \cdots, s_n)$ belongs to $\widehat{\psi}_m(n)$.

The choice of this particular case is caused by theorem 2; that is that we have the description of monotone Boolean functions which correspond to the upper boundary vectors of minimum weight.

Theorem 3. $S = (m_1, s_2^{'} + s_2^{''}, \cdots, s_n^{'} + s_n^{''}) = (s_1, s_2, \cdots, s_n) \in \widehat{\psi}_m(n)$

Proof.

Suppose this is not the case and the vector $\widetilde{S} = (\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n)$, $\widetilde{S} > S$ serves as the associated vector of partitions for some monotone Boolean function (component wise comparison is used).

Consider cases:

a) $\widetilde{s}_1 = s_1$. Consider partition of E^n according to the value of x_1. We get monotone functions in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ with s_1 and $m-(s_1+1)$ ones respectively. Denote by $(s^*_2, \cdots, s^*_n)$ and $(s^{**}_2, \cdots, s^{**}_n)$ the corresponding associated vectors. Let $\widetilde{s}_2 = s_2, \cdots, \widetilde{s}_{i-1} = s_{i-1}$ and $\widetilde{s}_i > s_i$. We confirm that $s^*_2 = s'_2$ and $s^{**}_2 = s_2'', \cdots, s^*_{i-1} = s'_{i-1}$ and $s^{**}_{i-1} = s''_{i-1}$, otherwise $s^*_2 > s'_2$ (similarly $s^*_3 > s'_3, \cdots, s^*_{i-1} = s'_{i-1}$) implies $S' = (s'_2, \cdots, s'_n) \notin \widehat{\psi}_{m_1}(n-1)$ and $s^{**}_2 > s''_2$ (similarly $s^{**}_3 > s''_3, \cdots, s^{**}_{i-1} = s''_{i-1}$) implies $S'' = (s''_2, \cdots, s''_n) \notin \widehat{\psi}_{m_2}(n-1)$ due to the note above. $\widetilde{s}_i > s_i$ implies either $s^*_i > s'_i$ or $s^{**}_i > s_i''$. By the same reasoning each of them leads to contradiction.

b) $\widetilde{s}_1 > s_1$. Consider partition of E^n according to the value of x_1. We get monotone functions in $E^{n-1}_{x_1=1}$ and $E^{n-1}_{x_1=0}$ with $\widetilde{s}_1$ and $m-\widetilde{s}_1$ ones respectively. There exist $\widetilde{s}_1 - (m-\widetilde{s}_1)$ vertices in $E^{n-1}_{x_1=1}$ belonging to the set of ones of the function, for which the corresponding vertices in $E^{n-1}_{x_1=0}$ does not belong to the set of ones of the function. We move these vertices from $E^{n-1}_{x_1=1}$ to $E^{n-1}_{x_1=0}$. It follows from the case a) that $\widetilde{s}_2 = s_2, \cdots, \widetilde{s}_n = s_n$. To provide s_2 in E^n we need to provide s'_2 in $E^{n-1}_{x_1=1}$ and s_2'' in $E^{n-1}_{x_1=0}$, otherwise either $S' = (s'_2, \cdots, s'_n) \notin \widehat{\psi}_{m_1}(n-1)$ or $S'' = (s''_2, \cdots, s''_n) \notin \widehat{\psi}_{m_2}(n-1)$, etc. By theorem 2 the obtained sets are $D^{2,\cdots n}(m_1)$ and $D^{2,\cdots n}(m_2)$ respectively, which is contradiction since $m_1 \geq m_2$.

As a corollary we get that for any $\widetilde{m}$, $m/2 \leq \widetilde{m} \leq m$, a vector of $\widehat{\psi}_m(n)$ exists with a coordinate which equal to $\widetilde{m}$.

We may also mention the topic of construction of m_1-subsets of vertices with the given associated vector of partitions $S = (m_1, s'_2 + s''_2, \cdots, s'_n + s''_n) = (s_1, s_2, \cdots, s_n)$.

Given that the sets $D^{2,\cdots n}(m_1)$ and $D^{2,\cdots n}(m_2)$ can be constructed by the “interval bisection” method [SA, 2006], then a construction of the required set can be provided in the following way: construct first column putting m_1 consecutive ones and ($m - m_1$) consecutive zeros; continue the construction in this 2 sections by the same “interval bisection” method.

The weight of S and the values of coordinates can be calculated.

Conclusion

Generating numerical characteristics of partitions by means of the same characteristics in $n-1$ dimensional unit cube for a particular case, leads to more boundary elements. The corresponding subsets can be easily constructed.

Acknowledgement

The paper is partially financed by the project ITHEA XXI of the Institute of Information Theories and Applications FOI ITHEA and the Consortium FOI Bulgaria (www.ithea.org, www.foibg.com).

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