Decision Making

PROCEDURES OF SEQUENTIAL ANALYSIS AND SIFTING OF VARIANTS FOR THE LINEAR ORDERING PROBLEM

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Abstract: This paper investigates the procedures of sequential analysis and sifting of unpromising variants after restrictions and after the restriction on the goal function. On the basis of the modified procedure W and the general scheme of sequential analysis, the algorithm of solving the linear ordering problem is developed for the problems of discrete optimization.

Keywords: the linear ordering problem, the sequential analysis of variants.

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Introduction

The linear ordering problem (LOP) is NP-hard combinatorial optimization problem with wide application: in collective decision making, economics, archaeology and scheduling [Reinelt, 1985]. Many works are devoted to the development of efficient algorithms for solving this problem [Grotschel, 1984], [Chanas, 1996], [Laguna, 1999], [Campos, 2001].

Problem formulationa

LOP can be formulated as follows. Consider a set of alternatives $A = \{A_1,...,A_n\}$ and permutation $\pi : A \to A$. Each permutation $\pi = (\pi_1,...,\pi_n)$ uniquely determines some linear ordering of alternatives. Denote by e_{ij}, $i, j \in N = \{1,...,n\}$ the cost of disposition of alternative A_i to alternative A_j in linear order, and by E the n-square matrix of costs. Then LOP consists in finding such permutation π, in which the maximum total cost is achieved

$$E(\pi) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} e_{\pi_i \pi_j} \qquad (1)$$

Evidently, that (1) is the sum of elements above the main diagonal of matrix P, its' elements p_{ij} are the result of permutation of π lines and columns of matrix E, that is $P = XEX^T$, where X is the matrix, that corresponds to the permutation π [Reinelt, 1985].

As in most combinatorial optimization problems, LOP has many alternative formulations. In the paper [Grotschel, 1984] the linear ordering problem is considered in the equivalent raising of linear programming problem with Boolean variables.

$$E(x)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}d_{ij}x_{ij}\to\max \qquad (2)$$

$$x_{ij}+x_{jk}-x_{ik}\le 1,\ 1\le i<j<k\le n, \qquad (3)$$

$$-x_{ij}-x_{jk}+x_{ik}\le 0,\ 1\le i<j<k\le n, \qquad (4)$$

$$x_{ij}\in X_{ij},\ 1\le i<j\le n, \qquad (5)$$

where $d_{ij}=e_{ij}-e_{ji}$, $X_{ij}=\{0,1\}$ are the set of possible variants of values of component x_{ij}, $X=\prod_{j>i}X_{ij}$ is the set of all possible variants.

Approach to solve the problem

The basis for developing an algorithm to solve the problem (2) – (5) is the method of sequential analysis, elimination and creation of variants [Mikhalevich, 1965], [Volkovich, Voloshin, 1978, 1984,1993].

Procedures of sequential analysis

For the discrete optimization problems the basis of schemes of sequential analysis and elimination of variants without step-by-step constructing solution is the procedure W of consistent elimination (exclusion from the consideration, generally speaking, on the current step) of values of variables. The procedure W, in its turn, consists of two procedures W_1 and W_2. Specify them for our case.

Let's choose any indexes $i,j,k\in N$, but such as $k>j>i$. Following the general scheme of sequential analysis and elimination of variants for the problem with linear restriction, we receive the following criteria of elimination after the restrictions (procedure W_1 on the s-step, $s\in\{1,...,s_0\}$):

for fixed $j>i$, if

$$\left[\begin{array}{l}\bar{x}_{ij}>1-\min\limits_{X_{jk}^{(s)}}x_{jk}+\max\limits_{X_{ik}^{(s)}}x_{ik}\\ \bar{x}_{ij}<-\max\limits_{X_{jk}^{(s)}}x_{jk}+\min\limits_{X_{ik}^{(s)}}x_{ik}\end{array}\right. \qquad (6)$$

at least for one $k:k>j>i$, then the component x_{ij} of permissible solution of problem (2) – (5) can't take the value equal to $\bar{x}_{ij}\in X_{ij}^{(s)}$,where $X_{ij}^{(s)}$ is the set of possible variants of component x_{ij} on the step s, $s\in\{1,...,s_0\}$;

for fixed $k>j$, if

$$\left[\begin{array}{l}\bar{x}_{jk}>1-\min\limits_{X_{ij}^{(s)}}x_{ij}+\max\limits_{X_{ik}^{(s)}}x_{ik}\\ \bar{x}_{jk}<-\max\limits_{X_{ij}^{(s)}}x_{ij}+\min\limits_{X_{ik}^{(s)}}x_{ik}\end{array}\right. \qquad (7)$$

at least for one $i:k>j>i$, then the component x_{jk} of permissible solution of problem (2) – (5) can't take the value equal to $\bar{x}_{jk}\in X_{jk}^{(s)}$;

for fixed $k>i$, if

$$\left[\begin{array}{l}\overline{x}_{ik} < \min\limits_{X_{ij}^{(s)}} x_{ij} + \min\limits_{X_{jk}^{(s)}} x_{jk} - 1 \\ \overline{x}_{ik} > \max\limits_{X_{ij}^{(s)}} x_{ij} + \max\limits_{X_{jk}^{(s)}} x_{jk}\end{array}\right. \tag{8}$$

at least for one $j : k > j > i$, then the component x_{ik} of permissible solution of problem (2) – (5) can't take the value equal to $\overline{x}_{ik} \in X_{ik}^{(s)}$.

1) Let $\left|X_{ij}^{(s)}\right| \cdot \left|X_{jk}^{(s)}\right| \cdot \left|X_{ik}^{(s)}\right| \geq 4$. Realized the selection of all possible situations and analysed them by relevant criteria (6) – (8), it is easy to make sure, that the elimination won't be in this case.

2) Let $\left|X_{ij}^{(s)}\right| \cdot \left|X_{jk}^{(s)}\right| \cdot \left|X_{ik}^{(s)}\right| = 2$. Let $\left|X_{ij}^{(s)}\right| = 2$. It is easy to see, when $X_{jk}^{(s)} = \{0\}$ i $X_{ik}^{(s)} = \{1\}$, we'll have the elimination of value $\overline{x}_{ij} = 0$. Similarly if $X_{jk}^{(s)} = \{1\}$ and $X_{ik}^{(s)} = \{0\}$, we'll have the elimination of value $\overline{x}_{ij} = 1$. In case when $\left|X_{jk}^{(s)}\right| = 2$ and if $X_{ij}^{(s)} = \{0\}$, $X_{jk}^{(s)} = \{0,1\}$, $X_{ik}^{(s)} = \{1\}$, the value $\overline{x}_{jk} = 0$ will be eliminated, and in case $X_{ij}^{(s)} = \{1\}$, $X_{jk}^{(s)} = \{0,1\}$, $X_{ik}^{(s)} = \{0\}$ the value $\overline{x}_{jk} = 1$ will be eliminated. At $\left|X_{ik}^{(s)}\right| = 2$, when we have $X_{ij}^{(s)} = \{0\}$, $X_{jk}^{(s)} = \{0\}$, $X_{ik}^{(s)} = \{0,1\}$, the value $\overline{x}_{ik} = 1$ will be eliminated after the criterion (8), and if $X_{ij}^{(s)} = \{1\}$, $X_{jk}^{(s)} = \{1\}$, $X_{ik}^{(s)} = \{0,1\}$, then $\overline{x}_{ik} = 0$ will be eliminated. Like to the previous case, realized the selection of all other possible situations (related to the case examined by us), it is easy to make sure, that the elimination after the criteria (6) – (8) will never be.

3) Let $\left|X_{ij}^{(s)}\right| = \left|X_{jk}^{(s)}\right| = \left|X_{ik}^{(s)}\right| = 1$. If $X_{ij}^{(s)} = \{0\}$, $X_{jk}^{(s)} = \{0\}$ and $X_{ik}^{(s)} = \{1\}$, then, after the proper criterion of elimination, we receive $X_{ij}^{(s)} = \varnothing$. If $X_{ij}^{(s)} = \{1\}$, $X_{jk}^{(s)} = \{1\}$ and $X_{ik}^{(s)} = \{0\}$, then we'll also receive $X_{ij}^{(s)} = \varnothing$. Such variants make it impossible to build any allowed variant of problem (2) - (5) and describe the situation, when on the s-step we have an abnormal end of procedure W_1. In all other cases, evidently, it won't be the elimination.

From the total scheme follows that the criteria of elimination after the restriction on the objective function (procedure W_2 on the s-step ($s = 1,\dots,s_0$)) will be:

If

$$d_{ij}\overline{x}_{ij} < e_s^* - \max_{X^{(s)}/X_{ij}^{(s)}} \left\{ \sum_{\substack{t>k \\ (k \neq i) \vee (t \neq j)}} d_{kt}x_{kt} \right\} = e_s^* - \sum_{\substack{t>k \\ (k \neq i) \vee (t \neq j)}} \max_{X_{kt}^{(s)}} \{d_{kt}x_{kt}\},\ (j > i) \tag{9}$$

then the component x_{ij} of permissible solution of problem (2)-(5) can't take the value equal to $\overline{x}_{ij}$, where e_s^* some value taken from the interval $[\min\limits_{X^{(s-1)}} E(x), \max\limits_{X^{(s-1)}} E(x)]$.

The follow cases are possible during the application of procedure W_2.

Case 1. It wasn't any elimination. Then the contraction of set of possible variants may occur, if reinforce the inequality (9), which, evidently, can be done only by increasing restrictions on the objective function, selecting, for example, the value e_{s+1}^* by the method of dichotomy from the interval $[e_s^*, e_{max}^{(s)}]$, where

$$e_{max}^{(s)} = \sum_{j>i} \max_{X_{ij}^{(s-1)}} \{d_{ij}x_{ij}\}$$

is the maximum possible value of objective function on the step s.

Case 2. It was the elimination, but the reduced set of possible variants $X^{(s)}$ is empty or $X^{(s)} \neq \varnothing$, but there is no allowable variant of problem (2) - (5). In this case, the abnormal end of procedure W_2 is carried out. It is also known [Volkovich, Voloshin, 1978] that, in this case, any allowable variant of problem (2) - (5) satisfies the conditions $E(x_D) < e_s^*$. Then the extension of set $X^{(s)}$ can be obtained by reducing the value e_s^*, thus weakening the further restriction on the objective function, by choosing $e_{s+1}^* < e_s^*$ after the rule of dichotomy from the interval $[e_{min}^{(s)}, e_s^*]$, where $e_{min}^{(s)} = \sum_{j>i} \min_{X_{ij}^{(s-1)}} \{d_{ij} x_{ij}\}$ is the minimum possible value of objective function on the step s.

Statement 1. The value $x_{ij}^{(max)} = \arg\max_{X_{ij}^{(s-1)}} \{d_{ij} x_{ij}\}$, $\forall\ i, j \in N$, $j > i$ and $\forall\ s \in \{1, ..., s_0\}$ can't be eliminated for LOP as a result of procedure W_2 work

Proof. Suppose the opposite. Let $\overline{S} \subseteq \{1, ..., s_0\}$ is the set of all steps for which our supposition is correct. And let $\tilde{s} = \arg\min_{s \in \overline{S}} s$ is the first of these steps (exactly at this step W_2 the elimination will be firstly done after the supposition), on which at least one pair of indexes $j > i$ will be found such that

$$d_{ij} x_{ij}^{(max)} < e_{\tilde{s}}^* - \sum_{\substack{t>k \\ (k \neq i) \vee (t \neq j)}} \max_{X_{kt}^{(\tilde{s})}} \{d_{kt} x_{kt}\} \tag{10}$$

Let $N_i^{(\tilde{s})}$, $N_j^{(\tilde{s})}$ are the sets that completely describe all pair of indexes $j > i$, $i \in N_i^{(\tilde{s})}$, $j \in N_j^{(\tilde{s})}$ for which on the step $\tilde{s}$ is executed (10). At first, let take the first of these pairs $j_{\tilde{s}} > i_{\tilde{s}}$ (exactly for this pair W_2 the elimination will be firstly done after the supposition), that is $i_{\tilde{s}} = \arg\min_{i \in N_i^{(\tilde{s})}} i$ and $j_{\tilde{s}} = \arg\min_{j \in N_j^{(\tilde{s})}} j$. In this case the rule of choice $\tilde{s}$ and pair $j_{\tilde{s}} > i_{\tilde{s}}$ gives an opportunity to rewrite the inequality (10) in the following equivalent form:

$$\max_{X_{i_{\tilde{s}} j_{\tilde{s}}}^{(\tilde{s}-1)}} \{d_{ij} x_{ij}\} + \sum_{\substack{t>k \\ (k \neq i_{\tilde{s}}) \vee (t \neq j_{\tilde{s}})}} \max_{X_{kt}^{(\tilde{s}-1)}} \{d_{kt} x_{kt}\} < e_{\tilde{s}}^*,$$

from where we get correlation

$$e_{max}^{(\tilde{s})} < e_{\tilde{s}}^*. \tag{11}$$

But (11) contradicts the rule of choice on any step s of value e_s^* from the interval $[e_{min}^{(s)}, e_{max}^{(s)}]$. By analogical reasoning, through the finite number of steps firstly we are convinced of the impossibility of our assumption for the step $\tilde{s}$, and then after a similar reasoning for the other steps we arrive at the correctness of statement. The statement is proved.

Corollary 1. For LOP the first situation described in case 2 can't arrise.

Corollary 2. For LOP the elimination of values of the component x_{ij} after the restriction on the objective function can occur only when $\left|X_{ij}^{(s-1)}\right| = 2$, thus the values $x_{ij}^{(min)} = \arg\min_{X_{ij}^{(s-1)}} \{d_{ij} x_{ij}\}$, $\forall\ i, j \in N$, $j > i$ and $\forall\ s \in \{1, ..., s_0\}$ can only be eliminated.

Corollary 3. For LOP the condition of elimination through the objective function can be rewritten in the following equivalent form:

If

$$d_{ij}x_{ij}^{(min)} < e_s^* - e_{max}^{(s)} + \max_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\} \quad (12)$$

then the component x_{ij} of permissible solution of problem (2) - (5) can't take value equal to $x_{ij}^{(min)}$.

From (12) we get:

$$e_{max}^{(s)} + \min_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\} - \max_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\} < e_s^* \quad (13)$$

Of the form (13) follows that for a given value e_s^* if the value $x_{i_1j_1}^{(min)}$ eliminates, we have

$$\min_{X_{i_1j_1}^{(s-1)}}\{d_{i_1j_1}x_{i_1j_1}\} - \max_{X_{i_1j_1}^{(s-1)}}\{d_{i_1j_1}x_{i_1j_1}\} \geq \min_{X_{i_2j_2}^{(s-1)}}\{d_{i_2j_2}x_{i_2j_2}\} - \max_{X_{i_2j_2}^{(s-1)}}\{d_{i_2j_2}x_{i_2j_2}\},$$

then the value $x_{i_2j_2}^{(min)}$ will be also eliminated. Then in order not to have the second situation of case 2, evidently, it is necessary to permit the elimination of all values $x_{i^*j^*}^{(min)}$,

where

$$(i^*, j^*) \in \operatorname{Arg} \min_{(i,j):i<j}\{\min_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\} - \max_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\}\}.$$

Remark 1. Note that the elimination of all such values can be obtained by using strict limit on the objective function $E(x) > e_s^*$ and by choosing

$$e_s^* = e_{max}^{(s)} + \min_{(i,j):i<j}\left(\min_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\} - \max_{X_{ij}^{(s-1)}}\{d_{ij}x_{ij}\}\right).$$

Algorithm

Step 1. Calculation of values $e_{max}^{(s)}$ and e_s^*.

Go to the next step.

Step 2. Application of procedure W_2, that is the elimination of all values $x_{i^*j^*}^{(min)}$, where

$$(i^*, j^*) \in \operatorname{Arg} \max_{(i,j):\begin{cases} i<j \\ |X_{ij}^{(s-1)}|=2 \end{cases}} |e_{ij} - e_{ji}|.$$

Step 3. Application of procedure W_1.

At the abnormal end of procedure W_1 the restoring of all eliminated on a current and previous step values and the end of algorithm work are realized.

Otherwise, if $\prod_{j>i} |X_{ij}^{(s)}| = 1$,

then it is the end of algorithm, otherwise it is the passage to the step 1.

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Conclusion

From point of view of realization the proposed algorithm is simple. If the found variant of solution $\tilde{x}$ satisfies the condition $E(\tilde{x}) > e^*_{s_0}$, it is an optimum variant [Volkovich, Voloshin, 1978, 1984, 1993]. The variant $\tilde{x}$ can be taken as an approximate solution if $E(\tilde{x}) \le e^*_{s_0}$. In case of abnormal end of procedure W_1 we receive the reduction of set of possible variants. The algorithm work efficiency is investigated on the real test dataset containing 49 matrices «production cost» from the previous years of some European countries and is popular in Internet [LOLIB]. For 10 of 49 examples an optimum value was found.

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Bibliography

[Campos, 2001] Campos V., Glover F., Laguna M., Marti R. An experimental evaluation of a scatter search for the linear ordering problem// Journal of Global Optimization. – 2001.– vol. 21.–№ 4.– P. 397-414.

[Chanas, 1996] Chanas S., Kobylanski P. A new heuristic algorithm solving the linear ordering problem// ComputationalOptimization and Applications.– 1996.– vol. 6.– P. 191-205.

[Grotschel, 1984] Grotschel M., Jünger M., Reinelt G. A cutting plane algorithm for the linear ordering problem// Operations Research. – 1984. – vol. 2.– №6.– P. 1195-1220.

[Laguna,1999] Laguna M., Marti R., Campos V. Intensification and diversification with elite tabu search solutions for the linear ordering problem// Computers \& Operations Research.–1999. –vol. 26.- P. 1217-1230.

[LOLIB] http://www.iwr.uni-heidelberg.de/groups/comopt/software/LOLIB.

[Mikhalevich, 1965] Mikhalevich V.S. Consecutive optimization algorithms and their application. I. II// Cybernetics. – 1965. – №1. – P. 45-55; – №2. – P. 85-88.

[Reinelt, 1985]. Reinelt G. The linear ordering problem: algorithms and applications. Research and Exposition in Mathematics. – Berlin, Germany: Heldermann Verlag, 1985.

[Volkovich, Voloshin, 1978] Волкович В.Л., Волошин А.Ф. Об одной схеме метода последовательного анализа и отсеивания вариантов//Кибернетика,1978,№4.-С.99-105.

[Volkovich, Voloshin, 1993] Волкович В.Л., Волошин А.Ф. и др.Модели и методы оптимизации надежности сложных систем.- Киев: Наукова думка,1993.-312с.

[Volkovich, Voloshin,, 1984] Волкович В.Л., Волошин А.Ф. и др. Методы и алгоритмы автоматизированного проектирования сложных систем управления. – Киев: Наукова думка, 1984. – 216 с.

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