MULTICRITERION PROBLEMS ON THE COMBINATORIAL SET OF POLYARRANGEMENTS

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Abstract: The multicriterion problem of discrete optimization on the feasible combinatorial set of polyarrangements is examined. Structural properties of feasible region and different types of efficient decisions are explored. On the basis of development of ideas of Euclidean combinatorial optimization and method of general criterion possible approaches for the solution of multicriterion combinatorial problem on the set of polyrrangements is developed and substantiated.

Keywords: multicriterion optimization, discrete optimization, polyarrangements, Pareto-optimal solution, weakly and strongly efficient solutions, combinatorial set of polyarrangements.

ACM Classification Keywords: G 2.1 Combinatorics (F2.2), G 1.6 Optimization

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Introduction

The multicriteria problems of optimization on different sets continue to attract attentions of many researchers [1-10]. The models of discrete combinatorial optimization are widely used at the decision of important problems of the geometrical planning, economy, placing of objects, control process of treatment of information, acceptance of decisions and others. Lately in the area of research of different classes of combinatorial models, developments of new methods of their decision great attention is paind to the methods which are based on the use of structural properties of combinatorial sets [2, 8-15].

A new and actual problem which unites multicriterion of alternatives and feasible sets of decisions having different combinatorial characteristics is formulated and is explored in this work.

It is of common knowledge, most combinatorial optimization problems can be taken to the problems of the integer programming, but it is not always justified, as an opportunity account of combinatorial properties of problems [2] is lost here.

The systematic study of properties of Euclidean combinatorial sets and their research is described in many works Along with well - known Euclidean combinatorial sets of transpositions, placing, combinations, breaking up more complicated structures are polycombsnatorial sets are selected. Interest to such sets is conditioned by the different applied problems, as their certain number is well described by polycombinatorial constructions [12, 14].

t should be noted that problems of Euclidean combinatorial optimization on the polycombinatorial sets are combined with combinatorial polyhedrons and their properties which are the protuberant shells of such sets. The promoted interest to combinatorial and polycombinatorial configurations is conditioned by researches of the last years in the area of computer technologies at creation of modern algorithms and programs for solution of optimization problems. So polycombinatorial criteria is fated by the necessities of practice. The paper continues the studies of multicriterion problems over combinatorial and polycombinatorial sets presented in [8, 9]. The interrelation established between multicriterion problems over combinatorial sets and optimization problems over a continuous feasible set is used to study some structural properties of the feasible domain and to formulate and prove a number of theorems on the optimality conditions for different types of efficient solutions of the problems considered. We propose a polyhedral approach to solving vector combinatorial problems over a set of polyarrangements. It is based on the methods of principal criterion, cutting planes, and relaxation.

1.Problem statement. Basic definitions

The multicriterion problems are examined:

$$Z(\Phi, P_{qk}^{ns}(A,H)): \quad \max\left\{\Phi(a) \mid a \in P_{qk}^{ns}(A,H)\right\},$$

consisting of maximization of vectori criterion $\Phi(a)$ on the Euclidean combinatorial set of polyarrangements, where $\Phi(a)$ $\Phi_i : R^n \to R^1, i \in N_l$.

For statement of material we use the concept of multiset A, which is determined by foundation $S(A)$ and multiple of elements $k(a)$.

Let a multiset $A = \left\{a_1, a_2, ..., a_q\right\}$, its base $S(A) = \left\{e_1, e_2, ..., e_k\right\}$, where $e_j \in R^1$ $\forall j \in N_n$, and the multiplicity of elements $k(e_j) = r_j, j \in N_k$, $r_1 + r_2 + ... + r_k = q$, be given.

Take an arbitrary $n \in N_q$. Call the ordered n-selection from the multiset A as the set

$$a = \left(a_{i_1}, a_{i_2}, ..., a_{i_n}\right), \tag{1}$$

where $a_{i_j} \in A$ $\forall i_j \in N_n$, $\forall j \in N_n$, $i_s \neq i_t$, if $s \neq t$ $\forall s \in N_n$.

Let $P_{qk}^n(A)$ be a general combinatorial set of n-arrangements, induced by q>n elements from the multiset A, *k* its elements being different. Denote by $E(A)$ the image of the set $P_{qk}^n(A)$ mapped into R^n. Any point $x \in E(A)$ is such that its coordinates take different values from the multiset A of real numbers, i.e., $x = \left(x_1, x_2, ..., x_n\right)$, where $x_j = a_{i_j}$ $a_{i_j} \in A$ $\forall i, j \in N_n$.

Let us represent a set N_q as an ordered partition into *s* (where *s*<*q*) nonempty pairwise disjoint subsets $J_1, ..., J_s$, i.e., such that $J_i \cap J_j = \varnothing$, $J_i \neq \varnothing$, $J_j \neq \varnothing$, $\forall i, j \in N_s$, and an ordered decomposition of the number n into s terms $n_1, n_2, ..., n_s$, that satisfies the condition $1 \leq n_i \leq q_i$, $\forall i \in N_s, \left|J_i\right| = q_i$. Obviously, $q_1 + q_2 + ... + q_s = q$ and $n_1 + n_2 + ... + n_s = n$.

Denote by *H* a set of elements of the form $h = (h(1), ..., h(n)) = (h^1, ..., h^s)$, where $h(j) \in N_n$, $j \in N_n$, a, and h^i is an arbitrary permutation of elements of the set J_i $\forall i \in N_s$.

Let a submultiset A^i of the multiset A consist of the elements of A whose numbers belong to the set J_i: $A^i = \{a_1^i, ..., a_{n_i}^i\}$, $\left|J_i\right| = n_i$.

Definition 1. [12] A set

$$P_{qk}^{ns}(A,H) = \left\{(a_{h(1)}, ..., a_{h(n)}) \middle| a_{h(i)} \in A \ \forall i \in N_n, \forall h \in H\right\} \subset R^n$$

is called a general set of polyarrangements.

Without loss of generality, let us arrange elements of the multiset A in nondecreasing order: $a_1 \leq a_2 \leq \ldots \leq a_n$. Obviously, this arrangement also remains for each submultiset A^i $i \in N_s$, of A.

2. Properties of Euclidean set of polyarrangements

The convex hull of a set of polyarrangements $P^n_{qk}(A)$ is a polyhedron of polyarrangements $\Pi^{ns}_{qk}(A,H) = \text{conv}\, P^{ns}_{qk}(A,H)$, whose set of vertices consists of elements of the set of polyarrangements: $\text{vert}\, \Pi^{ns}_{qk}(A,H) = P^{ns}_{qk}(A,H)$.

Theorem 1. A polyhedron of polyarrangements $\Pi^{ns}_{qk}(A,H)$ is defined by the set of all solutions of the following system of inequalities:

$$\begin{cases} \sum_{j=1}^{n_i} x_j \le \sum_{j=1}^{n_i} a^i_j,\ i \in N_s, \\ \sum_{j=1}^{m_i} x_{\alpha_j} \ge \sum_{j=1}^{m_i} a^i_j, \quad m_i \in N_{q_i-1}, \alpha_j \in J_i, \forall i \in N_s \end{cases} \tag{2}$$

$$\alpha_j \ne \alpha_t, \forall j \ne t, \forall j,t \in J_i.$$

Let us consider some properties of the polyhedron $\Pi^{ns}_{qk}(A,H)$ and its relationship with the general set of polyarrangements.

Obviously, s subsystems of linear inequalities describing polyhedra of arrangements $\Pi^{n_i}_{q_i k_i}(A^i)$, being convex combinations of the sets of arrangements $a_{h^i}, i \in N_s$, can be separated out from the system of linear inequalities (2).

Therefore,

$$\Pi^{n_i}_{q_i k_i}(A^i) = \left\{ x \in R^{n_i} \left| \sum_{j=1}^{n_i} x_j \le \sum_{j=1}^{n_i} a^i_{q_i - j}, \sum_{j=1}^{m_i} x_{\alpha_j} \ge \sum_{j=1}^{m_i} a^i_j \right. \right\},$$

$$m_i \in N_{q_i-1}, \alpha_j \in J_i, \alpha_j \ne \alpha_t, \forall j \ne t, \forall j,t \in J_i, \forall i \in N_s.$$

The product of polyhedra $M_1,\dots,M_s$, is known to be a set $\underset{i=1}{\overset{S}{\otimes}} M_i = \left\{ x \in R^{d_1+\dots+d_s} \mid x = (x_1,\dots,x_s),\ x_i \in M_i \quad \forall i \in N_s \right\}$, where M_i is an d_i-dimensional polyhedron.

Lemma. 1) The product of polyhedrons is a polyhedron;

2) $\dim(\underset{i=1}{\overset{S}{\otimes}} M_i) = \sum_{i=1}^{S} \dim M_i$, where $\dim M$ is dimension of set M;

3) k-measured the verges of polyhedron $\underset{i=1}{\overset{S}{\otimes}} M_i$ form the set with the elements of kind $\underset{i=1}{\overset{s}{\otimes}} F_i$, where $F_i -$ k_i is the measured verge of polyhedron M_i and $k_1 + \dots + k_s = k$.

By Statement 3.2 [12],

$$\underset{i=1}{\overset{S}{\otimes}} \Pi^{n_i}_{q_i k_i}(A^i) = \left\{ x \in R^{d_1+\dots+d_s} \mid x = (x_1,\dots,x_s),\ x_i \in \Pi^{n_i}_{q_i k_i}(A^i) \quad \forall i \in N_s \right\},$$

i.e., a point $x \in \overset{s}{\underset{i=1}{\otimes}} \Pi^{n_i}_{q_i k_i}(A^i)$ satisfies each of s subsystems of the system (2). Hence, we may state that if a_{h^i} is a vertex of the polyhedron $\Pi^{n_i}_{q_i k_i}(A^i)$, then $a(h) = \overset{s}{\underset{i=1}{\otimes}} a_{h^i}$, $a(h)=a_{h^1},...,a_{h^s}$, where $a(h) \in P^{ns}_{qk}(A,H)$.

Next theorems are just [14].

Theorem 2. $\Pi^{ns}_{qk}(A,H) = \overset{s}{\underset{i=1}{\otimes}} \Pi^{n_i}_{q_i k_i}(A^i)$.

Theorem 3. For n<q, the polyhedron of polyarrangements $\Pi^{ns}_{qk}(A,H)$ is combinatorially equivalent to the polyhedron of polypermutations $\Pi^{s}_{qk}(A,H)$ of dimension n.

The vertices of the polyhedron $\Pi^{ns}_{qk}(A,H)$ are elements of the set of polyarrangements $P^{ns}_{qk}(A,H)$.

Theorem 4. A vertex $a(h) \in \text{vert}\,\Pi^{ns}_{qk}(A,H)$ is adjacent to a vertex $a(z) \in \text{vert}\Pi^{ns}_{qk}(A,H)$ if and only if $a(z)$ can be formed from $a(h)$ by a permutation of two unequal components a^i_i и a^i_j, $j \in J_{q_i - 1}$, $i \in N_s$.

Note that the total number *p* of linear inequalities appearing in the system (2) and describing the polyhedron of polyarrangements $\Pi^{ns}_{qk}(A,H)$ is very high. It can be reduced in some cases.

Statement 1. If only k out of n coordinates $x_j, i \in N_n$, of a point $x \in R^n$ are different, then the number of inequalities of the system that describe the convex polyhedron $\Pi^{ns}_{qk}(A,H)$ can be reduced by excluding $N = \sum_{i=1}^{s} N_i$ inequalities, where $N_i = 1 + q_i + \sum_{j=i+1}^{q_i} C^j_{q_i}$.

Proof. Let us call an aggregate of inequalities of the subsystem for a subset J_i, $i \in N_s$, of the system (2), having equal values m_i of the upper limit of summation, the m_i th group of inequalities of this subsystem. Each mi m_i th group includes $C^{m_i}_{q_i}$ inequalities. Hence, the total number of inequalities describing the polyhedron $\Pi^{n_i}_{q_i k_i}(A^i)$ is $p_i = \sum_{i=0}^{q_i} C^{m_i}_{q_i} = 2^{q_i}$, $i \in N_s$. Since there are k_i different coordinates a^i_j, $j \in J_i$, out of q_i, then some inequalities can be excluded from the ith subsystem of inequalities (2). In view of the condition $a_1 \le a_2 \le \ldots \le a_q$ for any $j \in N_{m_i - 1}, m_i \le q_i, i \le N_s$, the following equality holds: $a^i_j = a^i_{j+1}$. Therefore, if the inequalities of the first group in the subsystem (2) hold, the inequalities of the second, third, ... m_i th, $i \in N_s$, groups will also hold. Indeed, since $x_j \ge a^i_1$, $j \in J_i$, $i \in N_s$, the condition $\sum_{j=1}^{m_i} x_{\alpha_j} \ge m_i a^i_1$ is satisfied for any $m_i \in N_n$. Hence, the inequalities of the second, third, ..., m_i th, $i \in N_s$, groups can be excluded from each subsystem of system (2) describing the polyhedron of polyarrangements $\Pi^{ns}_{qk}(A,H)$, and the total number of inequalities in the m_i th subsystem will be $N_i = 1 + q_i + \sum_{j=i+1}^{q_i} C^j_{q_i}$. If the set of numbers

$\left(a_1^i, a_2^i, ..., a_n^i\right)$ possesses the property $a_j^i = a_{j+1}^i \;\; \forall j \in N_{n_i-1} \setminus N_{n_i-m_i}$, $i \in N_s$, the reasoning may be similar. Then it will suffice to leave only the inequalities of the first, second, …, $(m_i - j)$th group in the subsystem of the system (2). Therefore, $N = \sum_{i=1}^{s} N_i$ inequalities can be excluded from the system (2).

Let a set of polyarrangements $P_{qk}^{ns}(A,H)$ be mapped into the Euclidean space R^n and let us formulate the problem $Z(F,X)$ of maximizing a vector criterion $F(x)$ on a feasible set X:

$$Z(F,X): \max\{F(x) \mid x \in X\}.$$

To each point a $a \in P_{qk}^{ns}(A,H)$, there corresponds a point $x \in X$ such that $F(x) = \Phi(a)$, where $F(x) = (f_1(x), f_2(x), ..., f_l(x))$, $f_i : R^n \to R^1, i \in N_l$, X is a nonempty set defined as follows: $X = \text{vert}\, \Pi_{qk}^{ns}(A,H)$, where $\Pi_{qk}^{ns}(A,H) = \text{conv}\, P_{qk}^{ns}(A,H) = \Pi$. Let the problem $Z(F,X)$ contain also convex constraints that form a convex closed set $D \subset R^n$ of the form $D = \{x \in R^n \mid Bx \le d\}$. Then the feasible set $X = \text{vert}\, \Pi_{qk}^{ns}(A,H) \bigcap D$.

In multicriterion optimization problems, the traditional concept of optimality is replaced with Pareto optimality (efficiency), weak efficiency (Slater optimality), and strong efficiency (Smale ptimality). Thus, by solutions of the problem $Z(F,X)$ we will mean elements of the following sets: $P(F,X)$ of efficient (Pareto optimal) solutions, $Sl(F,X)$ of weakly efficient (Slater optimal), and $Sm(F,X)$ of strongly efficient (Smale optimal) solutions. According to [4–7], the following statements are true for each feasible $x \in X$:

$$x \in Sl(F,X) \Leftrightarrow \{y \in X \mid F(y) > F(x)\} = \varnothing \tag{3}$$

$$x \in P(F,X) \Leftrightarrow \{y \in X \mid F(y) \ge F(x), F(y) \ne F(x)\} = \varnothing \tag{4}$$

$$x \in Sm(F,X) \Leftrightarrow \{y \in X \mid y \ne x, F(y) \ge F(x)\} = \varnothing. \tag{5}$$

$$Sm(F,X) \subset P(F,X) \subset Sl(F,X).$$

Since $|X| < \infty$, the set $P(F,X) \ne \varnothing$ and is externally stable [15].

3. Structural properties and optimality conditions of different sets of efficient solutions

Theorem 5. The elements of set $Sm(F,X)$ - strictly efficient, $P(F,X)$ - Pareto-optimal, and $Sl(F,X)$- weakly efficient solutions of a multicriteria combynatorial problem over polyrranqements of the form $Z(F,X)$ are located at the vertices of polyhedron of polyrranqements $\Pi_{qk}^{ns}(A,H)$.

Proof. Taking into account correlation (6) between the introduced sets of efficient solutions and also according to fact that the set of feasible solutions X is a subset of the set of vertices of the general polyhedron of polyrranqements, that $\Pi_{qk}^{ns}(A,H)$, and $x \subset \text{vert}\, \Pi_{qk}^{ns}(A,H)$ we come to $x \subset \text{vert}\, \Pi_{qk}^{ns}(A,H)$ the justice of inclusions $Sm(F,X) \subset P(F,X) \subset Sl(F,X) \subset \text{vert}\, \Pi_{qk}^{ns}(A,H)$ takes place. The theorem is proved.

Let the functions of vectorial criterion $f_i(x), i \in N_l$, are linear, that is $F(x)$ Structural properties of feasible region and sets of different types of efficient decisions, marked in the theorem 5, and also linear of functions of vectorial criterion allows to take the decisions of problem $f_i(x)=\langle c_i,x\rangle, i\in N_l$. X to the X decision of the problem $G=\Pi_{qk}^{ns}(A,H)\bigcap D$.

Statement 2. The following inclusions are true for sets of optimal solutions of the problem $Z(F,X)$: $Sm(F,X)\subset P(F,X)\subset Sl(F,X)\subset \text{vert}\,\Pi_{qk}^{ns}(A,H)$.

Let us represent the polyhedron Π_{qk}^{ns} as $\Pi_{qk}^{ns}(A)=\{x\in R^n \mid \langle \pi_i,x\rangle\le\gamma_i, i\in N_p\}$.

Introduce a set $N(y)=\{i\in N_q \,|\pi_i,y=\gamma_i\}$, $0^+Q(y)=\{x\in R^n \mid \pi_i x\le 0, i\in N(y)\}$ is a cone that can be constructed for all points $y\in\text{vert}\,\Pi_{qk}^{ns}(A,H)$. Obviously, if $N(y)=\varnothing$, then $X\subseteq y+0^+\Pi(y)$.

The structural properties of the feasible domain X and of the sets of various types of efficient solutions mentioned in Statement 3 and the linearity of the functions of vector criterion allow reducing the problem $Z(F,X)$ to the problem $Z(F,G)$ defined on the continuous feasible set $G=\Pi_{qk}^{ns}(A)\cap D$.

Theorem 6. The following inclusions are true: $P(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)\subset P(F,X)$, $Sl(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)\subset Sl(F,X)$, and $Sm(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)\subset Sm(F,X)$.

Proof. Since $\text{vert}\,\Pi_{qk}^{ns}(A,H)\cap D\subset G$, we have

$$P(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)\cap D\subset P(F,G\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)\cap D)=P(F,X).$$

Similarly, we can prove the relationships

$$Sm(F,X)=Sm(F,D\cap\text{vert}\,\Pi_{qk}^{ns}(A,H))\supset Sm(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H).$$

$$Sl(F,X)=Sl(F,\text{vert}\,\Pi_{qk}^{ns}(A,H)\cap D)\supset Sl(F,G)\cap\text{vert}\,\Pi_{qk}^{ns}(A,H).$$

Let the functions $f_i(x), i\in N_l$, of the vector criterion $F(x)$ be linear, i.e., $f_i(x)=\langle c_i,x\rangle, i\in N_l, C\in R^{n\times l}$ is a matrix and a linear mapping $C:R^n\to R^l$, and c_i is its row vector, $i\in N_l$. Denote by $K=\left\{x\in R^n \,|Cx\ge 0\right\}$ the cone of perspective directions [4] of the problem $Z(F,X)$, $K_0=\left\{x\in R^n \,|Cx=0\right\}$ is the kernel of the mapping C, and $\text{int}\,K=\left\{x\in R^n \,|Cx>0\right\}$ is the interior of the cone K.

As follows from (3)–(5), the statements below are true $\forall x\in X$:

$$x\in Sl(C,X)\Leftrightarrow (x+\text{int}\,K)\cap X=\varnothing, \tag{7}$$

$$x\in P(C,X)\Leftrightarrow x+(K\setminus K_0)\cap X=\varnothing \tag{8}$$

$$x\in Sm(C,X)\Leftrightarrow (x+K)\cap X\setminus\{x\}=\varnothing. \tag{9}$$

Theorem 7. If the feasible set X of the problem $Z(F,X)$ contains no constraints that describe the convex polyhedral set D, or $\Pi_{qk}^{ns}(A,H)\subseteq D$, i.e., $X=\text{vert}\,\Pi_{qk}^{ns}(A,H)$, then the following equalities are true $\forall x\in R^n$:

$$Sl(F,\Pi_{qk}^{ns}(A,H))\cap\text{vert}\,\Pi_{qk}^{ns}(A)=Sl(F,X),\quad P(F,\Pi_{qk}^{ns}(A,H))\cap\text{vert}\,\Pi_{qk}^{ns}(A,H)=P(F,X),$$

$$Sm(F,\Pi_{qk}^{ns}(A))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A)=Sm(F,X)$$

Proof. As follows from Theorem 6 and conditions of this theorem, the statements below are true $\forall x\in R^n$:

$$x\in Sl(F,\Pi_{qk}^{ns}(A,H))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H)\Rightarrow x\in Sl(F,X),$$

$$x\in P(F,\Pi_{qk}^{ns}(A,H))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H)\Rightarrow x\in P(F,X),$$

$$x\in Sm(F,\Pi_{qk}^{ns}(A,H))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H)\Rightarrow x\in Sm(F,X).$$

Let us prove inverse implications. Let $x\in Sl(F,X)$, whence $x\in \text{vert}\,\Pi_{qk}^{ns}(A,H)$ by Statement 3. Suppose by contradiction that $x\notin Sl(F,\Pi_{qk}^{ns}(A,H))$. Since functions $f_i(x), i\in N_l$, of the vector criterion $F(x)$ are linear, the condition $\text{int}\,K\bigcap(\Pi(x)-x)\neq\varnothing$ is satisfied by Theorem 5 [6], i.e., the cone $(x+\text{int}\,K)$ contains some points of the boundary of the polyhedron $\Pi_{qk}^{ns}(A,H)$. Therefore, there exists a vertex $\Pi_{qk}^{ns}(A)$ belonging to this cone. By virtue of (7), this means that $x\notin Sl(F,X)$, which leads to a contradiction with the condition of the theorem. Other statements of the theorem can be proved similarly.

Corollary 1. Under the conditions of Theorem 7, the following statements are true $\forall x\in X$:

$$x\in Sl(F,X)\Leftrightarrow x\in Sl(F,\Pi_{qk}^{ns}(A,H))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H),$$

$$x\in P(F,X)\Leftrightarrow x\in P(F,\Pi_{qk}^{ns}(A,H))\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H),$$

$$x\in Sm(F,X)\Leftrightarrow x\in Sm(F,\Pi_{qk}^{ns}(A,H)\bigcap \text{vert}\,\Pi_{qk}^{ns}(A,H).$$

If the feasible domain $X=\text{vert}\,\Pi_{qk}^{ns}(A,H)$, then the necessary and sufficient optimality conditions obtained in [6] for all the above-mentioned types of efficient solutions are true for any point $x=\text{vert}\,\Pi_{qk}^{ns}(A,H)$ of the problem $P(F,X)$. If $\Pi_{qk}^{ns}(A)\bigcap D\neq\Pi_{qk}^{ns}(A)$, then only sufficient optimality conditions are true.

Theorem 8. For an arbitrary $x=\text{vert}\,\Pi_{qk}^{ns}(A)$,

$$x\in P(F,\Pi_{qk}^{ns}(A))\bigcap D\Rightarrow x\in P(F,X),$$

$$x\in Sl(F,\Pi_{qk}^{ns}(A))\bigcap D\Rightarrow x\in Sl(F,X),$$

$$x\in Sm(F,\Pi_{qk}^{ns}(A))\bigcap D\Rightarrow x\in Sm(F,X).$$

Proof. Since $G=\Pi\bigcap D,$ the following implications are true:

$$\forall x\in \text{vert}\,\Pi_{qk}^{ns}(A,H): x\in P(F,\Pi_{qk}^{ns}(A,H))\bigcap D\Rightarrow x\in P(F,\Pi_{qk}^{ns}(A,H)\bigcap D)=P(F,G)\Rightarrow x\in P(F,X),$$

$$x\in Sl(F,\Pi_{qk}^{ns}(A,H))\bigcap D\Rightarrow x\in Sl(F,X),\ x\in Sm(F,\Pi_{qk}^{ns}(A,H))\bigcap D\Rightarrow x\in Sm(F,X).$$

Thus, Theorems 5 – 8 establish an interrelation between the problem $Z(F,X)$ and the problem $Z(F,G)$ defined over a continuous feasible set. It enables to apply the classic methods of continuous optimization to the decision of vectorial combinatorial problems on polyrrangements and on this basis develop new original methods of decision, using properties of combinatorial sets and their protuberant shells.

If the problem $Z(F,X)$ does not contain linear limitations forming a convex polyhedral set $D \subset R^n$, or if we have $\Pi \subseteq D$, i.e. $X = \mathrm{vert}\,\Pi$, then, taking into account necessary and sufficient optimality conditions (theorem 7), the process of its solution is reduced to the search for efficient solutions of the problem $Z(F,G)$ over the continuous feasible set $G = \Pi_{qk}^{ns}(A,H)$ with the subsequent choice of only the solutions thatf are vertices of the permutable polyhedron of polyrrangements $\Pi_{qk}^{ns}(A,H)$.

Analyzing theorems 6 and 8, we obtain the following relationships between the problems $Z(F,X)$ and $Z(F,G)$: if we have $x \in R(F,G) \cap vert\Pi_{qk}^{ns}(A,H),$ then $x \in R(F,X),$ and if we have $x \notin R(F,G) \cap vert\Pi_{qk}^{ns}(A,H),$ then this does not imply that $x \notin R(F,X)$, where $R(F,X)$ denotes the set set $P(F,X),$ $Sm(F,X)$ or $Sl(F,X)$.

4. Main approach of the task contains additional linear limitationds, the following approach its decision is offered.

If the problem $Z(F,X)$ contains additional linear constraints, then the following approach to its solve is proposed.

1. We find the efficient solutions of the problem $Z(F,\Pi_{qk}^{ns}(A,H))$.

2. We check their membership in the set D. If we have $x \in P(F,\Pi_{qk}^{ns}(A,H)) \cap D,$ then $x \in P(F,X).$

3. We will consider feasible solutions $x \in X$ to the problem $Z(F,X)$ that are inefficient in the problem $Z(F,\Pi)$, i.e. $x \in X \setminus P(F,\Pi_{qk}^{ns}(A,H)) \cap D$, and check them for efficiency. To this and, use the necessary and sufficient conditions formulated in [15].

Statement 3. A feasible solution x^0 is efficient if and only if it is an optimal solution of the following problem:

$$Z^1(F,X): \max\left\{\sum_{i=1}^{m} f_i(x) \mid x \in X,\, f_i(x) \geq f_i(x^0),\, i \in N_m\right\}.$$

If the solution x^0 is inefficient, then, as a result of solution of this problem, we find an efficient solution x^* that is more preferable than x^0, i.e., we have $F(x^*) \geq F(x^0).$

Continuing investigations and developing the results of [1, 5, 6, 8-13], we propose an approach to the solution of the problem $Z(F,X)$ on the basis of linear convolution (aggregation) of its partial criteria and the further reduction of the search for solutions of the initial problem to the solution of a series of scalar (one-criterion) problems and the check of the obtained solutions for optimality. The method of solution of one-criterion problems is based on the ideas of decomposition, Kelly 's cutting-planes, and relaxation.

Next, we consider a method whose realisation takes into account the fact that the number of constraints is sufficiently large. Then it is expedient to use a relaxation procedure or temporary rejection of some constraints and the solution of a problem over a wider domain, i.e., under remained constraints.

At the initial stage of construction of the sought-for algorithm, we should determine the initial point. We will consider a one-criterion problem without constraints that describe a polyhedron *D* and call them additional constraints.

Statement 4. If, for the elements of the multiset *A* and coefficients $c_j,\ j\in N_n$, of the objective function of the problem a maximum functions $f(x)$, conditions $c_{i_1}\le c_{i_2}\le\ldots\le c_{i_n}$ and $a_1\le a_2\le\ldots\le a_n$, respectively, are satisfied, on the admissible set is attained at a point $x^*=(x^*_{i_1},\ldots,x^*_{i_n})\in \operatorname{vert}\Pi^{ns}_{qk}(A,H)$ that is specified as follows:

$$x^*_{i_j}=a_j\ \forall\ j\in N_n, \tag{10}$$

and its minimum is accordingly attained at a point $y=(y_{i_1},y_{i_2},\ldots,y_{i_n})$, where

$$y_{i_{j+1}}=a_{n-j}\ \forall\ j\in N_{n-1}\cup\{0\}.$$

For description and basing of method of decision of the problem we will introduce next denotations. We write down the feasible region of the problem $Z(F,X)$ in the form $G=\{x\in R^n \mid Hx\le g\}$, $g=(g_1,g_2,\ldots,g_q)$ is matrix, which is used for the matrix-vectorial form of record of limitations of the form (2) and linear inequalities describing a polyhedron D, where all limitations are taken to one $\le$ kind of inequalities. We will designate N_q the set, the elements of which determine the numbers of limitations of the system (2) and additional limitations describing the protuberant many-sided set $D:=N_q$.

We define sets $G_i=\{x\in R^n \,|\, \langle h_i,x\rangle\le g_i\}, i\in N_q$ and, for an arbitrary $x^s\in R^n$, define sets $N^a(x^\nu)=\{i\in N_q \,|\, \langle h_i,x^\nu\rangle=g_i\}$ – accordingly active and nonactive limitations in the point; x^ν - accordingly $i\in N_q$ the vector-line of matrix and i component of vector H.

We will submit a problem in to consideration, the problem where $Z(F,G^\nu):\max\{F(x)\,|\,x\in G^\nu\}$ is set of indexes of limitations, describing the feasible region of problem $G^\nu=\{x\in R^n \,|\, \langle h_i,x\rangle\le g_i,\ i\in Q_\nu\subset N_q\}$, , which is solved on м step of algorithm, $Z(F,G^\nu)$, is set of numbers of limitations which were not included in this problem on м step.

Definition 5. We call the quantity $r_i(x)=\langle h_i,x\rangle-g_i,\ i\in N_q$, a deviation of point $x\in R^n$ from the boundary of a set G_i and the quantity $r(x)=\max\{r_i(x)\,|\, i\in N_q\}$ a deviation of point $x\in R^n$ from the boundary of the set G. It is obvious that, for $i\in N_p$, we have

$$r_i(x)=\sum_{j=1}^{i}x_{\alpha_j}-\sum_{j=1}^{i}a^i_j\ , \tag{11}$$

and for $i\in N_q\setminus N_p$, we have

$$r_i(x)=\langle b_i,x\rangle-d_i\ , \tag{12}$$

where b_i is the i th row vector of the matrix $B, d_i\in R$.

Theorem 9. An efficient (Pareto-optimal, weakly efficient, and strictly efficient) solution x^0 of problem $Z(F, G^{\nu})$ is an efficient (in the same sense) solution of the problem $Z(F, G)$ if and only if the condition $r(x) \le 0$ is true.

Proof. The necessity of this statement is obvious since the feasible solution x^0 of the problem $Z(f, G^{\nu})$ is a feasible solution of the problem $Z(F, G)$ if and only if the condition $r(x) \le 0$ is satisfied. The sufficiency of this statement follows from the construction of the problem $Z(F, G)$ and the definition of $r(x)$.

The approach to solving the class of vector problems proposed here is to reduce the original multicriterion problem to an optimization problem with one criterion $f_r(x), r \in N_l$, which is declared principal or basic provided that the values of all other criteria should be no less than some prescribed (threshold) values $t_i, i \in N_l \setminus \{r\}$. Thus, we have the problem

$$Z(f_r, X(t_i)) : \max \left\{ f_r(x) \middle| f_i(x) \ge t_i, i \in N_l \setminus \{r\}, x \in X \right\}.$$

The optimal solution x^0 of this problem is always weakly efficient, and if it is unique (up to equivalence $\sim_f$), it is also efficient. If the solution x^0 is efficient, then it is a unique (up to equivalence $\sim_f$) solution of the problem $Z(f_r, X(t_i))$ for any fixed $r \in N_l$ and $t_i = f_i(x^0), i \in N_l \setminus \{r\}$. Choosing a criterion as the principal one does not limit the choice of the optimal solution. To determine the threshold values t_i, $i \in N_l \setminus \{r\}$, Statement 4 can be used, which makes it feasible to establish the upper and lower bounds for the criteria $f_i(x)$, $i \in N_l$, on a set of polyarrangements. We propose two approaches to solve the original problem $Z(F, X)$. The first approach assigns the minimum values of criteria $f_i(x)$, $i \in N_l$, on the set of polyarrangements to the thresholds $t_i \in N_l \setminus \{r\}$ followed by the reduction of the feasible set of the problem $Z(f_r, X)$ by choosing the values of thresholds $t_i \in N_l \setminus \{r\}$, arranged in increasing order, next to the minimum values of the criteria. The second approach searches for the optimal solution of the problem $Z(f_r, X)$ by assigning the greatest feasible values to the criteria $f_i(x)$, $i \in N_l$, and then expanding its feasible domain if the original problem appears inadmissible; if it is admissible, an efficient or weakly efficient solution is found.

The procedure of assigning constraints to a series of threshold values t_i is simple in both approaches. Using Statement 4 and ordering the coefficients of the criteria, we reduce this procedure to computing the scalar product of two vectors, i.e., to finding values of linear criteria. Taking into account the structural features of the set of polyarrangements, it is feasible to compute t_i more efficiently, using permutations of the elements of each i th, $i \in N_s$, subset of the multiset A.

The general idea of the method proposed to solve the problem $Z(F, X)$ consists in successive inclusion of constraints of the problem that describe the feasible region.

1. Reduce the multicriterion problem $Z(F, G)$ to a one-criterion problem $Z(f, G)$ using the principal-criterion method. Put $\nu = 0$.

2. Select constraints of the original system of linear inequalities that describe the feasible set $G^{\nu} \subset G$ of the problem $Z(f, G^{\nu})$ and use the simplex method to find its optimal solution x^{ν}.

3. If the optimal solution x^V is an element of the set of polyarrangements, then, at the point x^V, check the constraints that have not been taken into account. Obviously, they can only be constraints that describe the convex closed set D. If the solution x^V does not satisfy some constraints, then supplement constraints of the feasible region of the problem $Z(f, G^V)$, with the worst-satisfied constraint from the convex closed set *D*. If the solution x^V satisfies the above-mentioned constraints, it is an efficient solution of the problem $Z(F, G)$ and, hence, of the problem $Z(F, X)$.

4. If the solution x^V is not a point of the set of polyarrangements, make a cutoff through adjacent vertices that cuts off a vertex not being feasible (i.e., polyarrangement). Add this cutoff to constraints of the problem $Z(f, G^V)$.

5. Compare the value $f(x^V)$ of the objective function with its value found at the previous step. If it decreases, reject the constraints insignificant at the point x^V. If the value of $f(x^V)$ does not change, do not reject the constraints. Use the changed feasible region go to Step 2 to solve the problem $Z(f, G^V)$.

Obviously, the algorithm results in an efficient solution of the problem $f(x^V)$ or establishes its unsolvability by solving a finite number of subproblems of the form $Z(f, G^V)$.

Conclusions

A vector combinatorial problem has been analyzed using the information about the convex hull of the feasible region and the properties of polyhedra whose vertices determine a prescribed combinatorial set of polyrrangements. A method to solve complex multicriterion problems over this combinatorial set have been developed and substantiated. Using the structural properties of combinatorial polyhedra makes it possible to develop efficient algorithms to solve new classes of vector combinatoria optimization problems.

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Bibliography

[1] I. V. Sergienko, Mathematical Models and Methods to Solve Discrete Optimization Problems [in Russian], Naukova Dumka, Kyiv (1988).

[2] I. V. Sergienko and M. F. Kaspshitskaya, Models and Methods of Computer Solution of Combinatorial Optimization Problems [in Russian], Naukova Dumka, Kyiv (1981).

[3] I. V. Sergienko and V. P. Shilo, Discrete Optimization Problems: Challenges, Solution Techniques, and Investigations [in Russian], Naukova Dumka, Kyiv (2003).

[4] I. V. Sergienko, L. N. Kozeratskaya, and T. T. Lebedeva, Stability and Parametric Analyses of Discrete Optimization Problems [in Russian], Naukova Dumka, Kyiv (1995).

[5] I. V. Sergienko, T.T. Lebedeva, and N. V. Semenova, "Existence of solutions in vector optimization problems," Cybern. Syst. Analysis, 36, No. 6, 823–828 (2000).

[6] T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, "Optimality and solvability conditions in linear vector optimization problems with convex feasible region," Dop. NANU, No. 10, 80–85 (2003).

[7] T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, "Stability of vector problems of integer optimization: Relationship with the stability of sets of optimal and nonoptimal solutions," Cybern. Syst. Analysis, 41, No. 4, 551–558 (2005).

[8] N. V. Semenova, L. N. Kolechkina, and A. N. Nagirna, "An approach to solving discrete vector optimization problems over a combinatorial set of permutations," Cybern. Syst. Analysis, 44, No. 3, 441–451 (2008).

[9] N. V. Semenova, L. M. Kolechkina, and A. M. Nagirna, "Vector combinatorial problems in a space of combinations with linear fractional functions of criteria," Inform. Theor. Appl., 15, 240–245 (2008).

[10] V. V. Podinovskii and V. D. Nogin, Pareto-Optimal Solutions of Multicriterion Problems [in Russian], Nauka, Moscow (1982).

[11] Yu. G. Stoyan and S. V. Yakovlev, Mathematical Models and Optimization Methods of Geometric Design [in Russian], Naukova Dumka, Kyiv (1986).

[12] Yu. G. Stoyan, O. O. Yemets, and E. M. Yemets, Optimization over Polyarrangements: Theory and Methods [in Ukrainian], RVTs PUSKU, Poltava (2005).

[13] O. O. Yemets and L. M. Kolechkina, Combinatorial Optimization Problems with Linear Fractional Objective Functions [in Ukrainian], Naukova Dumka, Kyiv (2005).

[14] V. A. Emelichev, M. M. Kovalev, and M. K. Kravtsov, Polyhedra, Graphs, and Optimization [in Russian], Nauka, Moscow (1981).

[15] K. Aardal and S. Hoesel, "Polyhedral techniques in combinatorial optimization. II: Computations," Statist. Neerlandica, 53, 131–177 (1999).

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