DISTANCES ON ANTIMATROIDS

Yulia Kempner, Vadim E. Levit

Abstract: *An antimatroid is an accessible set system $(U, \mathcal{F})$ closed under union. Every antimatroid may be represented as a graph whose vertices are sets of $\mathcal{F}$, where two vertices are adjacent if the corresponding sets are differ by one element. This graph is a partial cube. Hence an antimatroid with the ground set U of size n may be isometrically embedded into the hypercube $\{0,1\}^n$. Thus the distance on an antimatroid considered as a graph coincides with the Hamming distance. A poset antimatroid is an antimatroid, which is formed by the lower sets of a poset. We consider different definitions of the distance between elements of an antimatroid, and give a new characterization of poset antimatroids.*

Keywords: *antimatroid, partial cube, zigzag distance, Hamming distance.*

ACM Classification Keywords: *G.2*

MSC: *05C12; 52B40*

Introduction

An antimatroid is an accessible set system closed under union. There are two equivalent definitions of antimatroids, one as set systems and the other as languages [Korte et al., 1991]. An algorithmic characterization of antimatroids based on the language definition was introduced in [Boyd & Faigle, 1990]. Later, another algorithmic characterization of antimatroids which depicted them as set systems was developed in [Kempner & Levit, 2003]. Antimatroids can be viewed as a special case of either greedoids or semimodular lattices, and as a generalization of partial orders and distributive lattices. While classical examples of antimatroids connect them with posets, chordal graphs, convex geometries etc., [Glasserman & Yao, 1994] used antimatroids to model the ordering of events in discrete event simulation systems. In mathematical psychology, antimatroids are used to describe feasible states of knowledge of a human learner [Eppstein et al., 2008]. There are also rich connections between antimatroids and cluster analysis [Kempner & Muchnik, 2003].

Let U be a finite set. A *set system* over U is a pair $(U, \mathcal{F})$, where $\mathcal{F}$ is a family of sets over U, called *feasible* sets.

Definition 1. *[Korte et al., 1991] A finite non-empty set system $(U, \mathcal{F})$ is an antimatroid if $(A1)$ for each non-empty $X \in \mathcal{F}$, there exists $x \in X$ such that $X - x \in \mathcal{F}$*

$(A2)$ for all $X, Y \in \mathcal{F}$, and $X \nsubseteq Y$, there exists $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

Any set system satisfying $(A1)$ is called *accessible.*

Proposition 2. *[Korte et al., 1991] For an accessible set system $(U, \mathcal{F})$ the following statements are equivalent:*

(i) $(U, \mathcal{F})$ is an antimatroid

(ii) $\mathcal{F}$ is closed under union ($X, Y \in \mathcal{F} \Rightarrow X \cup Y \in \mathcal{F}$)

A set system $(U, \mathcal{F})$ satisfies the *chain property* [Kempner & Levit, 2010] if for all $X, Y \in \mathcal{F}$, and $X \subset Y$, there exists a chain $X = X_0 \subset X_1 \subset ... \subset X_k = Y$ such that $X_i = X_{i-1} \cup x_i$ and $X_i \in \mathcal{F}$ for $0 \leq i \leq k$.

It is easy to see that this chain property follows from $(A2)$, but these properties are not equivalent. Examples of chain systems include antimatroids, convex geometries, matroids and other hereditary systems (matchings, cliques, independent sets, etc.).

Definition 3. *[Korte et al., 1991] The set system $(U, \mathcal{F})$ is a poset antimatroid if U is the set of elements of a finite partially ordered set (poset) P and $\mathcal{F}$ is a family of lower sets of P. The maximal chains in the corresponding poset antimatroid are the linear extension of P.*

A game theory gives a framework, in which poset antimatroids are considered as permission structures for coalitions [Algaba et al., 2004]. The poset antimatroids can be characterized as the unique antimatroids which are closed under intersection [Korte et al., 1991]. The feasible sets in a poset antimatroid ordered by inclusion form a distributive lattice, and any distributive lattice can be built in this way. Thus, antimatroids can be seen as generalizations of distributive lattices.

Distance on graphs and antimatroids

Definition 4. *For each graph $G = (V, E)$ the distance $d_G(u, v)$ between two vertices $u, v \in V$ is defined as the length of a shortest path joining them.*

Definition 5. *If G and H are arbitrary graphs, then a mapping $f : V(G) \to V(H)$ is an isometric embedding if $d_H(f(u), f(v)) = d_G(u, v)$ for any $u, v \in V(G)$.*

Let $U = \{x_1, x_2, ...x_n\}$. Define a graph $H(U)$ as follows: the vertices are the finite subsets of U, two vertices A and B are adjacent if and only if the symmetric difference $A \bigtriangleup B$ is a singleton set. Then $H(U)$ is the *hypercube Q_n on U* [Djokovic,1973]. The hypercube can be equivalently defined as the graph on $\{0, 1\}^n$ in which two vertices form an edge if and only if they differ in exactly one position. The shortest path distance $d_H(A, B)$ on the hypercube $H(U)$ is the Hamming distance between A and B that coincides with the symmetric difference distance: $d_H(A, B) = |A \bigtriangleup B|$. A graph G is called a *partial cube* if it can be isometrically embedded into a hypercube $H(U)$ for some set U.

Definition 6. *[Doignon & Falmagne, 1997] A family of sets $\mathcal{F}$ is well-graded if any two sets $P, Q \in \mathcal{F}$ can be connected by a sequence of sets $P = R_0, R_1, ..., R_n = Q$ formed by single-element insertions and deletions ($|R_i \bigtriangleup R_{i+1}| = 1$), such that all intermediate sets in the sequence belong to $\mathcal{F}$ and $|P \bigtriangleup Q| = n$.*

Any set system $(U, \mathcal{F})$ defines a graph $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$, where $E_{\mathcal{F}} = \{\{P, Q\} \in \mathcal{F} : |P \bigtriangleup Q| = 1\}$.
Since a family $\mathcal{F}$ of every antimatroid $(U, \mathcal{F})$ is well-graded, each antimatroid is a partial cube ([Ovchinnikov, 2008]) and may be represented as a graph $G_{\mathcal{F}}$ that is a subgraph of the hypercube $H(U)$. Thus the distance on an antimatroid $(U, \mathcal{F})$ considered as a graph coincides with the Hamming distance between sets, i.e. $d_{\mathcal{F}}(A, B) = |A \bigtriangleup B|$ for any $A, B \in \mathcal{F}$.

Poset antimatroids and zigzag distance

For an antimatroid $(U, \mathcal{F})$ denote $C_k = \{X \in \mathcal{F} : |X| = k\}$ a family of feasible sets of cardinality k. A *lower zigzag* is a sequence of feasible sets $P_0, P_1, ..., P_{2m}$ such that any two consecutive sets in the sequence differ by a single element and $P_{2i} \in C_k$, and $P_{2i-1} \in C_{k-1}$ for all $0 \leq i \leq m$. In the same way we define an *upper zigzag* in which $P_{2i-1} \in C_{k+1}$. Each zigzag $P_0, P_1, ..., P_{2m}$ is a path connecting P_0 and P_{2m}, and so the distance on the zigzag $d(P_0, P_{2m}) = 2m$ is always no less than the distance $d_{\mathcal{F}}(P_0, P_{2m})$ on an antimatroid $(U, \mathcal{F})$.
Figure 1(a) shows two sets ($A = \{1, 2, 3, 5\}$ and $B = \{1, 3, 4, 5\}$) that are connected by a lower zigzag, such that the distance on the zigzag is 4, while $|A \bigtriangleup B| = 2$. Note, that the distance on the upper zigzag is indeed 2. For two sets $X = \{1, 2, 5\}$ and $B = \{3, 4, 5\}$ the distance on the lower zigzag and on the upper zigzag is equal to 6, while $|X \bigtriangleup Y| = 4$. In order that the distance on zigzags be equal to the distance on an antimatroid, the antimatroid have to be poset antimatroid. This property gives a new characterization of poset antimatroids.

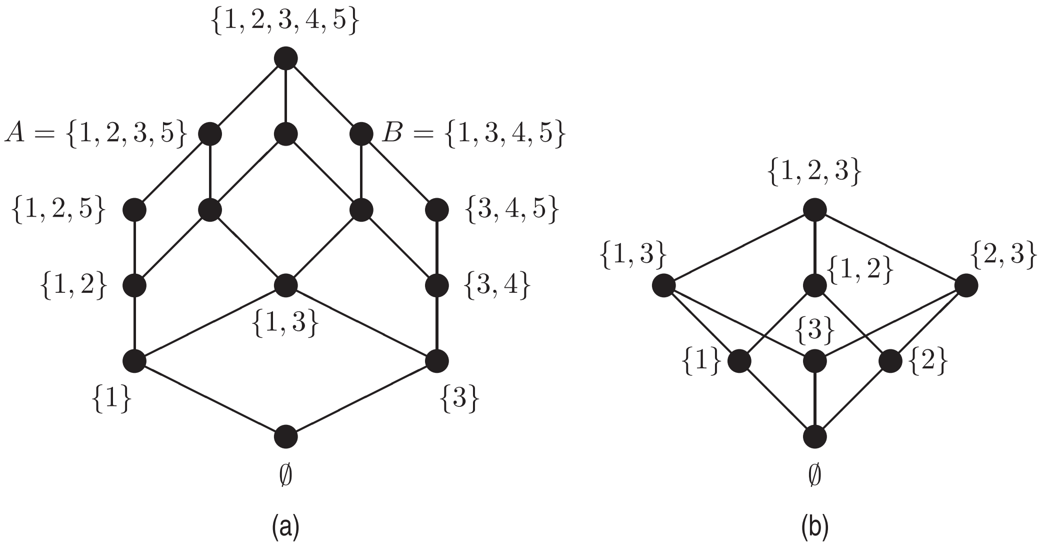

Figure 1: (a) An antimatroid without distance preserving zigzags and (b) a poset antimatroid without total distance preserving zigzags.

Theorem 7. *An antimatroid $(U, \mathcal{F})$ is a poset antimatroid if and only if every two feasible sets A, B of the same cardinality k can be connected by a lower and by an upper zigzags such that the distance between these sets $d_{\mathcal{F}}(A, B)$ coincides with the distance on the zigzags.*

Proof. The proof of the sufficiency may be found in [Kempner & Levit, 2012]. To prove the necessity we show that the antimatroid $(U, \mathcal{F})$ is closed under intersection, i.e., for each $A, B \in \mathcal{F}$ the set $A \cap B \in \mathcal{F}$. If $A \subseteq B$ or $B \subseteq A$ the statement is obvious. So we consider only incomparable sets. We use induction on $d_{\mathcal{F}}(A, B)$.

If $d_{\mathcal{F}}(A, B) = 1$ then the sets are comparable, so we begin from $d_{\mathcal{F}}(A, B) = 2$. Since the sets are incomparable, we have $A = (A \cap B) \cup a$ and $B = (A \cap B) \cup b$. So the lower distance preserving zigzag connecting A and B must go via $A \cap B$, i.e., $A \cap B \in \mathcal{F}$.

Let $d_{\mathcal{F}}(A, B) = m$. If $|A| = |B|$ then there is a distance preserved lower zigzag connecting A with B. Hence there is $a \in A - B$ such that $A - a$ belongs to the zigzag, and $b \in B - A$ with $B - b$ on the zigzag, such that $d_{\mathcal{F}}(A - a, B - b) = |A \triangle B| - 2 = m - 2$. By the induction hypothesis $(A - a) \cap (B - b) = A \cap B \in \mathcal{F}$.

Let $|A| < |B|$. The definition of an antimatroid $(A2)$ implies that there exists $b \in B - A$ such that $A \cup b \in \mathcal{F}$. Since $d_{\mathcal{F}}(A \cup b, B) = d_{\mathcal{F}}(A, B) - 1$, by the induction hypothesis $(A \cup b) \cap B = (A \cap B) \cup b \in \mathcal{F}$. Since $A \nsubseteq B$ and $|A| < |B|$, then $1 \leq |A - B| < |B - A|$. Hence $d_{\mathcal{F}}(A, (A \cap B) \cup b) = |A - (A \cap B)| + 1 = |A - B| + 1 < |A - B| + |B - A| = d_{\mathcal{F}}(A, B)$. By the induction hypothesis $A \cap ((A \cap B) \cup b) = A \cap B \in \mathcal{F}$. □

Note that if for a zigzag $P_0, P_1, ..., P_{2m}$ the distance on the zigzag $d(P_0, P_{2m}) = 2m = d_{\mathcal{F}}(P_0, P_{2m}) = |P_0 \triangle P_{2m}|$ then the zigzag preserves the distance for each pair P_i, P_j, i.e., $d_{\mathcal{F}}(P_i, P_j) = d(P_i, P_j) = |j - i|$.

For poset antimatroids there are distance preserving zigzags connecting two given sets, but these zigzags are not obliged to connect all feasible sets of the same cardinality. In Figure 1(b) we can see that there is a poset antimatroids, for which it is not possible to build a distance preserving zigzag connecting all feasible sets of the same cardinality. To characterize the antimatroids with total distance preserving zigzag we introduce the following definitions.

Each antimatroid $(U, \mathcal{F})$ may be considered as a directed graph $G = (V, E)$ with $V = \mathcal{F}$ and $(A, B) \in E \Leftrightarrow \exists c \in B$ such that $A = B - c$. Denote *in-degree* of the vertex A as $deg_{in}(A) = |\{c : A - c \in \mathcal{F}\}|$, and *out-degree* as $deg_{out}(A) = |\{c : A \cup c \in \mathcal{F}\}|$. Consider antimatroids for which their maximum in-degree and maximum out-degree is at most p, and there is at least one feasible set for which in-degree or out-degree equals p. We will call such antimatroids *p-antimatroids*.

Theorem 8. *In p-antimatroid $(U, \mathcal{F})$ all feasible sets C_k of the same cardinality k can be connected by a lower zigzag and by an upper zigzag such that the distance between any two sets P_i, P_j in each zigzag coincides with distance on the zigzag $d(P_i, P_j) = |j - i|$ if and only if $p < 3$.*

The proof of the sufficiency may be found in [Kempner & Levit, 2012]. To prove the necessity let $(U, \mathcal{F})$ be a p-antimatroid with $p > 2$. Let the out-degree of some A be equal to p. Then there are some $a, b, c \in U$ such that $A \cup a, A \cup b, A \cup c \in \mathcal{F}$. So there is a sub-graph of $G_{\mathcal{F}}$ isomorphic to a cube (see Figure 1(b)), and hence it is not possible to build a distance preserving zigzag connecting all feasible sets of the same cardinality $k = |A| + 1$.

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Authors' Information

Yulia Kempner *- Department of Computer Science, Holon Institute of Technology, P.O. Box: 58102, Holon, Israel; e-mail: yuliak@hit.ac.il*
Major Fields of Scientific Research: Combinatorial Optimization

Vadim E. Levit *- Department of Computer Science and Mathematics, Ariel University Center of Samaria, Israel; e-mail: levitv@ariel.ac.il*
Major Fields of Scientific Research: Graph Theory