12 Techniques for Robust Bayesian Estimation

12.1 Introduction

The sharp increase in global demand for energy caused by the emergence of such major industrial economies as China and India accelerated approaching of time when sources of oil and natural gas will be exhausted. The alternative to those fuels are nuclear energy, coal, and renewable sources. Although coal is not a limiting source of energy today, it is highly polluting. Renewal energy sources are "clear" types of energy, but they are still largely undeveloped. Therefore, only nuclear energy can be a real alternative to fossil fuels today.

Unfortunately, the Three Mile Island and Chernobyl accidents caused the adverse public opinion that has practically brought nuclear programs in many countries to a standstill. On the other hand, these accidents caused an upheaval in nuclear safety analyses. Until the accident at Three Mile Island in 1979, the USA Atomic Energy Commission (now the Nuclear Regulatory Commission (NRC)), has generally regulated Nuclear Power Plants (NPP) based on deterministic approaches, which consider a set of challenges to safety and determine how those challenges should be mitigated.

The Three Mile Island accident has clear demonstrated how important it is to quantify both the probability and the consequences of an accident. These are the key quantities that should be a basis in regulatory and safety decisions. Besides, in the early 1960s F. R. Farmer introduced another fruitful idea that predetermined future investigations in nuclear safety analyses, namely the idea of reactor safety based on the reliability of consequence-limiting equipment [NUREG/CR-2300, 1983]. These ideas substantially changed the character of the nuclear safety analyses resulting in a systematic and comprehensive methodology to evaluate Severe Accident Risks known as Probabilistic Risk Assessment (PRA).

First, PRA was systematically applied in 1975 for the study of core meltdown accidents in two commercial nuclear power plants. Results of this study were reported in the Reactor Safety Study (RSS) [WASH-1400, 1975]. This study identified a collection of accident sequences potentially capable of resulting in significant releases of radioactive material from a NPP, estimated the radiological consequences of these events, and the probability of their occurrence, using a fault tree/event tree approach. This study indicated that the probabilities of such accidents were higher than previously believed but that the offsite consequences were significantly lower. The study was peer-reviewed by the "Lewis Committee" in 1977, which approved the methodology, but cautioned that the risk Figures were subject to large uncertainty. That led to an NRC policy statement on the use of PRA [WASH-1400, 1979].

To support increasing application of PRA techniques within the industry and the regulatory process, the PRA Procedures Guide [NUREG/CR-2300, 1983] was developed. This document was intended to provide an overview of the risk-assessment field and to identify acceptable techniques for the systematic assessment of the risk from nuclear power plants. This guidebook was very useful to support PRAs that were performed in the 1980 and 1990 and in fact, much of it is still relevant today.

In 1988, the NRC requested information on the assessment of severe accident vulnerabilities by each licensed nuclear power plant [NUREG-1150, 1990]. The assessment could be done with either PRA or other approved means. In response, virtually all licensees indicated that they intended to perform PRAs in their assessments. Results of such assessments in five commercial nuclear power plants of different design was summarizes in NUREG-1150 [NUREG-1150, 1990]. This document provides improved PRA techniques and perspectives on how the results may be used by the NRC staff in carrying out its safety and regulatory responsibilities. To this day, the NUREG-1150 PRAs represents the NRC's most outstanding contribution to the state-of-the-art of PRA.

Probabilistic risk assessment involves developing a set of possible accident sequences and determining their outcomes. Accident sequences are sequences of events, which challenge the safety of the NPP. These sequences are launched by initiating events, and are modeled with event trees, and fault trees (system models). After occurrence of an initiating event, some standby safety systems must be demanded, and other, normally operating, systems must continue operating to prevent or mitigate serious undesirable consequences. Event trees depict initiating events and combinations of systems successes and failures. Fault trees depict ways in which the system failures represented in the event tree can occur.

To provide a quantitative assessment of the risk from accidents in NPPs, accident sequences must be quantified. In order to quantify the frequencies of the accident sequences delineated in the event trees, failure rates are assigned to each system model and frequencies are assigned to each initiating event. Combining the appropriate system success and failure models with each class of initiating events yields a logical representation of each accident sequence.

For a system to fail to perform its mission, several components must either fail or be unavailable. The logic events in system models that represent these failures or modes of unavailability are called basic events. Initiating events and basic events are modeled as resulting from random processes. These models have one or more parameters that should be estimated using available data. In this paper we consider only probability models based on exponential distribution. Detailed description of the standard probability models for each event and statistical methods for estimating parameters of these models can be found in the handbook NUREG/CR-6823 [NUREG/CR-6823, 2003].

Initiating events are commonly modeled with Poisson process defined by a rate of frequency λ. This model is appropriate, if simultaneous (common-cause) events do not occur. Traditionally, PRA includes to the set of initiating events such events that directly affect NPP safety, for instance, unplanned reactor trips, plant-centered loss-of-offsite power, and through-wall leak. The handbook [NUREG/CR-6823, 2003] includes also some failures occurred in a continuously running systems (such as sensor/transmitter failures) to the set of initiating events, following the viewpoint of probability modeling, in which the important fact is not the consequence of the events, but the way that they occur randomly in time. Failures occurred in a continuously running system are quickly revealed and removed. In contrast to this type of failures, failures to run during mission are failures of components or systems that do not run continuously. They are demanded to start when initiating event occur, and then they should run during some mission time. If they fail during the mission, they are nonrepairable, that is, they cannot be repaired or replaced quickly.

In PRA, components failures in both standby and continuously running systems are modeled by exponential distribution with the single parameter λ. This distribution is inherently associated with the Poisson process. Namely, the time to first failure and the times between successive failures follow an exponential distribution and the number of failures in a fixed time interval follows a Poisson distribution.

In order to use these models for quantification of accident sequences, the reliability parameters should be estimated. These estimates are propagated through logical relations to produce an estimated frequency of the undesirable end state, such as core damage. Uncertainties in the parameter estimates must be also quantified and propagated through the PRA models to obtain the uncertainty in the final estimate of risk.

Statistical techniques that are usually used for estimation of reliability parameters are based on two main approaches: the classical sampling theory methods and the Bayesian approach. The sampling theory methods are the preferred methods, if sufficient plant-specific data are available. In this case, risk estimates derived from this information reflect the actual plant experience. However, sampling theory methods are inappropriate for treating scarce data samples, which are usually available when reliability parameters of highly reliable equipment are estimated. For some nuclear power plant components, no failures were registered during many reactor-years of commercial operation. In these cases, the use of the sampling theory methods in estimation procedures leads to significant errors and makes it meaningless to use existing methods of probabilistic analysis.

One of the ways to overcome this shortcoming of the sampling theory is to utilize not only raw data specific for current investigation of equipment reliability but incorporate also other relevant information available from other data sources. Some possible sources of additional data are operating data in different environments; engineering judgments and personnel experience; and operating experiences with similar equipment. Several generic data sources are currently available and used throughout the nuclear PRA industry as sources of additional data. As a rule, generic data sources represent compilations of raw data, which have been collected directly from various NPPs. In generic databases, the raw data have been statistically analyzed and generic values of reliability parameters have been estimated. Uncertainties in the parameters estimates are presented in form of the four characteristics of probability distributions: the mean, the median, 5th percentile, and 95th percentile of the distributions. Unfortunately, the sampling theory methods are inappropriate for incorporating this relevant information.

The weak place of the Bayesian approach is to select a true distribution function. Incorrect selection of a prior distribution can result to significant errors and affect the confidence to results of a safety investigation in PRA.

Here we, following to Robbins [Robbins, 1951] and Berger [Berger, 1984], take into account that Bayesian procedures actually deal with a family of priors Γ instead of a single prior distribution. The uncertainty in the priors generates uncertainty in the posterior values of interest (e.g. mean, variance) described by the range of the posterior values as the prior varies in Γ. Finding lower and upper bounds for this range, we can evaluate sensitivity of the posterior values to the choice of the prior from the class Γ.

To find exact lower and upper bounds for the range of the posterior values as the prior varies in Γ one should solve optimization problems with linear-fractional functional with respect to distribution functions (priors). Detailed mathematical statements of such problems are presented in [Golodnikov et al, 2004].

Methods of optimization in class of distribution functions has a long story and begins from works of P.L.Chebyshev, who developed a basic inequality of probability theory called Chebyshev's inequality, and used the latter inequality to give a very simple and precise demonstration of the generalized law of large numbers [Enc.Britannica, 2009]. A.A.Markov developed ideas of his teacher in [Markov, 1898]. Detailed information about methods of optimization in class of distribution functions can be found in fundamental works of M. G. Krein, and A. A. Nudelman [Krein and Nudelman, 1977] and S. Karlin, and V. Studden [Karlin and Studden, 1966]. These works investigated only linear functionals with respect to distribution functions and utilized analytical tools.

The first work suggested computer based approach to optimization in class of distribution functions was the paper of Y. Ermoliev [Ermoliev, 1970]. Detailed description of idea underlying this work, namely, use of modification of the revised simplex method, was presented in [Ermoliev, 1976]. These works stimulated development of numerical methods of optimization in class of distribution functions in two directions.

The first approach considers the original problem of optimization of some functional under several constraints on other functionals as the optimization problem of infinite dimension. First, it was supposed that functionals are linear with respect to distribution functions. Using the idea of limit extremal problems [Ermoliev and Nurminski, 1973], the original problem of infinite dimension is approximated by the sequence of linear programming sub-problems, which can be solved by standard simplex methods [Golodnikov and Stoikova, 1978a], [Golodnikov, 1979b]. This approach was a basis for building algorithms for optimizing linear-fractional functionals [Golodnikov and Stoikova, 1978b], [Golodnikov, 2007] and nonlinear functionals [Golodnikov, 1982] with respect to distribution functions. Using the idea of ε- Quasigradient Method [Nurminski and Zhelikhovski, 1977], numerical method was developed for solving minimax problem in which the "inner" problem of maximization is linear-fractional functional with respect to distribution functions [Golodnikov, 1979a]. Actually, this was the first algorithm for building robust Bayesian estimates. The next works suggested methods for building robust Bayesian estimates, [Lavine, 1991], [Lavine et al, 1993] has appeared more than ten years later.

The second approach is based on ideas of duality. In the works of Y. Ermoliev, A. Gaivoronski, and C. Nedeva [Ermoliev and Nedeva, 1982], [Ermoliev et al, 1985] the original linear problem of infinite dimension is reduced to dual problem which is formulated as finite dimensional minimax problem without concavity of "inner" problem of maximization. A vast amount of work has been done on minimax problems but virtually all of the existing numerical methods fail if the inner problem is nonconcave. To overcome this difficulty the work [Ermoliev et al, 1985] adopts an approach based on stochastic optimization techniques.

In [Golodnikov et al, 2007] a new approach to selecting the Gibbs distribution in models of objects to be recognized is proposed. This approach proposes to determine the lower and upper bounds for probabilities of the object under study. The distance between these bounds may be used as a measure of error in pattern recognition problems.

This chapter analyzes statistical techniques that are usually used for estimation of reliability parameters. It compares two main approaches: the classical sampling theory methods and the Bayesian approach. It is known that sampling theory methods are inappropriate for treating scarce data samples. In contrast to sampling theory methods, the Bayesian approach allows naturally to incorporate data from various sources in reliability parameters estimates by considering each source as a sample from the same population. However, the justification of a prior distribution frequently is a practical difficulty in the application of the Bayesian approach.

In cases when small datasets of past reliability data are available, it is desirable to estimate how far the calculated Bayesian estimate is from the true Bayesian estimate. Therefore, when only partial prior information is available it is necessary to search upper and lower bounds for Bayesian estimates which can be derived for any prior distribution satisfying the given partial prior information. The chapter considers models for searching such bounds.

12.2 Methods of estimating parameters in risk models based on classical sampling theory

One of the important steps of the currently used methodology of risk assessment is estimating parameters characterizing the equipment reliability (failure rates and probability of demand-related failures). The accuracy of parameter estimates obtained at this step significantly affects the accuracy of ultimate result: risk assessment of the object.

Statistical techniques that are usually used for estimation of reliability parameters are based on two main approaches: the classical sampling theory methods and the Bayesian approach. The classical approach allows obtaining satisfactory estimates when detailed data concerning each component are available.

12.2.1 Sample methods of point estimation of reliability parameters

Let $f(x;\theta)$ be a probability density function of the random value X, where θ is an unknown parameter to be estimated. Consider method of point estimation of this parameter. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x;\theta)$. The likelihood function for this random sample is the joint probability density function of $X_1, X_2, \ldots, X_n$:

$$L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i;\theta),$$

which is considered as a function of θ.

The maximum likelihood estimate of parameter θ is the value $\widehat{\theta}$ such that $L(\widehat{\theta}; x_1, x_2, \ldots, x_n) \geq L(\theta; x_1, x_2, \ldots, x_n)$ for every value of θ. The maximum likelihood estimate is a function of the observed random sample $x_1, x_2, \ldots, x_n$, i.e. it is random.

Method of moments equates the distribution moments $\mu'_j = E(X^j)$ to the sample moments

$$m'_j = \frac{\sum_{i=1}^{n} X_i^j}{n} \text{ for } j = 1,2,\ldots,k.$$

Using the method one can easily obtain the estimate for parameter p of the binomial distribution, which provides model for failures occurring when demand occurs. According to this model, the probability P that x failures occur when n demands occur is calculated as follows:

$$P(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x} \quad (1)$$

The estimator for this probability is [Martz and Waller, 1991]:

$$\widehat{p} = \frac{x}{n} \quad (2)$$

12.2.2 Sample methods of interval estimation of reliability parameters

Practice of implementation of sample estimators in investigating reliability parameters shows that in many cases it is not enough to consider only point estimators. In cases when medium or small samples are available, values of estimators may vary too greatly from experiment to experiment. For this reason, they cannot be used as a stable estimator. That is why in addition to point estimators of reliability parameters one should also consider their confidence intervals as a measure of uncertainty.

Two-sided confidence interval for estimator ε of level $1-\gamma$ is a random interval $(\varepsilon_1(x), \varepsilon_2(x))$ such that

$$P\{\varepsilon_1(x) < \varepsilon < \varepsilon_2(x)\} \geq 1-\gamma .$$

Consider confidence intervals for the binomial model widely used in the probabilistic safety analysis.

Lower and upper bounds, $\varepsilon_1(x), \varepsilon_2(x)$, of two-sided confidence interval for estimator of parameter p in the binomial model is calculated as follows [Martz and Waller, 1991]:

$$\varepsilon_1(x) = \frac{x}{x+(n-x+1)F_{1-\gamma/2}(2n-2x+2, 2x)} \quad (3)$$

$$\varepsilon_2(x) = \frac{(x+1)F_{1-\gamma/2}(2x+2, 2n-2x)}{(n-x)+(x+1)F_{1-\gamma/2}(2x+2, 2n-2x)} \quad (4)$$

where $F_{1-\gamma/2}(n_1, n_2)$ is the $(1-\gamma/2)$-quantile of the $F-$ distribution with (n_1, n_2) degrees of freedom.

12.2.3 Basic shortcoming of the sample methods of estimation of reliability parameters

However, sampling theory methods are inappropriate for treating scarce data samples, which are usually available when reliability parameters of highly reliable equipment are estimated. For some nuclear power plant components, no failures were registered during many reactor-years of commercial operation. In these cases, the use of the sampling theory methods in estimation procedures leads to significant errors and makes it meaningless to use existing methods of probabilistic analysis.

One of the ways to overcome this shortcoming of the sampling theory is to utilize not only raw data specific for current investigation of equipment reliability but incorporate also other relevant information available from other data sources. Some possible sources of additional data are operating data in different environments; engineering judgments and personnel experience; and operating experiences with similar equipment. Unfortunately, the sampling theory methods are inappropriate for incorporating this relevant information.

12.3 Bayesian approach to parameters estimation. Analysis and methods of investigation

In contrast to sampling theory methods, the Bayesian approach allows naturally to incorporate data from various sources in reliability parameters estimates by considering each source as a sample from the same population. While in the classical approach the parameters of underlying life distribution are assumed unknown constants to be determined, in the Bayesian approach the parameters are considered to be random values to which is assigned a prior probability density function.

The basic elements of the Bayesian approach are the sampling model $f(x_i \mid \theta)$ – the conditional probability density function of random value X_i which depends on parameter θ; the prior model $H(\theta)$ - the prior probability distribution of parameter θ with the density $h(\theta)$; the posterior model $G(\theta \mid x)$ – the posterior probability distribution of parameter θ given sample data x with the density $g(\theta \mid x)$, and the loss function $L(\theta,\delta)$ for the estimator δ.

12.3.1 Choice of a prior distribution function

The prior probability density function $h(\theta)$ accumulates the totality information available concerning the parameter θ prior to the observation of the sample data x. The prior information, which is transformed into a prior distribution, can be obtained from objective or subjective sources. Objective prior information includes operating data in different environments, observational data from experiments conducted earlier, and operating experiences with similar equipment. Subjective prior information may include the assessor's personal experiences and judgments, and design information.

In cases when no prior information concerning the parameter θ is available or such information is very limited, non-informative priors are selected. Such choice is based on the "Principle of Insufficient Reason" which was first stated by H. Jeffreys in [Jeffreys, 1961]. This principle requires the distribution on the finitely many events to be uniform unless there is some definite reason to consider one event more probable than another does.

Ordinarily, one deals with cases when only partial prior information concerning the parameter θ is available which is insufficient to specify exactly a prior distribution. In such cases, there is a possibility of incorporation of additional subjective unsuitable information during transformation of the partial prior information to a full prior.

Transformation of the partial prior information to a full prior can be carried out by using various techniques. The most popular techniques use the partial prior information to specify a particular member of a chosen family of distributions. According to this technique, a prior can be obtained by selecting a distribution family for the prior and fitting its parameters based on estimated moments or quantiles of the prior. For example, the Reactor Safety Study assesses upper and lower values of the failures rates of components and then fits a lognormal distribution by using these values as the 95-th and 5-th percentiles of the distribution [Mosleh and Apostolakis, 1982]. This example demonstrates that selection of the lognormal distribution family for the prior incorporates additional subjective information, which does not belong to the prior information, into the full prior distribution.

For transformation of the partial prior information into a full prior which is as non-informative as possible subject to partial prior information, the maximum-entropy method can be used [Jaynes, 1968], [Deeley et al, 1970], [Savchuk and Martz, 1994]. According to this method, the prior probability density function $h(\theta)$ of the parameter θ is selected such, which maximizes the Shannon-Jaynes entropy [Savchuk and Martz, 1994]:

$$I(g) = -\int h(\theta)\ln(h(\theta))d\theta \to \max \tag{5}$$

among all those distributions which satisfy the given constraint:

$$S(h(\theta)) = 0, \tag{6}$$

where $S(h(\theta))$ is functional constraint on $h(\theta)$, which reflects given partial prior information.

Frequently, assessor selects a prior distribution that possesses some useful properties to simplify mathematical inference of Bayesian estimates. Conjugate prior distributions are important class of such distribution functions. A conjugate prior distribution $h(\theta)$ for a given sampling distribution $f(x|\theta)$ is such that the posterior distribution $g(\theta|x)$ and the prior $h(\theta)$ are members of the same family of distributions [Martz and Waller, 1991]. For example, widely used choice of a prior distribution in the Bayesian estimating parameter p in the binomial model is the beta distribution with parameters x_0 and n_0:

$$h(p;x_0,n_0)=\frac{\Gamma(n_0)}{\Gamma(x_0)\Gamma(n_0-x_0)}p^{x_0-1}(1-p)^{n_0-x_0-1},\quad n_0>x_0>0. \tag{7}$$

With such prior probability density function, the posterior probability density function of p given sample data x is [Martz and Waller, 1991]:

$$g(p\,|\,x;x_0,n_0)=\frac{\Gamma(n+n_0)}{\Gamma(x+x_0)\Gamma(n+n_0-x-x_0)}p^{(x+x_0)-1}(1-p)^{(n+n_0-x-x_0)-1} \tag{8}$$

which coincides with the beta probability density function with parameters $x+x_0$, $n+n_0$. Thus, the beta-distribution is the conjugate prior distribution for the binomial failure model.

12.3.2 Bayesian methods of point estimation of reliability parameters

For a quadratic loss function, the Bayesian estimating procedure is to minimize the posterior risk given by

$$E\left[a(\theta-\hat{\theta})^2\,|\,x\right]=\int a(\theta-\hat{\theta})^2 g(\theta\,|\,x)d\theta \tag{9}$$

This risk is minimized when

$$\hat{\theta}=E(\theta\,|\,x)=\int \theta\, g(\theta\,|\,x)d\theta \tag{10}$$

Thus, for a quadratic loss function, the Bayesian point estimator coincides with the posterior mean of the posterior distribution.

If in Bayesian point estimating parameter p for the binomial failure model a beta distribution (7) is used as prior probability density function, and x failures occur when n demands occur, then, the posterior probability density function of p given sample data x is the beta probability density function (8) with parameters $x+x_0$, $n+n_0$, and the Bayesian point estimator, which is the posterior mean, is [Martz and Waller, 1991]:

$$\hat{p}=E(p\,|\,x;x_0,n_0)=\frac{x+x_0}{n+n_0}. \tag{11}$$

12.3.3 Bayesian methods of interval estimation of reliability parameters

In Bayesian estimation, unknown reliability parameter is considered to be a random value ε to which is assigned a prior probability density function $h(\varepsilon)$. For given sample data x the posterior probability density function $g(\varepsilon|x)$ of ε can be obtained. In Bayesian approach, analog of classical two-sided confidence interval is a $(1-\gamma)$ level two-sided Bayes probability interval $(\varepsilon_1,\varepsilon_2)$ for ε given by

$$P\{\varepsilon_1(x) < \varepsilon < \varepsilon_2(x)\} = \int_{\varepsilon_1}^{\varepsilon_2} g(\varepsilon \mid x) d\varepsilon \geq 1-\gamma \tag{12}$$

It can be found by solving the following two equations:

$$\Pr(\varepsilon \leq \varepsilon_1 \mid x) = \int_0^{\varepsilon_1} g(\varepsilon \mid x) d\varepsilon = \frac{\gamma}{2}, \tag{13}$$

and

$$\Pr(\varepsilon \geq \varepsilon_2 \mid x) = \int_{\varepsilon_2}^{1} g(\varepsilon \mid x) d\varepsilon = \frac{\gamma}{2} \tag{14}$$

for the lower limit ε_1 and the upper limit ε_2.

Consider the binomial failure model. Assume that a prior distribution of parameter p is the beta distribution with parameters x_0 and n_0, and x failures occur when n demands occur. Then, the lower limit $\varepsilon_1(x)$ and the upper limit $\varepsilon_2(x)$ of $(1-\gamma)$ level two-sided Bayes probability interval $(\varepsilon_1, \varepsilon_2)$ for parameter p are calculated as follows [Martz and Waller, 1991]:

$$\varepsilon_1(x) = \frac{x + x_0}{x + x_0 + (n + n_0 - x - x_0) F_{1-\gamma/2}(2n + 2n_0 - 2x - 2x_0, 2x + 2x_0).} \tag{15}$$

$$\varepsilon_2(x) = \frac{(x + x_0) F_{1-\gamma/2}(2x + 2x_0, 2n + 2n_0 - 2x - 2x_0)}{n + n_0 - x - x_0 + (x + x_0) F_{1-\gamma/2}(2x + 2x_0, 2n + 2n_0 - 2x - 2x_0).} \tag{16}$$

where $F_{1-\gamma/2}(n_1, n_2)$ is the $(1-\gamma/2)$-quantile of the $F-$distribution with (n_1, n_2) degrees of freedom.

12.3.4 Methods of fitting parameters of a prior distribution function

When the prior distribution is assumed a member of a certain parametric family of distributions the problem of fitting its parameters arises. It is unlikely that expert judgment can be quantified in terms of statement, which gives directly values of parameters of the prior distribution. However, the assessors are able to give more or less accurate values of quantiles and moments of the prior distribution in quantifying their beliefs. Frequently these characteristics of the prior distribution are used to specify a particular member of a chosen family of distributions. This can be easily done when the characteristics and the parameters of the distribution are related through a simple analytic relation [Martz and Waller, 1991].

Unfortunately, there are no simple analytic relations between the quantiles and parameters of the gamma and beta distributions, which are widely used as the prior in Bayesian estimation of reliability parameters [Martz and Waller, 1991]. For this reason graphical techniques [Waller et al, 1977], [Weiler, 1965] or tables of the incomplete gamma or beta functions [Pearson, 1957], [Pearson, 1968] are used. The existing graphical techniques cover a very limited range of quantiles or the parameters. Tables for determining the beta prior are presented in [Waterman et al, 1976] for cases when 1) the mean and 5th quantile or 2) the mean and the 95th quantile are given. The paper [Mosleh and Apostolakis, 1982] presents techniques that facilitate estimation of the parameters of a gamma or beta distribution when two percentiles or one percentile and the mean value are known. The problem of determination of the parameters of a beta distribution for the same types of available prior information is also considered in [Duran and Booker, 1988].

When there is observational data from past experiments, it can be used to estimate parameters of the prior distribution, which is assumed a member of a certain parametric family of distributions. Two basic methods are usually used for this purpose: the method of matching moments and method of maximum likelihood based on the marginal distribution [Martz and Waller, 1991]. The method of moments is to equate the sample moments to their expected values and solve for the parameters. The method of maximum likelihood chooses estimators that maximize the marginal likelihood function.

Consider estimators for parameters x_0 and n_0 of a beta distribution. Assume that we have results obtained in N binomial life test experiments conducted previously with the same or similar items. Let p_j denote the unknown value of the parameter p in j th experiment. We assume that the underlying sequence $p_1, p_2, \ldots, p_N$ is statistically independent realizations of a random value, which has beta distribution with constant x_0 and n_0 values for the sequence of tests [Martz and Waller, 1991]. Let for the j th experiment x_j failures occur out of sample size n_j. The classical estimator for the parameter p in the j th experiment is $\hat{p}_j = x_j / n_j$. Then, the unweighted moment estimators for n_0, x_0 are [Martz and Waller, 1991]:

$$\hat{n}_0 = \frac{N(\overline{p}_u - m_u^2)}{Nm_u^2 - K\overline{p}_u - (N-K)\overline{p}_u^2}, \qquad \hat{x}_0 = \hat{n}_0 \overline{p}_u, \tag{17}$$

where

$$\overline{p}_u = \sum_{j=1}^{N} \frac{\hat{p}_j}{N}, \qquad m_u^2 = \sum_{j=1}^{N} \frac{\hat{p}_j^2}{N}, \qquad K = \sum_{j=1}^{N} n_j^{-1}. \tag{18}$$

The marginal maximum likelihood estimates for parameters n_0, x_0 can be obtained by numerical solving the following system of equations [Martz and Waller, 1991]:

$$\begin{aligned} &\sum_{j=1}^{N} \sum_{i=0}^{x_j-1} \left(\frac{1}{x_0 + i} \right) - \sum_{j=1}^{N} \sum_{i=0}^{n_j - x_j - 1} \left(\frac{1}{n_0 - x_0 + i} \right) = 0, \\ &\sum_{j=1}^{N} \sum_{i=0}^{n_j - x_j - 1} \left(\frac{1}{n_0 - x_0 + i} \right) - \sum_{j=1}^{N} \sum_{i=0}^{x_j-1} \left(\frac{1}{x_0 + i} \right) = 0. \end{aligned} \tag{19}$$

12.3.5 Empirical Bayesian point estimation of reliability parameters

In order to avoid the necessity of identifying the unknown prior distribution empirical Bayesian estimation was suggested in [Robbins, 1955]. The difference between empirical Bayesian and ordinary Bayesian is that the former does not make explicit the form of the prior distribution in order to make possible a Bayesian solution. Instead, the empirical Bayesian method depends on existence of prior information in the form of past estimates of either parameter in question or some close variation of it [Tillman et al, 1982], [Tsocos and Canavos, 1972].

Empirical Bayesian procedures consists of 1) methods that attempt to approximate the Bayesian estimator without explicitly estimating the unknown prior distribution [Clemmer and Krutchkoff, 1968], [Lin, 1972], [Martz and Krutchkoff, 1969], [Miyawasa, 1961], [Nichols and Tsokos, 1972], and 2) methods in which the unknown prior distribution is explicitly estimated [Bennet and Martz, 1972], [Lemon and Krutchkoff, 1969], [Maritz, 1966; 1970].

One of the widely used empirical Bayesian estimators is the Copas estimator [Copas, 1972]. Consider a sequence of binomial sampling experiments in which the j th experiment results in x_j failures in n_j trials [Martz and Waller, 1991]. Let $y_j = x_j / n_j$. It is assumed that N-1 previous experiments have been conducted prior to the current or N-th experiment. For convenience let $x = x_N$, $n = n_N$, $y = x / n$ and $p = p_N$. Let μ and σ^2 denote the prior mean and variance, respectively. Then, the Copas estimator [Copas, 1972] is given by

$$\widehat{p}_C = (y + W_N \mu_{1N}) / (1 + W_N), \tag{20}$$

where

$$\mu_{1N} = \sum_{i=1}^{N} y_i / N,\ W_N = [\mu_{1N}(1-\mu_{1N}) - \sigma_{1N}^2] / n\sigma_{1N}^2, \tag{21}$$

$$\sigma_{1N}^2 = \max[0, \{\sum_{i=1}^{N}(y_i - \mu_{1N})^2 - k\mu_{1N}(1-\mu_{1N})\}/(N-k)],\ k = \sum_{i=1}^{N} n_i^{-1} \tag{22}$$

Another empirical Bayesian estimator of binomial parameter is the Lemon and Krutchkoff estimator [Lemon and Krutchkoff, 1969]:

$$\widehat{p}_L = [\sum_{i=1}^{N} \widetilde{p}_i^{x+1}(1-\widetilde{p}_i)^{n-x}] / [\sum_{i=1}^{N} \widetilde{p}_i^{x}(1-\widetilde{p}_i)^{n-x}], \tag{23}$$

where $\widetilde{p}_i$ is given by

$$\widetilde{p}_i = [\sum_{j=1}^{N} y_j^{x_i+1}(1-y_j)^{n_i-x_i}] / [\sum_{j=1}^{N} y_j^{x_i}(1-y_j)^{n_i-x_i}], \quad N \ge 2, \tag{24}$$

and

$$y_j = x_j / n_j. \tag{25}$$

12.3.6 Basic shortcomings of Bayesian and empirical Bayesian estimation of reliability parameters

A major shortcoming of the Bayesian approach is related to the ability of the investigator to select a true distribution function. In cases when no prior information concerning the parameter θ is available, the more appropriate selection is a non-informative prior distribution or the distribution, which maximizes the Shannon-Jaynes entropy [Jaynes, 1968], [Deeley et al, 1970]. Difficulties of implementation of the Bayesian procedures of estimation arise when the partial prior information is available.

Most techniques of identification of a prior distribution considered above consist of two stages. At the first stage, the family of distributions is selected. At the second stage, a particular member of this family is specified by means of estimating its parameters based on the available prior information. The second stage can be considered as justified and formalized since it is based on the available prior information. In contrast to the second stage, choice of the family of distribution functions on the first stage depends on a subjective factor. At this stage, additional subjective information can be introduced into decision procedure, which may not belong to the available prior information.

This shortcoming of the Bayesian approach is usually overcome by using empirical Bayesian decision process that does not require accurately estimate a prior distribution function. Empirical Bayesian decision procedure is an efficient tool for combining existing sets of either reliability data or reliability parameter estimates from various sources. One of the advantages of such methods is their asymptotic optimality. However, the rate of convergence of the empirical Bayesian risk to the minimum Bayesian risk can be quite slow. Therefore, in cases when small datasets of reliability data or reliability parameter estimates are available, the accuracy of the approximation of the Bayesian estimator by the empirical Bayesian estimators is never really known.

Errors in Bayesian estimating reliability parameters resulted from incorrect selecting a prior distribution can be significant and affect the confidence to results of a safety investigation of ecologically dangerous objects.

12.4 Statement of the problem of robust estimating reliability parameters

One of the ways to avoid arbitrariness in selecting prior distribution based on partially available prior information is to use Γ-minimax approach that was originally proposed by Robbins [Robbins, 1951] and was investigated by Berger [Berger, 1984]. The Γ-minimax approach takes into account that Bayesian procedures actually deal with a family of priors Γ instead of a single prior distribution. Any distribution from the class Γ has the same reasons to be used as a prior distribution. The uncertainty in the priors generates uncertainty in the posterior values of interest (e.g. mean, variance) described by the range of the posterior values as the prior varies in Γ.

This range is used as a measure of robustness [Vidakovic, 2000], [Ruggeri and Sivaganesan, 2000], [Carota and Ruggeri, 1994]. A small range means that uncertainty in the priors does not heavily affect the posterior value, and it is robust with respect to the choice of the prior from the class Γ. Conversely, a large range means that the posterior value is sensitive to the choice of the prior, i.e. it is not robust. In the latter case, the statistician should pay attention to the least favorable prior in Γ and use Γ-minimax approach to develop robust estimates.

In the Γ-minimax approach, the estimate that minimizes the supremum of the cost functional over distributions in Γ is selected as a robust [Vidakovic, 2000]. In case of scarce prior information, the class Γ is large and result obtained by using this approach is close to the result in minimax approach. In other case when complete prior information is available, the class Γ contains a single prior and the Γ-minimax approach is equivalent to the ordinary Bayesian approach.

The natural choices of the cost functional are

1. Bayes risk

$$r_H(\delta(x)) = \int_{\Theta}\int_{X} L(\theta, \delta(x)) f(x \mid \theta) dx dH(\theta) \tag{26}$$

2. posterior risk

$$\phi_H(\delta(x)) = \int_\Theta L(\theta,\delta(x))dG(\theta \mid x) = \frac{\int_\Theta L(\theta,\delta) f(x \mid \theta) dH(\theta)}{\int_\Theta f(x \mid \theta) dH(\theta)} \tag{27}$$

3. posterior mean of parameter θ given sample data x

$$\hat{\theta}_H = \frac{\int_\Theta \theta\, f(x \mid \theta) dH(\theta)}{\int_\Theta f(x \mid \theta) dH(\theta)} \tag{28}$$

From the point of view of the statistician a prior distribution, which maximizes Bayes (26) risk or posterior risk (27), is the least favorable prior in Γ since it corresponds to the worst estimation accuracy. For any other prior from class Γ the estimation accuracy can be only better.

According to conservative approach it is naturally to assume that the "true" prior always coincides with the least favorable prior in Γ. Therefore, the estimator that minimizes Bayes risk or posterior risk under the least favorable prior is robust.

The cost functional (28) corresponds to a Bayesian estimator when a squared-error loss function is used in Bayesian inference. Calculating values of Bayes estimates $\hat{\theta}_H$ for any prior $H \in \Gamma$ we obtain the range of possible values (θ_*, θ^*). Suppose that upper bound θ^* corresponds to some distribution $H^* \in \Gamma$, i.e. $\theta^* = \hat{\theta}_{H^*}$. From the point of view of risk theory, the conservative assumption concerning a prior is that H^* is the "true" prior. Use of any other Bayesian estimate $\hat{\theta}_H \in (\theta_*, \theta^*)$, which corresponds to any other distribution $H \in \Gamma$, can only diminish the assessment of failure probability and thus promote obtaining more optimistic result of probabilistic safety assessment. For this reason, the upper bound θ^* of the range of possible values (θ_*, θ^*) of posterior mean is appropriate for utilization as a robust Bayesian estimate in estimating reliability parameters for probabilistic safety assessment.

If the cost functional is posterior risk then, the problem of seeking robust Bayesian estimate given sample data x is reduced to the following stochastic minimax problem:

$$\min_\delta \sup_{H\in\Gamma} \phi_H(\delta, x) = \min_\delta \sup_{H\in\Gamma} \frac{\int_\Theta L(\theta,\delta) f(x \mid \theta) dH(\theta)}{\int_\Theta f(x \mid \theta) dH(\theta)} \tag{29}$$

The objective functional in (29) is the linear-fractional functional with respect to distribution functions. The inner problem in (29) is the problem of optimization of the linear-fractional functional in the space of distribution functions that belong to class Γ.

If the cost functional is posterior mean of parameter θ given sample data x then, the problem of seeking robust Bayesian estimate is reduced to the following stochastic programming problem of maximization of linear-fractional objective functional over distribution functions from class Γ:

$$\sup_{H\in\Gamma}\widehat{\theta}_H = \sup_{H\in\Gamma}\frac{\int_{\Theta}\theta\, f(x\mid\theta)dH(\theta)}{\int_{\Theta} f(x\mid\theta)dH(\theta)} \qquad (30)$$

12.5 Statement of the problem of seeking lower and upper bounds for Bayesian estimates of reliability parameters

In order to investigate sensitivity of Bayesian estimates with respect to selection of a prior distribution from class Γ it is necessary to find low and upper bounds for the range of possible values of the cost functional of interest. According to the Bayesian approach, if $H(\theta)$ is the "true" prior, the quality of the Bayesian point estimator $\widehat{\theta}_H(x)$ is measured in terms of the Bayesian risk $r(\hat{\theta}_H(x))$ (26), or the posterior risk $\phi_H(\widehat{\theta}_H(x))$ (27). Now consider the case when instead of a single prior $H(\theta)$, class Γ of such distribution functions is available. In this case the value $r_H(\widehat{\theta}_H(x))$ or $\phi_H(\widehat{\theta}_H(x))$ cannot be used for characterization of the quality of the Bayesian point estimator $\widehat{\theta}_H(x)$. For this purpose the range $(r_*(\widehat{\theta}_H(x)), r^*(\widehat{\theta}_H(x)))$ of possible values of the Bayesian risk or the range $(\phi_*(\widehat{\theta}_H(x)), \phi^*(\widehat{\theta}_H(x)))$ of possible values of the posterior risk, obtained by varying prior in Γ, are more appropriate.

In estimating reliability parameters for probabilistic safety assessment it is also of interest to know how wide is the range (θ_*, θ^*) of possible values of posterior mean of parameter θ given sample data x (Bayesian estimate), as the prior varies in Γ.

The problem of calculating low $r_*(\widehat{\theta}_H(x))$ or upper $r^*(\widehat{\theta}_H(x))$ bounds for the range $(r_*(\widehat{\theta}_H(x)), r^*(\widehat{\theta}_H(x)))$ of possible values of the Bayesian risk associated with the Bayesian point estimator $\widehat{\theta}_H(x)$ is reduced to the following optimization problems in the space of distribution functions:

$$r_*(\widehat{\theta}_H(x)) = \inf_{G\in\Gamma}\int_{\Theta}\int_{X} L(\theta,\widehat{\theta}_H(x)))f(x\mid\theta)dxdG(\theta) \qquad (31)$$

and

$$r^*(\widehat{\theta}_H(x)) = \sup_{G\in\Gamma}\int_{\Theta}\int_{X} L(\theta,\widehat{\theta}_H(x)))f(x\mid\theta)dxdG(\theta) \qquad (32)$$

In (31), (32) objective functionals are linear with respect to distribution functions.

The problem of calculating lower $\phi_*(\widehat{\theta}_H(x))$ or upper $\phi^*(\widehat{\theta}_H(x))$ bounds for the range $(\phi_*(\widehat{\theta}_H(x)), \phi^*(\widehat{\theta}_H(x)))$ of possible values of the posterior risk associated with the Bayesian point estimator $\widehat{\theta}_H(x)$ is reduced to the following optimization problems in the space of distribution functions:

$$\phi_*(\widehat{\theta}_H(x)) = \inf_{G\in\Gamma} \frac{\int_\Theta L(\theta,\widehat{\theta}_H(x)) f(x\mid\theta)dG(\theta)}{\int_\Theta f(x\mid\theta)dG(\theta)} \tag{33}$$

and

$$\phi^*(\widehat{\theta}_H(x)) = \sup_{G\in\Gamma} \frac{\int_\Theta L(\theta,\widehat{\theta}_H(x)) f(x\mid\theta)dG(\theta)}{\int_\Theta f(x\mid\theta)dG(\theta)} \tag{34}$$

In (33) and (34) objective functionals are linear-fractional with respect to distribution functions.

The problem of calculating lower θ_* or upper θ^* bounds of the range (θ_*,θ^*) of possible values of posterior mean of parameter θ given sample data x (Bayesian estimate), is reduced to the following minimization or maximization of linear-fractional objective functional (posterior mean) over distribution functions from class Γ :

$$\theta_* = \inf_{G\in\Gamma} \frac{\int_\Theta \theta\, f(x\mid\theta)dG(\theta)}{\int_\Theta f(x\mid\theta)dG(\theta)} \tag{35}$$

and

$$\theta^* = \sup_{G\in\Gamma} \frac{\int_\Theta \theta\, f(x\mid\theta)dG(\theta)}{\int_\Theta f(x\mid\theta)dG(\theta)} \tag{36}$$

Now consider the problem of seeking two-sided Bayes probability interval for the parameter θ. We assume that the parameter θ is one-dimensional and $\theta\in[a,b]$. In accordance with the Bayesian approach (13), (14), symmetric $(1-\gamma)$ level two-sided Bayes probability interval for the parameter θ can be calculated by solving the following two equations:

$$\Pr(\theta \le \theta_* \mid x) = \int_a^{\theta_*} dG(\theta\mid x) = \frac{\int_a^{\theta_*} f(x\mid\theta)dH(\theta)}{\int_a^b f(x\mid\theta)dH(\theta)} = \frac{\gamma}{2} \tag{37}$$

and

$$\Pr(\theta \ge \theta^* \mid x) = \int_{\theta^*}^{b} dG(\theta\mid x) = \frac{\int_{\theta^*}^{b} f(x\mid\theta)dH(\theta)}{\int_a^b f(x\mid\theta)dH(\theta)} = \frac{\gamma}{2} \tag{38}$$

for the lower limit θ_* and the upper limit θ^*. Thus

$$\Pr(\theta_* \le \hat{\theta} \le \theta^* \mid x) = 1-\gamma .$$

Now consider the case when available prior information is insufficient to accurately specify a single distribution function. Instead of a single prior $H(\theta)$ it defines the class Γ of such distribution functions, use of any of which as a prior does not contradict to available prior information. In this case, if for any distribution $H(\theta)\in\Gamma$ one determines symmetric $(1-\gamma)$ level two-sided Bayesian probability interval $(\theta_*(H),\theta^*(H))$ by solving equations (37), (38), then, there is no warranty that unknown parameter belong to it with the probability $1-\gamma$. In this case the generalized symmetric $(1-\gamma)$ level two-sided Bayes probability interval (θ_*,θ^*) for the parameter θ can be determined by solving the following optimization problems:

$$\theta_* \to \sup \tag{39}$$

subject to

$$\int_a^{\theta_*} dG(\theta \mid x) = \frac{\int_a^{\theta_*} f(x\mid\theta)dH(\theta)}{\int_a^{b} f(x\mid\theta)dH(\theta)} \le \frac{\gamma}{2} \tag{40}$$

for any $H(\theta)\in\Gamma$,

and

$$\theta^* \to \inf \tag{41}$$

subject to

$$\int_{\theta^*}^{b} dG(\theta \mid x) = \frac{\int_{\theta^*}^{b} f(x\mid\theta)dH(\theta)}{\int_a^{b} f(x\mid\theta)dH(\theta)} \le \frac{\gamma}{2} \tag{42}$$

for any $H(\theta)\in\Gamma$.

The problems (39)-(40) and (41)-(42) can be expressed in the following equivalent form:

$$\theta_* \to \sup \tag{43}$$

subject to

$$\sup_{H\in\Gamma} \frac{\int_a^{\theta_*} f(x\mid\theta)dH(\theta)}{\int_a^{b} f(x\mid\theta)dH(\theta)} \le \frac{\gamma}{2} \tag{44}$$

and

$$\theta^* \to \inf \tag{45}$$

subject to

$$\inf_{H\in\Gamma} \frac{\int_a^{\theta^*} f(x\mid\theta)dH(\theta)}{\int_a^b f(x\mid\theta)dH(\theta)} \geq 1-\frac{\gamma}{2} \quad (46)$$

12.6 Classes of distributions

Methods of solution the problems that were stated in sections (3), (4) depend on types of classes Γ of distributions. Commonly the following classes Γ are used in applications [Berger, 1994].

1. class of distributions of given shape, for example,
   Γ = {all symmetric, unimodal priors};
2. class of distributions that have given system of quantiles
   $\Gamma=\{H(\theta):\alpha_i \leq \Pr(\theta\in\Theta_i)\leq\beta_i,\ i=1,\ldots,m\}$;
3. class of distributions that have given system of moments;
4. contamination class
   $\Gamma=\{H=(1-\varepsilon)H_0+\varepsilon F,\ F\in Q\}$,
   where H is a base prior, Q – is the allowed class of contaminations;
5 mixture classes
   $\Gamma=\{g(\theta)=\int g(\theta\mid\alpha)dG(\alpha),\ G\in Q\}$

In our investigations, we consider the class of distribution functions that can be expressed by means of the following linear constraints:

$$\int_\Theta dH(\theta)=1 \quad (47)$$

$$\begin{aligned}&\int_\Theta f_i(\theta)dH(\theta)\leq d_i,\\ &i=1,2,\ldots,m_1,\\ &\int_\Theta g_j(\theta)dH(\theta)=\mu_j,\\ &j=1,\ldots,m_2,\end{aligned} \quad (48)$$

where functions $f_i(\theta),\ i=1,2,\ldots,m_1,\ g_j(\theta),\ j=1,2,\ldots,m_2,$ can be discontinuous.

In case when parametric space is one-dimensional, $\Theta=[a,b]$, and functions $g_j(\theta)=\theta^j$, $j=1,2,\ldots,m_2$, this class consists of all distribution functions that have given constraints on the first m moments. If $f_i(\theta),\ i=1,2,\ldots,m$, are indication functions for some intervals of the parametric space $\Theta=[a,b]$, then this class consists of all distribution functions that have given constraints on the quantiles.

12.7 Discussion

The approach considered in this paper overcomes one of the shortcomings in the modern methodology of evaluation of Severe Accident Risks on Nuclear Power Plants. Particularly, it provides techniques for quantification of uncertainty in estimates of reliability parameters resulted from insufficient data.

This approach was used in [Golodnikov et al, 2009], [Golodnikov, 2009] for conducting sensitivity analysis of Bayesian estimates of reliability parameters with respect to selection of a prior distribution in two widely used failure models: exponential and binomial.

For selected data we calculated lower and upper bounds for Bayesian estimates of Centrifugal pump failure rate and Isolation valves failure probability per demand assuming that only two quantiles are known about prior distribution.

Basing on results obtained we drew the following conclusions.

1. In the case of Centrifugal pump failure rate estimation (exponential model), Bayesian estimates of failure rates are essentially sensitive to the selection of a prior distribution function.
2. In the case of the Isolation valves failure probability per demand estimation (binomial model), strength of sensitivity of Bayesian estimates of failure probability to selection prior distribution depends on the number of failures occurred. When number of failures was less or equal 1 we observed slight sensitivity. Therefore, in this case any distribution, which has the same two quantiles, may be used as a prior. When number of failures exceeded 1, we observed rapid increasing of the sensitivity with the increase of number of failures.