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Polyhedral Coherent Risk Measures and their Application to Investment Decisions Support under Catastrophic Flood Risks

11.1 Introduction and some definitions

The results of managerial decisions are influenced by numerous factors, parameters, and processes that impart them a stochastic, or even, to a certain extent, an uncertain character. It is caused by as the fundamental impossibility to describe fully enough and to forecast future processes in advance, as the stochastic nature of some factors and parameters of these processes (for example, fluctuations in financial markets, natural cataclysms caused by the coincidence of some conditions and phenomena, and so on). This concerns in full degree various decisions in economic and financial fields, which are sensitive enough to numerous risks. Similar properties have a special significance in development and realization of important (global) decisions with long-term consequences, because appropriate uncertainties and risks essentially increase in a long-term prospect. Therefore, the development of a methodology for decision-making under risk and uncertainty, and appropriate models and methods of search of optimal (effective) decisions take a critical meaning in the present conditions of global interferences of various processes.

As a rule, risk arises in those cases where there is a possibility (probability) of adverse consequences (damages, losses), hence, where considered processes cannot be described by deterministic values. Besides, risk can be caused by essentially stochastic nature of parameters and characteristics of processes, or by the impossibility to predict and describe them in advance (uncertainty), or by imposing of both specified circumstances.

In various applications, especially in financial and economic ones, the choice of concept of efficiency is transparent, as a rule, it is a profit, return, utility, etc. At the same time, a question how to estimate risk remains methodologically difficult and ambiguous. Certain functions which describe risk quantitatively are called as risk functions, or, according to terminology [Artzner et al, 1999], as risk measures. What kind should be a risk measure? An inappropriate choice of such measure can lead to inconsistent decision-making results, to difficulties in searching optimal decisions for studied problems, etc.

In the process of development of the theory and applications to support of economic and financial decisions in conditions of risk and uncertainty, different functions were used as risk measures: a dispersion (deviation) [Markowitz, 1959], a semi-deviation [Ogryczak and Ruszczynski, 1999], VaR (Value-at-Risk) [Jorion, 1996], an expected absolute deviation [Konno and Yamazaki, 1991], and others.

However, none of them is perfect. For instance, the dispersion is a traditional measure in the theory of errors for the estimation of a deviation from the mean; however, for financial area measures characterizing downside deviations from some level (loss) are more suitable. Therefore, the dispersion [Markowitz, 1959] estimating both a downside deviation (losses), and an upside deviation (profit), is not an adequate risk measure for financial and economic problems. The same concerns the semi-deviation [Ogryczak and Ruszczynski, 1999] and the expected absolute deviation [Konno and Yamazaki, 1991].

Currently, VaR is the most popular risk measure in financial applications [Jorion, 1996]; it came there from the insurance area and was propagated by the RiskMetrics methodology. The measure has simple and clear interpretation for risk-managers; however, it has drawbacks as well. In particular, it is not subadditive that can lead to the following paradox: a portfolio diversification can increase its risk (in terms of VaR as the risk measure). Besides, it ignores risks of high losses. Therefore, in 2000 the Basel committee on bank supervision did not recommend to use VaR, as well as dispersion, for measuring risks.

Later, in [Artzner et al, 1999] four axioms have been formulated, to which from a theoretical point of view a risk function should satisfy to claim to be a successful measure of risk, and the corresponding class of risk functions was called as the class of coherent risk measures (CRM). We remind that such functions should be: 1) translation invariant; 2) subadditive; 3) positively homogeneous; 4) monotonous.

Then in [Rockafellar and Uryasev, 2000; 2002], CVaR (Conditional VaR) risk measure was proposed, which, on the one hand, is interpreted as integrated VaR (integral of the appropriate tail distribution), and, on the other hand, belongs to the class of CRM. Remarkable properties of this measure allow to reduce portfolio optimization problems with CVaR objective or/and constraint functions to linear programming problems (LPP).

Later, in [Kirilyuk, 2003; 2004a; 2004b], a class of polyhedral coherent risk measures (PCRM) was introduced, which, on the one hand, is a subset of CRM class (hence it possesses all theoretically attractive properties); on the other hand, it allows to reduce various portfolio optimization problems with such risk measures to LPP. The class contains CVaR and some other important risk measures as special cases.

Let us notice that a considerable interest to CVaR in the financial literature and attempts to consider it as reasonable alternative to VAR currently are observed. However, a question of a choice of the risk measure for concrete problem settings remains open. Use in important problems not only one risk measure, but also a whole set of measures, for instance, in constraints on risk levels which are described by these risk measures, looks quite reasonable.

In this paper we review some results of the PCRM theory from [Kirilyuk, 2003-2007], consider their applications to decision making support in conditions of risk, and develop numerical methods for searching optimal decisions. As a particular application, an investment decisions making under catastrophic flood risks is considered (section 5).

Let there be a certain amount of money (resources), which need to be distributed among several instruments (financial assets, branches, programs, etc.) in such a way that the total result obtained was optimal with respect to efficiency–risk ratio. Accordingly, such instruments are called as the portfolio components, and their total result is called as the portfolio result. Such problems often arise in financial and economic applications.

We consider decision-making models guided definitely by the scenario analysis. Suppose that there is some set of developed scenarios, which allow describing price evolution for each portfolio component. For example, this can be done from the analysis of corresponding prehistory (a statistical data analysis) and a certain forecast of future events. In case when probabilities of the developed scenarios are known, it is clear that distributions of all portfolio components are known as well. Accordingly to terminology [Knight, 1921], we say then that decisions are made in the condition of risk.

A situation, when scenario probabilities are not known but some of their estimates are available, is more difficult and requires certain additional methodological efforts. In this case, the situation is characterized as partially uncertain incomplete information on distributions).

Let us consider now discrete random variables (r.v.) taking n values, which are called scenarios. Then each scenario i = 1, …, n has some probability $p_i > 0$, that is, a vector of scenario probabilities $p_0 = (p_1^0,\dots,p_n^0), p_i^0 > 0, i = 1,\dots,n, \sum_1^n p_i^0 = 1$ is given, and a r.v. X is characterized by its distribution $x = (x_1, \dots, x_n) \in R^n$ on these scenarios, hence, it is identified with this n-dimensional vector.

Consider a function of the following form

$$\rho(x) = \max\{E_p[-X] \mid p \in P\} \qquad (1)$$

where $E_p[-X] = \langle -x, p\rangle$ is the mean value of r.v. (–X) with a discrete probability measure p, and P is a convex closed set of probability measures.

Let's remind that functions of form (1) which have a certain sense and where the set P is described in the form of a convex hull of a finite number of points, are called by the polyhedral coherent risk measure (PCRM) [Kirilyuk, 2003]. More exactly, if P is a set of the following form

$$P = \mathrm{co}\{p_i: i=1,\dots,k\},$$

Alternatively, it is equivalent,

$$P = \{p: B\,p \le c,\ p \ge 0\} \qquad (2)$$

where B and c are a matrix and a vector of appropriate dimensions, the relations (1)–(2) unambiguously describe a polyhedral coherent risk measure.

Note that as P is a set of probability measures, its description in the form of (2) should include the condition $\sum_{i=1}^{n} p_i = 1$, i.e., $\sum_{i=1}^{n} p_i \le 1, -\sum_{i=1}^{n} p_i \le -1$.

Hence, the first two rows of the matrix B are (1, …., 1) and –(1, …., 1), and the first two components of the vector c are 1 and –1.

Consider known risk measures, which satisfy this PCRM definition. Because of all of them are set in form of (1), for the full description of a concrete measure it is enough to describe the set P from relation (2).

1) WCR is the worst-case risk on all distribution of a r.v. [Artzner et al, 1999]. In the case, $WCR(x) = \max\{-x_i: i = 1,\dots, n\}$. Then, as it is easy to see, the set P has the following form

$$P_{WCR} = \{p = (p_1,\dots, p_n): p_i \ge 0,\ i = 1,\dots, n,\ \textstyle\sum_1^n p_i = 1\}.$$

2) $CVaR_\alpha$ is the conditional average of losses on α-tail of a r.v. distribution (Conditional VaR) [Rockafellar and Uryasev, 2000;, 2002]. It has the form of (1), where appropriate set P is described as

$$P_{CVaR_\alpha} = \{p = (p_1,\dots, p_n): p_i \le p_i^0 / \alpha,\ p_i \ge 0,\ i = 1,\dots, n,\ \textstyle\sum_1^n p_i = 1\},$$

where $p_0 = \left(p_1^0,\ldots,p_n^0\right)$ is the vector of initial scenario probabilities.

3) WCE $_\alpha$ is the worst conditional expectation of r.v. [Artzner et al, 1999]. Appropriate set P is

$$P_{WCE_\alpha} = \text{co}\{(p_1,\ldots,p_n)/ \text{ for } \textstyle\sum_1^m p_{i_j}^0 > \alpha\ ,\ p_{i_j} = p_{i_j}^0 / \sum_1^m p_{i_j}^0, j \le m; p_{i_j} = 0, j > m\}.$$

4) SCRM is the spectral coherent risk measure [Acerbi, 2002]. As it shown in [Kirilyuk, 2006],

$$SCRM(x) = \sum_1^m \lambda_j CVaR_{\alpha_j}(x)$$

therefore,

$$P_{SCRM} = \{p = (p_1,\ldots,p_n): p_i \le \left(\sum_1^m (\lambda_j / \alpha_j)\right) p_i^0,\ p_i \ge 0,\ i=1,\ldots,n,\ \sum_1^n p_i = 1\},$$

where $p_0 = \left(p_1^0,\ldots,p_n^0\right)$ is the vector of initial scenario probabilities.

As it shown in [Kirilyuk, 2004], the following measures are PCRM as well:

5) δ_S (x; r) is a measure on the semideviation from the mean value of a r.v. [Ogryczak and Ruszczynski, 1999], where

$$\delta_S(x; r) = -E[x] + r\,E[(E[x] - x)^+];$$

6) δ_A (x; r) is a measure on the absolute deviation from the mean value of a r.v. [Ogryczak and Ruszczynski, 1999], where

$$\delta_A(x; r) = -E[x] + r\,E[\,|x - E[x]|\,].$$

These measures can be presented in form of relations of (1)–(2) as well.

Besides, the PCRM class is invariant relatively to the following operations [Kirilyuk, 2004]: 1) convex combination; 2) maximum functions; 3) infimal convolution.

Thus, it is possible to generate new such risk measures by application of the specified operations to representatives of the class, and it is easy to describe the obtained measures, having specified appropriate sets P from (1). It means that the class PCRM is wide enough and includes all (known for authors) coherent risk measures.

Now in PCRM terms we consider models of support of portfolio optimal decisions on efficiency-risk (return-risk) ratio. These models are put in the following forms. Let a certain sum (set) of money (resources) which need to be distributed on tools (financial assets, branches, programs, other) which are called as portfolio components, is given. It is necessary to distribute it so that the total result obtained thus was optimal on efficiency-risk ratio. The following sections of the paper are devoted to the problem.

11.2 Models of support of optimal portfolio decisions on return-risk ratio in conditions of risk (known distributions)

Let distribution of efficiency (return) of portfolio component z_j, j=1,…, k be described by matrix H of n×k dimension where a j column describes the distribution of j-th component. A vector u = (u_1, …, u_k) which describes a portfolio structure, it is considered as a variable, where $\sum_1^k u_i = 1$, $u_i \ge 0$, i=1,…,k. It is necessary to find such portfolio structure u, which optimizes a total portfolio result on a efficiency-risk ratio. In case of known component distributions (in our problem statement it is described by matrix H of component distributions according to scenarios and by the known vector of probabilities of scenarios $p_0 = (p_1^0,...,p_n^0)$), as an efficiency indicator the average efficiency, and as a risk measure some PCRM are considered. It is clear that, the more the average efficiency and the less the portfolio risk level, the more preferable it to be for a decision maker (DM). However, these characteristics are much interconnected and, as a rule, improving one of them, simultaneously we worsen the second one. Suppose that a DM has certain knowledge about what risk levels are acceptable and what efficiency levels are desirable. He optimizes one of these criteria under constraints on another. On this way, the following two interconnected problem statements are possible.

Risk measure minimization problem under average efficiency guaranteed. Fix the lower admissible value of average efficiency $E_p[Hu]$ which the investigated portfolio should guarantee, by the value μ in the form of constraints, and minimize its risk measure $\rho(Hu)$:

$$\min_{\substack{\sum_1^n u_i=1, u_i\geq 0 \\ E_{p_0}[Hu]\geq\mu}} \rho(Hu) \tag{3}$$

Average efficiency maximization problem under risk measure constrained. Fix some level of risk measure $\rho(Hu)$ which the investigated portfolio should not exceed, by the value σ in the form of constraints, and maximize its average efficiency $E_p[Hu]$:

$$\max_{\substack{\sum_1^n u_i=1, u_i\geq 0 \\ \rho(Hu)\leq\sigma}} E_{p_0}[Hu] \tag{4}$$

Let remind the results obtained in [Kirilyuk, 2004], relatively to reduction of these problems to LPP.

Theorem 1. A solution of portfolio problem (3)–(1) is the component u of a solution (v, u) of the following LPP:

$$\min_{\substack{(v,u) \\ -B^T v - Hu\leq 0 \\ Au\leq b \\ v\geq 0, u\geq 0}} < c, v > \tag{5}$$

Theorem 2. A solution of portfolio problem (4), (1)–(2) is the component u of a solution (v, u) of the following LPP:

$$\max_{\substack{(v,u) \\ -B^T v - Hu\leq 0 \\ <c,v>\leq\sigma \\ \sum_1^n u_i=1 \\ v\geq 0, u\geq 0}} < H^T p_0, u > \tag{6}$$

Where

$$A = \begin{pmatrix} 11...1 \\ -1-1...-1 \\ -p_0^T H \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ -\mu \end{pmatrix}, B = \begin{pmatrix} 11...1 \\ -1-1...-1 \\ B_0 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -1 \\ c_0 \end{pmatrix}.$$

It is easy to see that the matrix B_0 and the vector c_0, which define some PCMR, are supplemented with technical restrictions from units and a minus of units (the first two rows) and in the form of B and c accordingly are used in LPP (5) and (6).

It is important sometimes that under the average efficiency maximization, a number of constraints on risk measures, which guarantees certain reliability of obtained solutions within the bounds of some concept of risk management, should be satisfied. For instance, it is important for a DM that certain risk measures should not be exceeded by some critical levels. In the case, problem (4) contains m constraints on risk measures (PCRM), it has the following form

$$\max_{\substack{\sum_1^n u_i = 1, u_i \geq 0 \\ \rho_i(Hu) \leq \sigma_i, i=1,\dots,m}} E[Hu] \tag{7}$$

where $\rho_i(.)$, i =1, …, m are appropriate risk measures:

$$\rho_i(x) = \max\{E_p[-X] \,/\, B_i\, p \leq c_i,\ p \geq 0\} \tag{8}$$

Then the result formulated in the form of the following theorem takes place**.**

Theorem 3. A solution of portfolio problem (7)–(8) is the component u of a solution (v_1, v_2, …, v_m, u) of the following LPP:

$$\begin{gathered} \max_{(v_1,\dots,v_m,u)} < H^T p_0, u > \\ -B_1^T v_1 - Hu \leq 0 \\ <c_1, v_1> \leq \sigma_1 \\ \dots\dots\dots\dots \\ -B_m^T v_m - Hu \leq 0 \\ <c_m, v_m> \leq \sigma_m \\ \sum_1^n u_i = 1 \\ v_1 \geq 0, \dots, v_m \geq 0, u \geq 0 \end{gathered} \tag{9}$$

Let us remark that in this case, appropriate LPP has more large dimension (in proportion to the number of constraints on risk measures), than the one in the conditions of theorem 2. That is, a charge for performance of all formulated constraints is the essential increase in dimension of the problem, which should be solved.

Let us notice that the PCRM concept supposes certain generalizations, which allow operating with a wider class of risk measures, without losing thus the basic remarkable property of this class as a possibility of reduction of portfolio optimization problems on return-risk (efficiency-risk) ratio to LPP. Besides, the mathematical technique, which is used, becomes complicated not essentially, and within certain technical details only [Kirilyuk, 2006].

11.3 PCRM in conditions of partial uncertainty

Essentially other looks a situation at which a matrix H of the component distributions in accordance with scenarios remains known and a vector of scenario probabilities $p_0 = (p_1^0, \dots, p_n^0)$ is not. As a rule, identification of probabilities of the future scenarios is essentially more a challenge, than a development of such scenarios of future events. Especially, it is important for so-called rare events [Ermoliev et al, 2000a,b]. For example, how many years should to fix flooding to guarantee, that a flooding of a certain capacity occurs only once in hundred years?

Nevertheless, it is clear that certain information on probabilities is available. Moreover, sometimes it is possible to estimate these scenario probabilities in certain bounds, or, more mathematically, in the form of an inclusion: $p_0 \in P$, where P is a polyhedron. However, at once such situation adds principal difficulties to the problem, because of it is impossible to operate even with average values (mathematical expectations) under unknown probabilities p_0.

Therefore, in [Kirilyuk, 2006; 2007] the following mathematical technique has been proposed.

Together with a risk function in the PCRM form

$$\rho(x) = \max\{E_p[-X] \,/\, p \in P\} \tag{10}$$

an efficiency (return) functional is considered as well

$$g(x) = \max\{E_p[X] \,/\, p \in P\} \tag{11}$$

where P is a convex closed polyhedral set of probability measures, which is described in the form of (2):

$$P = \{p: B\, p \le c,\ p \ge 0\} = co\{p_i: i=1,\ldots,k\} \tag{12}$$

Let's notice that, generally speaking, sets P from (10) and (11) can be different (various risk measures ρ_i (.) can use different sets P_i), however they are identical in the most simple case when these sets are interpreted as an estimation on scenario probabilities in the form $p_0 \in P$.

It is easy to see that

$$\rho(x) = \max\{E_p[-X] \,/\, p \in P\} = -\min\{E_p[X] \,/\, p \in P\} \tag{13}$$

and together with (11), the average value of a r.v. X can be estimated from above and from below by the interval $[-\rho(x), g(x)]$, since

$$-\rho(x) = \min\{E_p[X] \,/\, p \in P\} \le E_p[X] \le \max\{E_p[X] \,/\, p \in P\} = g(x).$$

Hence, it is clear that in terms of these criteria under decision-making it is necessary to shift the interval $[-\rho(x), g(x)]$ to the right, increasing both its ends (if X describes "positive" outputs: efficiency, return, others). Or, in terms of efficiency and risk, it is necessary to maximize the efficiency functional g(.) and to minimize the risk functional ρ(.).

Consider now certain aspects of construction of risk measures on set P_0 that estimates a vector of scenario probabilities p_0, and the appropriate mathematical technique for it from [Kirilyuk, 2008].

Address now to examples of risk measures from the PCRM class considered in section 1. As it is easy to see, sets of probability measures P for all them from the relation (2) depends on the vector of scenario probabilities p_0. That is, the set P from (2), generally speaking, is some set-valued map (s.v.m.) on the vector p_0.

Consider now a construction, in which set of probability measures P for design of risk measures ρ (.) is described by some s.v.m. a(.) on a vector of scenario probabilities p_0 with closed convex images:

$$\rho(x) = \sup\{E_p[-X] \,/\, p \in a(p_0)\} \tag{14}$$

As it is easy to see, since the image of a(.) is the subdifferential of the function ρ(.) accurate within sign, a(.) has been named the subdifferential s.v.m. [Kirilyuk, 2008].

Consider now a construction of risk measures for the case of partial uncertainty, that is, such construction on set P_0 that estimates a vector of scenario probabilities p_0. Let now there be some CRM, given on a known vector of scenario probabilities p_0 in the form of (14).

Definition 1. The risk measure induced by the initial CRM and the uncertainty set P_0, the following function is called as

$$\rho(x; P_0) = \sup\{E_p[-X] \,/\, p \in P(P_0)\} \tag{15}$$

where

$$P(P_0) = \overline{co}\left(a(P_0)\right),\ a(P_0) = \bigcup_{p_0 \in P_0} a(p_0) \tag{16}$$

Here co means the convex hull, $\overline{M}$ means the closure of set M. Obviously, ρ (x;P_0) is a CRM through its construction.

Remind that a s.v.m. a (.) is called as convex in a range of definition dom a, if

$$a(\lambda p_1 + (1-\lambda)p_2) \supseteq \lambda a(p_1) + (1-\lambda)a(p_2)\ \forall p_1, p_2 \in \mathrm{dom}\, a\ \forall \lambda \in (0,1).$$

Proposition 1. If P_0 is convex set, and a subdifferential s.v.m. a(.) of an initial risk measure ρ(.) is a convex s.v.m., then appropriate set P has the following form

$$P(P_0) = \overline{a(P_0)} \tag{17}$$

Definition 2. We will say that s.v.m. a(.) is quasilinear in a range of definition dom a, if

$$a(\lambda p_1 + (1-\lambda)p_2) = \lambda a(p_1) + (1-\lambda)a(p_2)\ \forall p_1, p_2 \in \mathrm{dom}\, a\ \forall \lambda \in (0,1) \tag{18}$$

Proposition 2. Subdifferential s.v.m. for risk measures WCR, $CVaR_\alpha$ and SCRM are quasilinear.

From quasilinear properties of s.v.m. a(.), as it is easy to see, the following proposition follows at once.

Proposition 3. If P_0 is a polyhedral set, i.e. it is described in the form of (12), and a subdifferential s.v.m. of an initial risk measure ρ(.) is a quasilinear s.v.m., the set from (17) has the following form

$$P(P_0) = \mathrm{co}\left\{a(p_i^0), i = 1,\ldots,k_0\right\} \tag{19}$$

Corollary 3. If in conditions of proposition 3 an initial risk measure is a PCRM, i.e. its subdifferential s.v.m. has polyhedral images in the extreme points of P_0:

$$a(p_i^0) = \mathrm{co}\left\{p_j(p_i^0), j = 1,\ldots,m(p_i^0)\right\}, i = 1,\ldots,k_0$$

then set from (19) has the following form

$$P(P_0) = \mathrm{co}\left\{p_j(p_i^0), j = 1,\ldots,m(p_i^0), i = 1,\ldots,k_0\right\}. \tag{20}$$

Corollary 4. For PCRM with a quasilinear subdifferential s.v.m. a(.), the composition operation is invariant on the class:

$$\rho(.) \equiv \rho_1 \circ \rho_2(.) \Leftrightarrow \rho(x) = \sup\{E_p[-X] : p \in a_1(a_2(p_0))\}.$$

Corollary 5. Corollaries 3 and 4 take place for WCR, $CVaR_\alpha$ and SCRM.

11.4 Models of support of optimal portfolio solution in conditions of partial uncertainty (incomplete information on scenario probabilities)

Turn now to the formulation of portfolio optimization problems on efficiency-risk ratio in terms of functionals of efficiency g(.) and risk ρ(.). Let, as before, distributions of efficiencies of portfolio components z_j, j=1,…, k be described by a matrix H, and a vector u = (u_1, …, u_k) which describes portfolio structure, be considered as a variable.

Risk measure minimization problem under average efficiency guaranteed. Begin from the minimization problem of the portfolio risk measure ρ(·) under guaranteed return values g(·)≥ g_0. The problem is formulated as follows

$$\min_{\substack{\sum u_i=1, u\geq 0\\ g(Hu)\geq g_0}} \rho(Hu) \tag{21}$$

Average efficiency maximization problem under risk measure constrained. Connected with the previous task, the maximization problem of the portfolio efficiency g(·) under constraints on its risk measure of risk in the form ρ (·) ≤ ρ_0 can be formulated in the following form

$$\max_{\substack{\sum u_i=1, u\geq 0\\ \rho(Hu)\leq \rho_0}} g(Hu) \tag{22}$$

Consider now possibilities of a reduction of portfolio optimization problems (21) and (22) to certain sequences of LPP obtained in [Kirilyuk, 2008].

Theorem 4. A solution of problem (21) is the part u of a solution (u, v) of the following problem

$$\min_{1\leq i\leq k}\{ \min_{\substack{(u,v)\\ \sum u_i=1, u\geq 0, v\geq 0\\ -B^T v-Hu\leq 0\\ <Hu, p_i>\geq g_0}} < c, v >\} \tag{23}$$

where p_i, i =1, …, k are the extreme points of set P from (12) and denotations:

$$\min_{\substack{(u,v)\\ \sum u_i=1, u\geq 0, v\geq 0\\ -B^T v-Hu\leq 0\\ <Hu, p_i>\geq g_0}} < c, v >= +\infty$$

if constraints of the subproblem are not fulfilled.

Theorem 5. A solution of problem (22) is the part u of a solution (u, v) of the following problem

$$\max_{1\leq j\leq k}\{ \max_{\substack{(u,v)\\ \sum u_i=1, u\geq 0, v\geq 0\\ -B^T v-Hu\leq 0\\ <c, v>\leq \rho_0}} < Hu, p_j >\} \tag{24}$$

where p_j, j = 1, …, l are the extreme points of set P from (12).

Remark on the case of different sets P which can be used to define functionals of risk measure ρ(.) and efficiency g(.) from (10) and (11) respectively. In theorem 4, p_i from LPP (16) designate the extreme points of set P for the risk measure ρ(.) from (10), and in the theorem 5, p_j from a problem (24) are the extreme points of set P for the efficiency functional g(.) from (11).

If there are various risk measures which keep the form of functional (10), but use various sets P_i, for example, for the purpose of more conservative behavior of a DM, etc (see the interpretation of risk measures in [Kirilyuk, 2008]), the situation can demand the efficiency functional maximization under constraints on these risk measures. The content of similar functionals in some sense can guarantee some robustness of the obtained solutions relatively to risk.

So, let there be the efficiency functional (11) and some risk measures

$$\rho i(x) = \max\{Ep[-X] / p \in Pi \} \tag{25}$$

where

$$P_i = \{p: B_i\, p \le c_i,\ p \ge 0\} = \mathrm{co}\,\{p^i_j,\ i = 1,\ldots, s_i\},\ 1 \le i \le m, \tag{26}$$

which should be taken into account in a process of search of optimal solutions.

Consider the return functional maximization problem g(·) under constraints of m risk measures $\rho_i(\cdot)$ from (25)–(26) by values ρ_i^0, $1 \le i \le m$ accordingly which is formulated as follows

$$\max_{\substack{\sum u_j = 1, u \ge 0 \\ \rho_1(Hu) \le \rho_1^0 \\ \ldots\ldots\ldots\ldots \\ \rho_m(Hu) \le \rho_m^0}} g(Hu) \tag{27}$$

The following theorem takes place.

Theorem 6. A solution of problem (27)–(25) is the part u of a solution $(u, v_1, \ldots, v_m)$ of the following problem

$$\max_{1 \le j \le m} \{ \max_{\substack{(u, v_1, \ldots, v_m) \\ \sum u_i = 1, u \ge 0, v \ge 0 \\ -B_1^T v_1 - Hu \le 0 \\ <c_1, v_1> \le \rho_1^0 \\ \ldots\ldots\ldots\ldots \\ -B_m^T v_m - Hu \le 0 \\ <c_m, v_m> \le \rho_m^0 \\ v_1 \ge 0, \ldots, v_m \ge 0}} < u, H^T p_j > \} \tag{28}$$

where p_j, $1 \le j \le m$ are the extreme points of set P from (12).

Various generalizations of these problem statements when, for example, various variants of efficiency functionals $g_i(\cdot)$ can be given and it is necessary to minimize risks under guaranteed values of these functionals, are possible as well. Similar problem statements were considered in [Kirilyuk, 2006].

At last, make the following two useful remarks. First, in the case, when the set P of efficiency functional g(.) is described as $P = \{p_0\}$, that is functional $g(.) = E_{p_0}[X]$ for known scenario probabilities p_0, as it is easy to see, theorems 1–3 immediately follow from theorems 4–6 respectively.

Secondly, the weighed sum of the upper and lower estimations of functionals g(.) and $-\rho(\cdot)$ can be quite a reasonable variant of the optimization criterion (so-called Gurwitz's criterion). Besides, sometimes it has sense to consider so-called convolution of criteria in a case of use of a certain set of different risk measures. Then it is easy to obtain the statements similar to theorems formulated above which reduce appropriate portfolio optimization problems to certain sequences of LPP [Kirilyuk, 2006].

It is quite transparent also, development of appropriate mathematical technique on the case of multicomponent efficiency criteria looks. Then it is natural to apply already stated results under a convolution of these criteria. According to author viewpoint, there is not any principal problem in a reformulation of these results for more difficult case of the search of weak portfolio optimums: optimization of a one of criteria under constraints on the guaranteed values of others, under ranging of criteria by their importance, etc. It is clear that the value of necessary calculations for this grow, however it is purely technical problems.

11.5 Decision making on investments allocation under catastrophic flood risks

Investments into the objects planned or located in valleys and near rivers, often appear more favorable, than investments into objects remote from rivers. It can be explained by a smaller slope of a landscape, the greater fertility of lands, availability of water as a resource, presence of infrastructure, roads, affinity to consumers and manufacturers of goods or services that imply smaller cost of construction and faster recoupment of projects. Volume of investments can take both continuous and discrete values. However, in decision-making on investments it is necessary to take into account risk of possible losses from flooding and risk of significant losses from catastrophic flooding, and so to provide mitigation measures against these risks, for example, such as diversification and insurance of the investments.

A similar problem arises before an insurance company aspiring geographically to diversify insurance contracts in areas subjected to catastrophic flooding. On one hand, the closer to water, the higher insurance tariffs, but on the other hand the greater risk of big dependent insurance claims.

Another example is an investment into structural flood mitigation measures (construction of dikes, pools, etc.) characterized by more or less certain costs and by saved different property values under possible future floods.

One of a complex methodological problems is that decisions are made today and the implementation money are spent today, but a catastrophic flood may happen as in the nearest future as not to happen at all. The problem is to compare today's expenses with losses in uncertain and possibly rather distant future. In the applied hydrology, this problem is considered for many years [USACE, 1992; 2000], [RAUFDRD, 2000]. Each decision (plan, portfolio of subdecisions, insurance coverage) under uncertainty is characterized by a spectrum (distribution) of future outcomes and the problem is to select decision corresponding to the most preferable outcome distribution. This problem setting assumes the existence of certain order (stochastic dominance) in the space of outcome distributions and has basically theoretical significance. In practice, decision making under uncertainty is made by means of some functionals (expected utility or profit functions, variance and other risk measures and etc.) on outcome distributions and thus defining corresponding order in the space of distribution. There is a number of results connecting these functionals with (first or second order) stochastic dominance [Ogryczak and Ruszczynski, 2001]. In financial theory and practice, many decision-making problems concern to the structure of financial portfolios with random return. In an aggregate form these portfolios are characterized by two criteria, mean and variance of return, and thus as a risk measure the variance is used [Markowitz, 1959]. A more natural risk measure is the down-side (risk) deviation of return from its mean value [Konno and Yamazaki, 1991]. Variance or down-side variance of a decision outcome is an adequate risk measure if random outcomes are grouped around mean outcome. However, if the decision maker concerns on large losses he has to take into account characteristics of tails of outcome distributions. In this context tail related risk measures are now used such as quantile, value at risk, conditional value at risk and others [Jorion, 1996], [Rockafellar and Uryasev, 2000, 2002], [Pflug and Romisch, 2007].

Decision making problem under uncertainty assumes modeling decisions and uncertainty structure. Decisions may include discrete and continuous components, and uncertainty may be represented either by discrete tree of possibilities (scenarios) with associated probabilities, or through random simulations. In most cases, this problem includes at least two-criteria, for example, expected outcome and some associated risk measures. Each of these criteria is a specific functional defined on outcome distribution, for instance, some expected value, conditional value at risk, etc. Remark that investment decision making problems under flood risk belong exactly to the latter case, small and frequent floods need not be taken into account, but a considerable risk is connected with catastrophic rare floods and thus with the tail of the decision outcome distribution. In this case, as a measure of risk one can take, for example, a mean value of losses from rare catastrophic floods. The arising stochastic programming problems admit a variety of modeling and solution techniques [Ruszczynski and Shapiro, 2003], in some cases they can be reduced to large scale (mixed-integer) linear programming problems [Rockafellar and Uryasev, 2000] and in case of very large (or continuum) number of scenarios adaptive Monte Carlo technique may be useful [Ermoliev et al, 2000a,b].

The goal of the project "Integrated system for hazardous flood modeling and risks reduction: case study for Tisza (Ukraine), Riony (Georgia) rivers" (2005-2007, Glushkov Institute of Cybernetics, Institute of mathematical machines and systems (Kiev, Ukraine), Tbilisi State University (Georgia) and Science and Technology Center in Ukraine (STCU)) was to develop contemporary tools to support non-structural measures for flood mitigation at mostly exposed to hazardous floods rivers Tisza (Ukraine) and Rioni (Georgia), estimating the risks for the insurance and investment allocation in the areas affected by catastrophic floods. Accordingly, the main project objectives were the following:

To develop a methodology and a prototype computer system for optimal investments allocation and optimal insurance coverage in the areas, exposed to risk of hazardous flooding, on the basis of the up-to-date technologies of stochastic risk optimization and numerical flood mapping;

To implement the developed methods and prototype software for the watersheds of Ukrainian part of Tisza basin and Rioni river basin, Georgia, providing by this way to the regional and national authorities the possibilities for mapping the hazardous floods and promotion of future investment activities and developments of insurance coverage system in Tisza and Rioni basins and, after the verification of the software and proposed methods, in other river basins of Ukraine and Georgia.

11.5.1 Structure of the modeling framework

The specifics of flood risks – unpredictable timing of flood occurrences, absence of spatially explicit information about potentially inundated areas and related losses, long-term flood re-occurrence patterns, complex dependencies between structural and financial flood defense and damage sharing measures, socio-economic heterogeneities of various agents (such as individuals, farmers governments, and insurers) – all these call for adequate model based approaches integrating socio-economic, topographic, geophysical, policy related data and knowledge for evaluation of flood prevention and reduction structural and financial measures.

Decision making under flood risks requires study of the influence of decisions on probability and propagation of floods, impacts on economies losses and on their ability to recover after floods. In turn, this requires integral (system) approach to flood modeling from their beginning, propagation, up to impact upon economic objects. At each stage of modeling one has to take into account inevitable uncertainties in data and knowledge, in particular a lack of data, uncertainty in the structure and parameters of processes and their models, stochasticity, and uncertainty of decisions outcomes. Adequate modeling and treatment of these uncertainties is a key issue for making sound decisions under risk of catastrophic flood losses.

Typically, flood management framework combines geographically explicit data on property values in the region with a stochastic flood risk scenario generator to give estimates of potential flood losses.

In the project this idea is implemented in the form of blocks (modules): (1) discharge/rainfall scenario generation block; (2) a river flow/rainfall-runoff module that assesses water flows in the region; (3) inundation module for evaluation of inundated areas; (4) property/vulnerability module that incorporates damage curves (structural and agricultural) for loss estimation; (5) decision support block.

In more details each module is described as follows.

✓ Discharge/rainfall scenario generation block

There exists three basic ways to generate stochastic input data for flood modeling [Blokhinov, 1974], [Kuchment and Gelfan, 1993; 2002], [Cameron et al, 1999]:

- sampling from historical data;
- discharge generation by means of maximal monthly/decade discharge distribution and standard hydrograph;
- runoff generation by means of stochastic extreme weather (rainfall) conditions simulation and their transformation into discharges by rainfall-runoff model.

In the project, all three options are utilized.

In particular, Institute of Mathematical Machines and Systems of the National Academy of Sciences of Ukraine procured observation data on catastrophic flood occurrences in Zacarpattye in 1998 and 2001 on Tisza river and its tributaries. These data have a form of time series on water level at a certain water gauge station. By means of specific for each site "stage-discharge" curves, these data are transformed into a hydrograph (discharge as a function of time). Since there are a rather detailed observation data on 1998 and 2001 floods in Zacarpattye these floods serve as a reliable examples for river flow model identification and input generation. Input hydrographs of different probability were simulated by scaling 1998 Tisza flood by means of maximal site discharge distribution. Remark that it is rather difficult adequately to estimate the particular floods of 1998 and 2001.

Georgian school of stochastic hydrology [Svanidze, 1977], [Grigolia, 1994] developed original methods for hydrological time series generation and for maximal discharge distributions. In particular, it was proposed to use in hydrological modeling four parametric Jonson's distribution with a bounded support. This method was tested on more than 200 world largest rivers and was recommended for application by "International handbook on basic hydrological characteristics calculation" (Leningrad, 1984; Paris, 1987). In the project, this methodology was utilized by Tbilisi state University for calculation of the maximal monthly discharges of different probabilities for Rioni river.

For flash floods modeling, it is possible to use maximal rainfall distributions with an appropriate within month allocation for stochastic input generation for rain-fall-runoff model. In the project, this approach was used at a certain site in a Tisza river valley.

The scenario generation block includes also the description of possible events leading to destruction of hydrological constructions. Each such event forms a separate flood scenario.

Each flood scenario generation method contains a large number of different uncertainties starting from incompleteness of our knowledge and models to errors in parameters and random factors estimates. So an important problem is the evaluation of the robustness of modeling conclusions with respect to these uncertainties. An example of uncertainty contemporary treatment is given in US Army Corps of Engs flood damage analysis system HEC-FDA [HEC-FDA, 1997].

✓ River flow block

In the river, flow module two physical processes are considered: rainfall-runoff due to precipitation and water flow in the river (channels) and the river dynamics process.

✓ Rainfall-runoff submodel

Simulation of a rainfall-runoff process is made based on the TOPKAPI model methodology. TOPKAPI belongs to the class of the so-called physically-based rainfall-runoff models and was developed by Prof. Todini in mid 1990th [Todini, 1995], [Ciarapica and Todini, 2002]. The model is based on the idea of combining the kinematic approach with the topography of the basin described by means of a lattice of square cells, generally increasing in size with the scale of the problem, over which the model equations are integrated. Each cell represents a computational node for the physical characteristics of the model, namely the mass balance and the momentum balance. The flow paths and slopes are evaluated from the DEM, according to a neighborhood relationship based on the principle of minimum energy, namely the maximum elevation difference which takes into account the links between the active cell and the eight surrounding cells connected along the edges or vertices; the active cell is assumed to be connected downstream with a sole cell. At present, the model version developed by UCEWP is structured around three modules, which represent the soil component, the overland flow component and flow through the drainage network respectively. The present version of the TOPKAPI model does not account for water percolation towards the deeper soil layers and for their contribution to the discharges; this will be introduced as an additional model layer in the future.

In the project rainfall-runoff model TOPKAPI-IPMMS (version by Institute of Mathematical Machines and Systems Problems, [Kivva and Zheleznyak, 2005]) was calibrated for the watersheds of Tisza river and its tributary Uzh. As input for the model precipitation time series registered at the closest weather stations were used.

✓ River dynamics submodel

A river dynamics submodel is developed to perform calculation of the flow for the specific river. Water flow in open channels is simulated using unsteady flow 1-D model within MIKE-11 software package developed by DHI Water & Environment, Denmark [Mike-11, 2004]. The basis objects of river network are

- a branch – simplest part of a river channel;
- a computational gridpoint – an element of a grid, in which flow related variables are calculated for every computational time step;
- a node or a junction – a connector of branches into more complex river network.

Therefore, each branch contains several computational gridpoints, and several branches are connected with nodes into network.

The mathematical model of a branch is based on Saint-Venant system of partial differential equations of 1-D flow mass and momentum conservation, which is solved numerically for every computational gridpoint on a branch. Additionally, in every node the mass balance and level balance equations are written. In a perfect case the information on river network structure and the cross-sectional profiles, representing water capacity of a gridpoint, is extracted from a detailed DEM, but in many cases, it is supplemented with manual river measurements. Additional information on structures along the river channel is introduced. The module transforms dynamics of input discharges into the flow dynamics using a representation of conservational laws. Due to this it is possible to get water levels and discharges along the river for the whole period of modeling.

✓ Inundation block

The calculated by the RIVER module water levels combined with DEM information are further used by the inundation module. For the calculated flood event dynamics, it is possible to reconstruct two types of maps:

- inundation maps that show the depth of standing water. These can relate either to a certain time or to the maximum level throughout the flood event;
- duration maps that represents for how long the water is standing on a floodplain.

For example, the module calculates inundation zones, in which the inundation level was 0-2 meters, 2-4 meters and more than 4 meters. Duration maps show zones, which were covered by water for less than 12 hours, 12-24 hours, 24-48 hours, and more than 48 hours. The maps can be generated to represent dynamics of inundation, say every 3 hours of flood event, but also the maximum inundated area can be estimated. Combination of inundation and duration maps gives time-depth-area detailization, which is used directly in the VULNERABILITY module for estimation of losses caused by a flood.

The module can be utilized within a Monte Carlo framework, giving on the output inundation/duration maps corresponding to flood events of different return period, for example 50-year flood, 100-year flood, 200-year flood, 500-year flood etc. All the computations are made within MikeGIS package (DHI software), built as a project in ArcView 3.x GIS software.

✓ Vulnerability block

Combination of inundation and duration maps with so-called vulnerability curves gives immediately estimation of losses. This task is performed by property/vulnerability module. The Vulnerability block produces estimates of losses for a given pattern of flood. Potential structural damages or losses associated with flood can be calculated in relative values for each type of the building in the region. Here losses are described as a certain decrease in percentage of the whole property value. This approach is especially applicable for the cases where detailed spatial information on property distribution is not available now, but can be obtained later. Vulnerability block utilizes vulnerability curves (Depth-Damage functions) – functions that represent losses depending on severity of catastrophe event and the type of structure. This can be agricultural losses depending on the inundation time, the crop and the time of the year; losses in building, depending on the depth and duration of flood as well as building material (wood, concrete, brick etc.). Usually, vulnerability curves are derived from historical observations, and are available for the modeling purposes. There are a number of works on the utilization of the DD functions; see [USACE, 1992; 2000], [Merz et al. 2004]. A nice discussion on the state-of-the-art of the problem is presented in [Messner and Meyer, 2005]. In the most papers, duration factor is not taken into account, there are only several approaches that consider a change of DD curve with increase of flood duration [Penning-Rowsell and Chatterton, 1977], [Penning-Rowsell et al, 2003]. In some cases, there is no detailed GIS information on structures' types within the modeling region. In this case, loss estimation can be done in relative or percentage terms. For example, in a certain sub-area of the region due the losses for wooden houses to flood event would be 50%, for brick houses – 40%, and for concrete houses with pillars – 10%. Once the GIS distribution of house types becomes available, the produced relative losses can be easily converted into absolute ones.

One of the direct applications of the Vulnerability block is estimation of effectiveness of a flood mitigation structural measure in terms of reduction of associated losses in "what-if" scenarios. From the other hand, as the Integrated System naturally supports Monte Carlo approach, the losses probability distribution information becomes available in terms of mean estimates, histograms etc.

Furthermore, once potential flood losses can be described in financial terms different financial oriented applications can be implemented, for example, investment allocation, insurance coverage planning, catastrophe fund design etc. The broad number of applications appears starting with estimation of the optimal premium for the fund, estimation of the current financial policies etc.

Options of input, editing and maintaining catalogue of the so-called "depth-damage relationships" were implemented in HEC-FDA (1997) system.

In the project own "depth-damage relationships" database was developed based on Excel and damage calculation module.

- ✓ **Decision support block**

Decision support block integrates information on the goals and constraints of agents that are involved in catastrophe management and may potentially suffer, share or mitigate the losses. These are households, farmers, local and central governments, flood defense offices, city planners, insurers, investors, financial markets, etc. The methodology has been already tested for the analysis of flood and seismic risks in Italy, Hungary, Poland, Russia, Japan, Ukraine [Baranov, 1999], [Galambos et al, 2000], [Amendola et al, 2000], [Ermoliev et al, 2000a,b], [Ermolieva et al, 2003], [Ermolieva and Ermoliev, 2005], [Linnerooth-Bayer and Amendola, 2003]. For example, procedures for flood reduction plans evaluation are implemented in US Army Corps of Engs HEC-FDA system [HEC-FDA, 1997].

In the project a special subsystem for investment/financial/insurance decision support was developed. The subsystem was designed to plan and evaluate structural and non-structural actions against flood damages to take into account a complex interaction between flood scenarios, river topography and flood countermeasures.

- ✓ **Model based flood damage estimation**

Within the proposed framework, we estimate potential flood-induced damages and explore feasible financial mechanisms to share the losses between the stakeholders. Implementation of the Vulnerability module is directed to estimate potential losses due to a flood. Once inundation maps are constructed, they can be directly combined with the Depth-Damage Functions (DD functions, DD curves) showing dependence of the structural damages caused by flooding. DD functions are derived by statistical analysis of losses measured in the past. The curves take into account material of the structure, depth of the flood and implicitly include the temporal flood pattern for the region in which they were estimated, see [USACE, 1992; 2000], [Merz et al, 2004].

Potential structural damages or losses associated with flood can be calculated in relative values for each type of the building in the region. Here losses are described as a certain decrease in percentage of the whole property value. This approach is especially applicable for the cases where detailed spatial information on property distribution is not available now, but can be obtained later.

One of the direct applications of the Vulnerability module is estimation of effectiveness of a flood mitigation structural measure in terms of reduction of associated losses in "what-if" scenarios. From the other hand, as the Framework naturally supports Monte Carlo approach, the losses probability distribution information becomes available in terms of mean estimates, histograms etc.

Furthermore, once potential flood losses can be described in financial terms different financial oriented Framework extensions can be implemented, for example, Insurance module, Catastrophe Fund module etc. The broad number of applications appears starting with estimation of the optimal premium for the Fund, estimation of the current financial policies etc.

Monte Carlo simulation of the Framework makes possible to move from "what-if" scenarios (passive changes) to optimization of financial instruments of flood regulation (active changes).

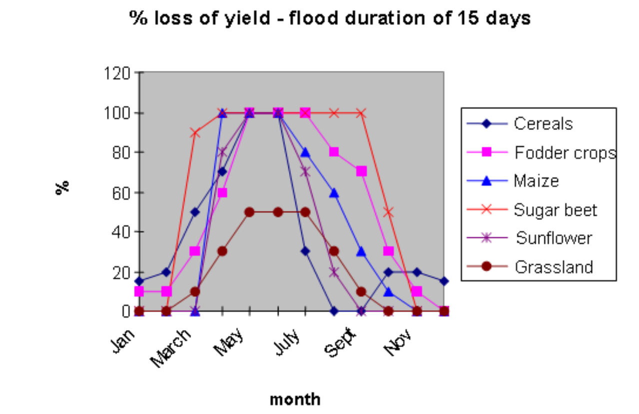

Figure 168. Functions "moment of flooding – yield losses"

✓ Catalogue of functions "flood moment – yield losses"

Losses for agricultural crops are determined basically by moment and duration of flood and are calculated by means of empirical functions "moment and duration of flood – percentage losses for yield". A number of such functions for fixed flood duration are presented on Figure 168.

✓ Catalogue of functions "depth – damage to structure"

For calculation of flood losses functions "depth – percentage losses to a structure" are used. A number of such functions is presented on Figure 169.

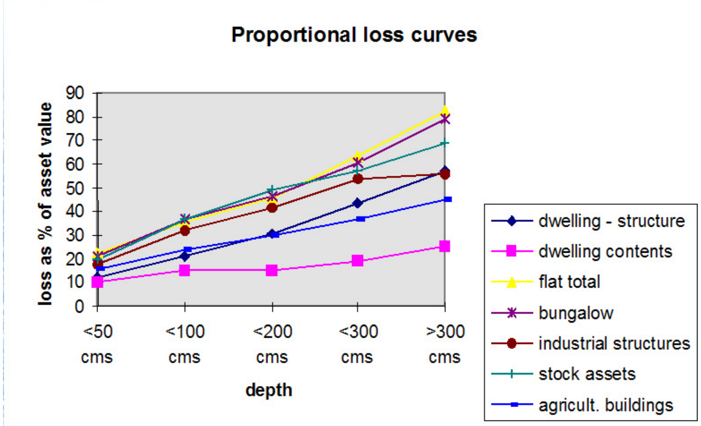

Figure 169. Graphs of percent property losses dependent on structure inundation depth

✓ Methodology for optimal investment allocation

Optimal investment allocation problem is set as follows: choose an investment plan (portfolio) $u=(u_1,...,u_n)$, where component u_i designate the volumes of investments in i-th object, in such a way that the obtained investment portfolio is optimal with respect to return – risk criteria.

Let for a given river, a set (ω_s, p_s), $s=1,...,S$, of flood scenarios has been developed, where ω_s is the description of scenario s, for instance as a hydrograph (an input for modeling), p_s is the probability of scenario s. Assume that each scenario is mapped into flood zone indexed s. According to flood zones, and classifications of investment objects, percent damages to every object i can be found for every scenario s. Let every object i is described by investment return θ_i and potential percent damage $l_i(\omega_s)$ for every scenario s: $i = 1,...,n$, $s = 1,...,S$. Thus, return distribution of every object i looks like: $(\theta_i(\omega_s) - l_i(\omega_s), p_s), s = 1,...,S$, and return distribution of total portfolio is described as:

$$x(u,\omega_s) = \sum_{i=1}^{n} (\theta_i(\omega_s) - l_i(\omega_s))u_i \tag{29}$$

where variable ω_s describes realizations of corresponding scenario s with probability p_s, $s = 1,...,S$.

Under insurance the return function $x(u,\omega)$ of insured portfolio looks like

$$X(u,v,\omega_s) = x(u,\omega_s) + g(u,v,\omega_s) \tag{30}$$

where

$$g(u,v,\omega_s) = -\sum_{i=1}^{n} \pi_i(v_i) + \sum_{i=1}^{n} \min\{l_i(\omega_s)u_i, v_i\} \tag{31}$$

and $\pi_i(v_i)$ designates the insurance premium for compensation of damages with a maximum compensation level v_i.

Then the following optimization problems can be considered:

1) minimization of a risk measure under a guaranteed level of the return expectation;

2) maximization of expected return under constrains on a risk measure level, where as risk measure it is proposed to use the polyhedral coherent risk measures (PCRM).

Another possible setting [Norkin, 2006] is to make such decisions that give maximal outcome in normal conditions (no catastrophe scenario) and limit losses in abnormal scenarios (catastrophes of different severity).

The class of polyhedral coherent risk measures (PCRM) contains the following risk measures (see [Kirilyuk, 2004a,b; 2008]): 1) risk of an inexact estimation of scenario probability; 2) worst case risk (WCR); 3) conditional loss expectation on α-tail distribution (CVaR$_\alpha$); 4) worst conditional expectation (WCE$_\alpha$); 5) spectral coherent risk measure (SCRM); 6) measure, based on semi-deviation (absolute deviation) on expected return etc.

Portfolio optimization problems on return-risk ratio **1**) and **2**) are described as

$$F(u,v) = \max_{Bp \le c,\, p \ge 0;\, \sum_{s=1}^{S} p_s x(u,v,\omega_s) \ge \mu;\, \sum_{i=1}^{n} u_i = \bar{u}.} \left\{ -\sum_{s=1}^{S} x(u,v,\omega_s) p_s \right\} \to \min_{u \ge 0, v \ge 0};$$

$$G(u,v) = \sum_{s=1}^{S} p_s x(u,v,\omega_s) \to \max_{u \ge 0, v \ge 0},$$

$$\max_{Bp \le c,\, p \ge 0} \left\{ -\sum_{s=1}^{S} p_s x(u,v,\omega_s) \right\} \le \sigma,$$

$$\sum_{i=1}^{n} u_i = \bar{u},\ u_i \ge 0.$$

Here, depending on problem setting (with or without insurance) instead of $x(\cdot)$ either formula (29), or (30)-(31) are used, and matrix B and vector c describe the chosen risk measure.

Concerning mathematical methods**,** these problems were reduced to corresponding linear programming (LP) problems [Kirilyuk, 2004a,b, 2008]. The obtained result allows solving very large problems by standard LP technique by means of available software.

Besides, optimal insurance coverage models (without and with reinsurance) were proposed**.** The difference with respect to optimal investment allocation and optimal insurance coverage problems consists in constrains on portfolio variables: $u_i \geq 0, \sum_{i=1}^{n} u_i = \overline{u},$ for the first problem, and $u_i \in [0,1]$ or $u_i \in \{0,1\}$ – for the second one. Concerning optimum insurance coverage models with reinsurance, they are not convex and require special mathematical technique.

- ✓ **Planning of investments in commercial objects and countermeasures in the areas of catastrophic flood risks**

Let index $i = 1,...,n$ marks position (including geographical location and type of the project) of possible allocation of objects of investments. With every position some cost of investments c_i, profitableness θ_i on a unit of cost of object i under normal operational conditions and relative losses l_{ij} in catastrophic scenario j are tied-up. In this setting different types of objects of investing which are located in one geographical place have different indexes i. An investor can fully invest a project, or not to build it at all. A total volume of investments is limited by C. Let us introduce a Boolean variable $x_i \in \{0,1\}$, which means that at $x_i = 1$ object i is built (invested) and at $x_i = 0$ the object is not built. Let $\{\lambda_j\}$ is an admissible rate of losses of investments under catastrophic scenario j. Then the task of making optimal decisions reads as [Norkin, 2006]:

To maximize over $x = \{x_i\}$ function

$$F(x) = \sum_i \theta_i c_i x_i$$

subject to constraints

$$G_j(x) = \sum_i l_{ij} c_i x_i \leq \lambda_j \sum_i c_i x_i \; ; \; j = 1,...,m \; ; \; \sum_i c_i x_i \leq C \; ; \; x_i \in \{0,1\}, \; i = 1,...,n \, .$$

More general mathematical model (nonlinear discrete optimization), which takes into account a possibility of countermeasures, has the following form:

To maximize over investment plans $x = \{x_i \in \{0,1\}, i = 1,...,n\}$ and countermeasures $y = \{y_s \in \{0,1\}, s = 1,...,k\}$ profitableness function in normal conditions

$$F(x,y) = \sum_i \theta_i c_i x_i - \sum_s d_s y_s$$

subject to constraints on relative losses in the case of a catastrophic event

$$\sum_i l_{ij}(y) c_i x_i + \sum_s d_s y_s \leq r_j \sum_i c_i x_i \; , \; j = 1,...,m;$$

and constraints on available resources

$$\sum_i c_i x_i + \sum_s d_s y_s \leq C \, ,$$

where θ_i is index of profitableness of object i;

c_i is cost of object i;

d_s is cost of countermeasure s;

$l_{ij}(y)$ is a coefficient of losses of investments in an object i in scenario j under condition of implementation of plan of countermeasures y;

C is a total volume of investments;

$r_j,\ j=1,...,m,$ is a rate of relative risk.

✓ Algorithm of the problem solution

1. For the plan of countermeasures y and every catastrophic scenario j to model a corresponding flood and to build inundation maps.
2. For every potential object i, plan of countermeasures y and scenario j to calculate inundation levels and coefficients of losses $l_{ij}(y)$.
3. For every feasible plan of countermeasures y to find the optimum plan of investments $x(y)$ and a corresponding profitableness $F(x(y),y)$.
4. To find the plan of countermeasures y^*, which corresponds to the maximal profitableness $F(x(y^*),y^*)=\max_{y:\sum_s d_s y_s} F(x(y),y)$.

✓ A continuous investments allocation task in the risky agricultural areas

Planning of agricultural production in the areas of risky agriculture (back-waters of the rivers, non-irrigated droughty territories, mountain slopes) is an actual task. Thus there always is more costly alternative to develop a production in the protected or irrigated territories. A problem consists in the choice of sound compromise between a costly and reliable technology and cheap, but risky one. Let there is only one type of investments, for example, in sowing of agricultural culture of certain kind. Let x_i designate sowing area in region i, c_i is a cost of growing of the culture on a unit square in region i, θ_i is the productivity of the culture in region i at normal conditions, l_{ij} is an expertly determined coefficient of losses of sowing areas in region i at catastrophic scenario j, m_{ij} is an experimental coefficient of losses of the productivity in region i at catastrophic scenario j, b_i is the maximal sown area in region i, λ_j are admissible relative losses of sowing areas for scenario j, μ_j are admissible relative losses of the total harvest for scenario j, I is a total volume of investments. Then the task of investments allocation reads as follows [Norkin, 2006]:

To maximize over $x=\{x_i\}$ a function

$$F(x)=\sum_i \theta_i x_i$$

Subject to constraints

$$F_j(x)=\sum_i m_{ij}\theta_i x_i \le \mu_j \sum_i \theta_i x_i\ ,\ j=1,...,m;$$

$$G_j(x)=\sum_i l_{ij}x_i \le \lambda_j \sum_i x_i\ ,\ j=1,...,m;$$

$$\sum_i c_i x_i \le I\ ,\ 0\le x_i \le b_i\ ,\ i=1,...,n.$$

✓ Software for decision making under catastrophic flood risks

Besides hydrological modeling, the proposed modeling framework allows estimating potential flood damages for regional economies and private investors; it can help to select investment allocation plan, insurance and reinsurance arrangements in a catastrophic flood risk zone. For this, flood damage models and flood loss mitigation (investment diversification, insurance, and reinsurance) models are developed for study regions. In particular, V.M.Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine developed a catalogue of "depth-damage" functions, and a software Decision Support subSystem "Catastrophic Flood Risk Manager" (DSS CFRM) for investment/insurance decision support in a flood prone area. The developed software supports data preparation and solution of a number of tasks related to optimal investment allocation (investment portfolio), insurance coverage (insurance portfolio) planning and counter measures selection by calculation and analysis of the risk-return relationships. The inputs to the subSystem are data on potential commercial objects/projects and their inundation levels of different probabilities obtained because of flood scenarios generation and simulation by means of hydrological models of the modeling framework. The outputs of the subsystem are dependences of decision outcomes and measure of risk. In the decision support subsystem contemporary approaches to decision making under catastrophic risks are implemented, which numerically are reduced to solution of a number of large scale linear and mixed integer nonlinear programming problems. The decision support subsystem operates as a standalone MS Windows application with help, diagnostics, and graphics facilities; data for the subsystem are prepared or imported through Excel, output results are presented in a graphical form in output windows and are also put into Excel tables.

For solution of such problems within the modeling framework, the following preparation steps are fulfilled:

By means of the scenario generation block a number of discharge scenarios of different annual exceedance probabilities (p = 0.5, 0.2, 0.1, 0.04, 0.02, 0.01, 0.004, and 0.002) are formed;

By means of the river flow block for each flood scenario and a structural flood counter measure (if such measures are potentially planned), flood wave propagation is simulated and maximal water stage profile along a given river section is obtained;

By means of the inundation block water stage profiles are transformed into inundation maps of different annual probabilities for the site of interest;

By means of geo-information system ArcView the site inundation maps are viewed, a set of places of existing or potential structures is indicated and structure inundation levels corresponding to different flood scenarios are calculated;

By means of Excel the existing or potential structure inventory is updated with structure information (structure occupancy type and attributes);

By means of the vulnerability block (bank of depth-damage curves) and structure characteristics percent damage to structure value and to its content is calculated.

As a result, coefficients of the value loss for each structure, each flood scenario, and each counter flood measure are obtained. This information is an input data for setting and solution of a variety of decision-making problems under catastrophic flood risks.

For the analysis of decision making problems a special subsystem "CATASTROPHIC FLOOD RISKS MANAGER" was developed.

Decision Support System CATASTROPHIC FLOOD RISKS MANAGER (DSS CFRM) is a standalone computer system designated for decision support under catastrophic flood risks. Decision making concerns careful resources allocation and countermeasures planning accounting for trade-offs between costs, benefits and possibility of catastrophic losses. The subsystem implements decision-making methodology, described in [Norkin, 2006, 2007], [Ermoliev et al, 2000, 2001], [Kirilyuk, 2003-2008].

DSS CRM (Version 1.0) supports solution of the following tasks:

1. Selection of objects in a flood prone area for insuring (and levels of insurance) to get maximum total premium under reasonable exposure to flood claims risks.
2. Selection of a reinsurance level and objects in a flood plain for insuring to get maximum total revenue accounting for high reinsurance tariffs and flood claims risks.
3. Selection of potential projects in a flood plain to invest to get maximum total revenue under reasonable exposure to flood damage risks.
4. Selection of flood mitigation measures and potential projects in a flood plain to finance to get maximum total revenue under reasonable exposure to catastrophic flood damage risks.
5. Minimization of a certain (coherent) portfolio risk measure subject to a bound (from below) on the portfolio means revenue.
6. Maximization of a financial portfolio expected return subject to a bound (from above) on a certain (polyhedral coherent) portfolio risk measure and guaranteed mean revenue.
7. Maximization of a financial portfolio expected return subject to several constraints (from above) on certain (polyhedral coherent) portfolio risk measures.

The class of polyhedral coherent risk measures (PCRM)contains the following risk measures (see [Kirilyuk, 2004a,b, 2008]: 1) risk of an inexact estimation of scenario probability; 2) worst case risk; 3) conditional loss expectation on α-tail distribution; 4) worst conditional expectation; 5) spectral coherent risk measure; 6) measure, based on semi-deviation (absolute deviation) from expected return etc.

Concerning solution technique**,** these problems were reduced to mixed integer linear and nonlinear programming problems.

Tasks data are prepared and edited in Excel, results of tasks solution are put in Excel files and also in output windows. The data describe characteristics of possible decisions, normal and catastrophic scenarios, volumes of presently available resources, and acceptable levels of their deficits, parameters of risk measures. Some (loss) data are calculated within the system based on objects/projects data, scenarios hitting (inundation) levels and catalogue of "hitting factor-damage functions" also are accessible for analysis in Excel.

The results represent dependences of decisions outcomes as functions of some risk parameter. As a risk parameter can serve a level of reserve deficit, level of reinsurance, value of some specific risk measure.

Interactive algorithm of solution of any task consists of the following steps:

- selection of a task;
- data/example/template selection and preparation/editing in Excel;
- attempts to solve the task and correction of (reaction on) errors in data;
- visual analysis of results in monitor windows and through Excel charts;

– change in data/parameters and repeated solutions.

The look of the decision support system interface is presented on Figure 170.

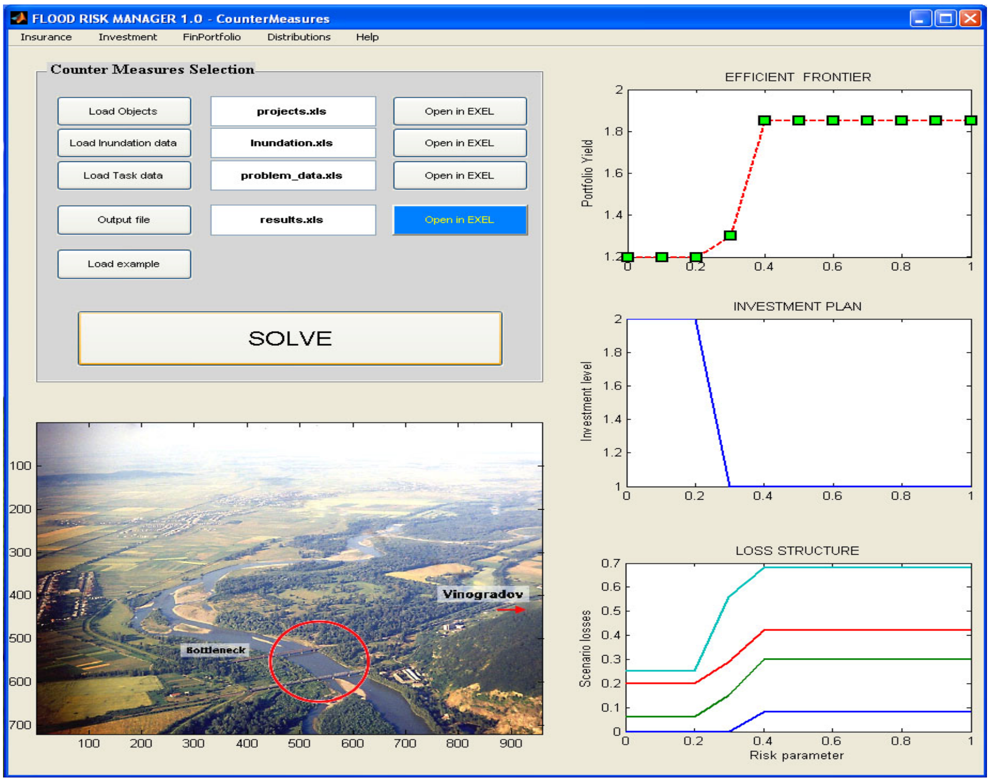

Figure 170. The look of the decision support system interface