Economics Decision Support Systems

COLLECTIVE PRODUCT COST SHARING IN CONDITIONS OF MANAGED ECONOMY

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Abstract: The collective product cost-sharing problems in the conditions of the managed economy are being observed. The fuzzy generalization and "dual" approach to the solution of this problem is offered.

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Introduction

The distribution of costs, profits and resources (financial, energy e.t.c) is one of the most important problems existing today. Thus, in 2007 the Nobel Prize in economy was granted to U.S. scientists Leonid Hurwicz, Eric Maskin and Roger Myerson for research of cost sharing processes in the conditions of managed economy [Nobelprize].

Liberal economists proclaim that state shall not interfere into economic relations. However, the classic liberal economic model with its cult of liberation of economic relations from the state's interference proved to be unable to cope effectively with multiple relevant challenges. Unregulated economic relations are abstract. Maximizing their profits, major economic players (monopolists and monopsonists [Ponomarenko, Perestyuk, Burym 2004]) do not pay much attention to solving social problems (including ecological). At the same time, existence of monopolies creates certain advantages. For example, in the situation when the production technology has increasing returns to scale, monopoly is more efficient at the production. Even under decreasing returns to scale, the case for joint production frequently arises (the workers of the cooperative jointly own its machines, and the partners of the law firm share its patrons [Moulin, 1991]). Considering the above, state regulation of economic activities is becoming a very important issue. The symbiosis of market and state methods of regulation is very popular, especially nowadays during the world economy crisis.

Collective product manufacturing problem

The model of production of collective product is examined [Voloshin, Mashchenko, 2006]:

$$\begin{gathered} u_i(M_i - x_i, y) \to \max, i \in N = \{\overline{1,n}\}, n \ge 2, \\ \sum_{i=1}^{n} x_i = c(y), \end{gathered} \tag{1}$$

M_i - initial amount of money of the agent *i*; x_i - his contribution to the production of *y* units of the collective product; *c(y)* – the cost of producing y units of collective product; u_i - his utility function. Let us assume that cooperation is efficient: $c(y) \le \sum_{i=1}^{n} M_i$. We impose on functions c and u_i terms that are ordinary for the models of the microeconomics [Ponomarenko, Perestyuk, Burym 2004]: $c(0) = 0$, c is non-decreasing and convex; u_i are cuasi-convex functions, increasing on every variable ($m_i = M_i - x_i$ and y) for all agents; c and u_i are differentiable. The economy has two goods, one pure public good and one private good (money).

Using additive aggregation $u = \sum_{i=1}^{n} u_i$ (which refers to the utilitarian approach [Voloshin, Mashchenko, 2006]), we can find Pareto optimal allocations (in assumptions of the (1) problem) from next necessary and sufficient condition (Samuelson's condition [Moulin, 1991]):

$$\sum_{i=1}^{n} u_{iy} / u_{im_i} = c'(y) , \tag{2}$$

where u_{iy} (u_{im_i}) – is the partial derivative of each agent's utility function w.r.t the public (private) good. $c'(y)$ - is the derivative of the production function w.r.t. the collective product.

Assume that all preferences are additively separable in the input and output goods and linear on the output: $u_i(M_i - x_i, y) = b_i(y) + (M_i - x_i)$, where $b_i(y)$ is the monetary equivalent of *y* units of public good.

Under quasi-linearity of utilities, Samuelson's condition (2) simplifies to $\sum_{i=1}^{n} b_i'(y) = c'(y)$, namely, the sum of marginal benefits equals the marginal cost of the public good. Note that the input levels x_i disappeared from the formula. One of the approaches to the solving of the (1) problem is modeling it as the following transferable utility (TU) cooperative game: for all $S \subseteq \{1,...,n\}$: $v(S) = \max_{y \ge 0} \left\{ \sum_{i \in S} M_i + \sum_{i \in S} b_i(y) - c(y), 0 \right\}$ (the reflection v puts in accordance to every coalition its surplus)

A feasible allocation $(x_1, x_2, ..., x_n, y)$ is then in the core of our TU public good economy if and only if $\sum_{i \in S} (b_i(y) + (M_i - x_i)) \ge v(S)$ for all of the coalitions. Note that this inequality for $S = N$ means precisely that the level *y* is efficient (i.e., maximizes total surplus).

In [Moulin, 1991] a numerical example is examined. The technology has constant returns to scale $c(y) = \frac{3}{2} y$.

There are two agents with quasi-linear preferences: $b_1(y) = \ln(1 + y), \ \ b_2(y) = 2\sqrt{y}$. Initial money values are ignored (accepted equal to zero). Unique maximum y^* is computed by solving the Samuelson's condition (2): $y^* = 1$. The corresponding surplus is $v(12) = (b_1 + b_2 - c)(y^*) = 1{,}19$. Costs for production of y^* are equal $c(y^*) = 1{,}5$. The surplus that single agent coalitions can achieve are $v(1) = 0$ and $v(2) = 0{,}67$.

The core allocations divide arbitrarily the cooperative surplus $\Delta v(12) = v(12) - v(1) - v(2) = 0{,}53$ between the two agents. Particularly, in [Moulin, 1991] two allocations are proposed. At one extreme, agent 1 keeps all the cooperative surplus, implying the following cost shares: $x_1 = 0{,}17$ and $x_2 = 1{,}33$. At the other extreme, agent 2 keeps the cooperative surplus, and hence the cost shares: $x_1' = 0{,}69$ and $x_2' = 0{,}81$.

"Dual" approach to the solution of problem (1)

The "dual" approach to the solution of problem (1) with quasi-linear preferences is proposed. Three cost-sharing principles are a priori set between agents [Voloshin, Mashchenko, 2006]:

1. Costs equality ($x_i = \dfrac{c(y)}{n}$ for all *i*);
2. "Free money" equality ($x_i = b_i - \dfrac{M - c(y)}{n}$);
3. Cost proportionality (to initial money amounts $x_i = (M_i / M)/c(y)$).

Note that in this model, agents can freely transfer the input (money), so these cost-sharing principles are not ideal. Thus, when using the cost equality principle there can be a situation, when some agent's cost share will exceed his initial wealth. Also, when using "free money" equality principle there can be a situation when some agents will receive a cash payment for using the public good. In such cases the will to co-operation decreases. The cost proportionality principle always leads to the allocations from the core of the game. But there can be a situation, when some of the agents will not be satisfies by his part of the collective surplus – in this case he might give up the co-operation.

Let's illustrate these cost-sharing principles on the above-mentioned example. Initially the agents are endowed with following amounts of money: $M_1 = 0{,}19$, $M_2 = 1{,}81$. Application of the cost equality principle leads us to the following allocation: $x_1 = x_2 = 0{,}75$. As we can see, cost share of the agent 1 exceeds his initial wealth, so he won't co-operate. Maximal coalition surplus in this case equals $v(12) = (M + b_1 + b_2 - c)(y^*) = 3{,}19.$

Using "free money" equality principle gives us the next allocation: $x_1 = \text{-}0{,}06$ and $x_2 = 1{,}56$. In this case the agent 1 receives a cash payment of 0,06. It's natural, that the agent 2 will refuse the co-operation.

The cost proportionality principle gives us a following cost sharing: $x_1 = 0{,}1425$, $x_2 = 1{,}3575$. Agents' surpluses are $b_1(y^*) + (M_1 - x_1) = 0{,}7405$ and $b_2(y^*) + (M_2 - x_2) = 2{,}4525$. If agent 1 will not be satisfied with his surplus, he can give up co-operation and coalition will disintegrate.

Let us consider the case, when all preferences are additive in the input and output goods: $u_i(m_i, y) = a_i \ln(m_i + M) + b_i \ln y$ ($m_i + M > 0$, because we suppose that the expenses of each agent can not exceed the sum of money in the economy), where a_i, b_i are positive constants, $i = \overline{1, n}$.

In this case there are difficulties with the search of the optimal cost share allocation, because in general we have two equations with *n+1* variables. The above-mentioned cost-sharing principles allow avoiding these difficulties. Beforehand setting some cost-sharing principle and putting the proper cost share expressions in the Samuelson's condition we can get an equation with one variable - *y*.

Let us consider an economy with constant returns to scale: $c(y) = \alpha y$. In this case the optimal amount of the public good y^* can be find analytically (in general it can be found numerically [Bandy, 1988]).

We find the expressions for the partial derivatives of each agent's utility function:

$$u_{iy}(m_i, y) = \frac{b_i}{y}, \qquad u_{im_i}(m_i, y) = \frac{a_i}{m_i + M} \text{ (где } m_i = M_i - x_i,\ \forall i \in N\text{)}.$$

As the technology has constant returns to scale, $c'(y) = \alpha$.

Putting these expressions in equation (2), we get:

$$\sum_{i=1}^{n}\frac{b_i}{y}\frac{M_i - x_i + M}{a_i} = \alpha ,$$

$$\sum_{i=1}^{n}\frac{b_i}{a_i}\left[M_i - x_i + M\right] = \alpha y .$$

Let’s examine the case of equal cost-sharing ($x_i = {\alpha y}/{n}$, $\forall i \in N$). Then

$$\sum_{i=1}^{n}\frac{b_i}{a_i}\left[M_i + M\right] - \frac{\alpha y}{n}\sum_{i=1}^{n}\frac{b_i}{a_i} = \alpha y , \qquad \sum_{i=1}^{n}\frac{b_i}{a_i}\left[M_i + M\right] = y\left(\frac{\alpha}{n}\sum_{i=1}^{n}\frac{b_i}{a_i} + \alpha\right)$$

From the last expression we can get the formula for y^*:

$$y^* = \frac{\sum_{i=1}^{n} {b_i}/{a_i}\left[M_i + M\right]}{\alpha\left[1 + n^{-1}\sum_{i=1}^{n}\left(b_i/a_i\right)\right]}$$

In a similar manner we can get formulas for finding the optimal value of the public good in case of other cost-sharing principles:

– $$y^* = \frac{\sum_{i=1}^{n} {b_i}/{a_i}\left[M(1 + n^{-1})\right]}{\alpha\left[1 + n^{-1}\sum_{i=1}^{n}\left(b_i/a_i\right)\right]}$$ (in case of “free money” equality);

– $$y^* = \frac{\sum_{i=1}^{n} {b_i}/{a_i}\left[M_i + M\right]}{\alpha\left[1 + \sum_{i=1}^{n}\left(b_i M_i/a_i M\right)\right]}$$ (in case of cost proportionality to initial money amounts).

Let’s consider a numerical example. There are two agents with initial wealth $M_1 = 0{,}19$, $M_2 = 1{,}81$ and additive preferences:

$$u_1(m_1, y) = 2\ln(m_1 + M) + \ln y , \quad u_2(m_2, y) = 3\ln(m_2 + M) + \ln y .$$

Technology has constant returns to scale: $c(y) = \frac{3}{2}y$.

For different cost-sharing principles we will get different optimal values for public good production. Thus, using cost equality principle gives us $y^* = 1{,}113$ with corresponding cost share $x_1 = x_2 = 0{,}835$ and surplus $v(12) = 4{,}093$. Using “free money” equality principle we get the next values: $y^* = 1{,}176$, $x_1 = 0{,}072$, $x_2 = 1{,}692$, $v(12) = 4{,}077$. In case of cost proportionality $y^* = 1{,}169$. Corresponding allocation is $x_1 = 0{,}167$ and $x_2 = 1{,}586$. Cooperative surplus in this case equals $v(12) = 4{,}119$. As we see, cost proportionality principle gives the biggest cooperative surplus. In the case of equal cost sharing agent 1 will refuse the co-operation, because his cost share exceeds his initial wealth.

Fuzzy generalization of problem (1)

Let us consider a fuzzy generalization of problem (1) [Voloshyn, Laver, 2008]. Assume the agents are normalized in order by their initial wealth M_i. Then *N* can be presented as a union of sets: $N = N_1 \cup N_2 \cup N_3$, where

$N_1 = \{\overline{1, n_1}\}$ - “poor” agents; $N_2 = \{\overline{n_1 + 1, n_2}\}$ - “middle class”; $N_3 = \{\overline{n_2 + 1, n}\}$ - “wealthy” agents, $n_1 \geq 1,\ n_2 \geq 2,\ n_1 < n_2 < n.$

Assume that there is some optimal value of the public good production y^*, found from the distinct problem using some cost-sharing principle. There can be a situation that “poor” agents will not have enough money to cover their cost share. On the other hand, “wealthy” agents can consent to cover the cost deficit, to guarantee the issue of optimum amount of public good.

Agents' cost shares satisfy the following terms: $0 \leq x_i \leq M_i,\ \forall i \in N$. Each agent sets a membership function $\mu_i : [0, M_i] \to [0,1]$, which characterizes the agent's satisfaction with his cost share.

Then the optimal allocation *($x_1, x_2, \dots, x_n$)* can be found as a solution of the following problem:

$$\begin{cases} \mu_A(x) \to \max; \\ \sum_{i=1}^{n} x_i = c, \end{cases} \quad \text{where } \mu_A(x) = \min\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\},\ c = c(y^*). \qquad (3)$$

(3) can be equivalently offered as:
$$\begin{cases} \alpha \to \max; \\ \mu_i(x_i) \geq \alpha\ \ (\forall i \in N); \\ \sum_{i=1}^{n} x_i = c. \end{cases} \qquad (4)$$

Let us examine a numerical example. There are three agents with the following initial amounts of money: $M_1 = 2,\ \ M_2 = 5,\ M_3 = 9$. Collective product worth *c=9* has to be produced. Using equal cost-chare we will get the following allocation: $x_1 = x_2 = x_3 = 3$. Agent 1 will refuse the co-operation, as his cost share exceeds his initial wealth. We can avoid this problem if the agents' cost-share satisfy the following terms $0 \leq x_i \leq M_i,\ \forall i \in N$.

One of the possible cost-sharing methods in this case is the head tax [Voloshin, Mashchenko, 2006] with corresponding allocation $x_1 = 2, x_2 = 3{,}5, x_3 = 3{,}5.$ Assume that the agents agree to deviate from their cost shares and their priorities are described with following membership functions:

$$\mu_1 = \begin{cases} 1, x_1 \in [0;0{,}8]; \\ -0{,}6x_1 + 1{,}48, x_1 \in [0{,}8;1{,}3]; \\ -x_1 + 2,\ x_1 \in [1{,}3;2]; \end{cases} \qquad \mu_2 = \begin{cases} 1, x_2 \in [0;3]; \\ -0{,}2x_2 + 1{,}6, x_2 \in [3;4]; \\ -0{,}5x_1 + 2{,}8, x_2 \in [1{,}3;2]; \end{cases}$$

$$\mu_3 = \begin{cases} 1, x_3 \in [0;4{,}5]; \\ -0{,}05x_3 + 1{,}225, x_3 \in [4{,}5;6{,}5]; \\ -0{,}6x_3 + 4{,}8, x_3 \in [6{,}5;8]; \\ 0, x_3 \in [8;9]. \end{cases}$$

Optimal allocations can be found as the solution of the next problem:

From next system we find the high boundaries of the agents' maximum permissible cost share changes:

$$\begin{cases} \alpha \to \max; \\ -0{,}6x_1 + 1{,}48 \geq \alpha; \\ -0{,}2x_2 + 1{,}6 \geq \alpha; \\ -0{,}05x_3 + 1{,}225 \geq \alpha; \\ \sum_{i=1}^{n} x_i = c. \end{cases} \qquad \begin{aligned} x_1 &\leq \frac{1{,}48 - \alpha}{0{,}6}; \\ x_2 &\leq \frac{1{,}6 - \alpha}{0{,}2}; \\ x_3 &\leq \frac{1{,}225 - \alpha}{0{,}05}. \end{aligned}$$

Summing up the right parts of these inequalities and comparing them to *c*, we get the equation for finding the α :

$$\frac{1{,}48 - \alpha}{0{,}6} + \frac{1{,}6 - \alpha}{0{,}2} + \frac{1{,}225 - \alpha}{0{,}05} = 9, \text{ so } \alpha \approx 0{,}97$$

Optimal (with 0,97 degree) allocation we find from the following terms:

$$\begin{cases} x_1 \leq 0{,}85; \\ x_2 \leq 3{,}15; \\ x_3 \leq 5{,}1; \\ x_1 + x_2 + x_3 = 9. \end{cases}$$

One of the possible allocations is $x_1 = 0{,}85; x_2 = 3{,}15; x_3 = 5.$

Conclusion

The considered models enable to adequately describe the process of the collective good production and cost sharing processes. Unevenness of money allocation in society and individual preferences are taken in account.

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