AUTOMATIC CONTROL SYSTEMS

POSITIVE STABLE REALIZATIONS OF CONTINUOUS-TIME LINEAR SYSTEMS

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Abstract: *The problem for existence and determination of the set of positive asymptotically stable realizations of a proper transfer function of linear continuous-time systems is formulated and solved. Necessary and sufficient conditions for existence of the set of the realizations are established. Procedure for computation of the set of realizations are proposed and illustrated by numerical examples.*

Keywords: *positive, stable, realization, existence, procedure, linear, continuous-time, system.*

Introduction

Determination of the state space equations for given transfer matrix is a classical problem, called realization problem, which has been addressed in many papers and books [Farina and Rinaldi 2000, Benvenuti and Farina 2004, Kaczorek 1992, 2009b, 2011c, 2012, Shaker and Dixon 1977]. An overview on the positive realization problem is given in [Farina and Rinaldi 2000, Kaczorek 2002, Benvenuti and Farina 2004]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [Kaczorek 2004, 2006a, 2006b, 2006c, 2011a, 2011b, 2011c] and the positive realization problem for discrete-time systems with delays in [Kaczorek 2004, 2005, 2006c]. The fractional positive linear systems has been addressed in [Kaczorek 2008a, 2009a, 2011c]. The realization problem for fractional linear systems has been analyzed in [Kaczorek 2008b] and for positive 2D hybrid systems in [Kaczorek 2008c]. A method based on similarity transformation of the standard realization to the discrete positive one has been proposed in [Kaczorek 2011c]. Conditions for the existence of positive stable realization with system Metzler matrix for transfer function has been established in [Kaczorek 2011a]. The problem of the existence and determination of the set of Metzler matrices for given stable polynomials has been formulated and solved in [Kaczorek 2012].

It is well-known that [Farina and Rinaldi 2000, Kaczorek 1992, 2002] that to find a realization for a given transfer function first we have to find a state matrix for given denominator of the transfer function.

In this paper necessary and sufficient conditions for existence of the set of positive stable realizations of a proper transfer function of linear continuous-time systems are established and a procedure for computation of the set of realizations is proposed.

The paper is organized as follows. In section 2 some preliminaries concerning positive linear systems are recalled and the problem formulation is given. Problem solution for systems with real negative poles of the transfer function is presented in section 3. The problem of the existence and computation of the set of positive asymptotically stable realizations for systems with complex conjugate poles is addressed in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), M_{ns} - the set of $n\times n$ asymptotically stable Metzler matrices, I_n - the $n\times n$ identity matrix, A^T - transpose of the matrix A, $\Re^{n\times m}(s)$ - the set of $n\times m$ rational matrices in s.

Preliminaries and the problem formulation

Consider the continuous-time linear system

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{2.1a}$$

$$y(t)=Cx(t)+Du(t) \tag{2.1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 2.1. [Farina and Rinaldi 2000, Kaczorek 2002] The system (2.1) is called (internally) positive if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x(0)=x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 2.1. [Farina and Rinaldi 2000, Kaczorek 2002] The system (2.1) is positive if and only if

$$A \in M_n,\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}. \tag{2.2}$$

Definition 2.2. [Farina and Rinaldi 2000, Kaczorek 2002] The positive system (2.1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for any } x_0 \in \Re_+^n. \tag{2.3}$$

Theorem 2.2. [Farina and Rinaldi 2000, Kaczorek 2002]The positive system (2.1) is asymptotically stable if and only if all coefficients of the polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{2.4}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

Definition 2.3. [Kaczorek 2002] A matrix $P \in \Re_+^{n\times n}$ is called the monomial matrix (or generalized permutation matrix) if its every row and its every column contains only one positive entry and its remaining entries are zero.

Lemma 2.1. [Kaczorek 2002] The inverse matrix A^{-1} of the monomial matrix A is equal to the transpose matrix in which every nonzero entry is replaced by its inverse.

Lemma 2.2. If $A_M \in M_n$ then $\bar{A}_M = PA_M P^{-1} \in M_n$ for every monomial matrices $P \in \Re_+^{n\times n}$ and

$$\det[I_n s - \bar{A}_M] = \det[I_n s - A_M]. \tag{2.5}$$

Proof. By Lemma 2.1 if $P \in \Re_+^{n\times n}$ then $P^{-1} \in \Re_+^{n\times n}$ and $\bar{A}_M = PA_M P^{-1} \in M_n$ if $A_M \in M_n$. It is easy to check that

$$\begin{aligned} \det[I_n s - \bar{A}_M] &= \det[I_n s - PA_M P^{-1}] = \det\{P[I_n s - A_M]P^{-1}\} \\ &= \det P \det[I_n s - A_M] \det P^{-1} = \det[I_n s - A_M] \end{aligned} \tag{2.6}$$

since $\det P \det P^{-1} = 1$.

The transfer matrix of the systems (2.1) is given by

$$T(s) = C[I_n s - A]B + D. \tag{2.7}$$

The transfer matrix is called proper if

$$\lim_{s\to\infty} T(s) = K \in \Re^{p\times m} \tag{2.8}$$

and it is called strictly proper if $K = 0$.

Definition 2.4. Matrices (2.2) are called a positive realization of transfer matrix $T(s)$ if they satisfy the equality (2.7).

The realization is called asymptotically stable if the matrix A is an asymptotically stable Metzler matrix (Hurwitz Metzler matrix).

Theorem 2.3. [Kaczorek 2002] The positive realization (2.2) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{2.9}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

Lemma 2.3. The matrices

$$\overline{A}_k = PA_kP^{-1} \in M_{ns},\ \overline{B}_k = PB_k \in \Re_+^{n\times m},\ \overline{C}_k = C_kP^{-1} \in \Re_+^{p\times n},\ \overline{D}_k = D_k \in \Re_+^{p\times m},\quad k = 1,...,N \tag{2.10}$$

are a positive asymptotically stable realization of the proper transfer matrix $T(s) \in \Re^{p\times m}(s)$ for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices

$$A_k \in M_{ns},\ B_k \in \Re_+^{n\times m},\ C_k \in \Re_+^{p\times n},\ D_k \in \Re_+^{p\times m},\ k = 1,...,N \tag{2.11}$$

are a positive asymptotically stable realization of $T(s) \in \Re^{p\times m}(s)$.

Proof. By Lemma 2.1 if P is a monomial matrix then $P^{-1} \in \Re_+^{n\times n}$ is also monomial matrix. Hence $\overline{A}_k \in M_{ns}$ if and only if $A_k \in M_{ns}$, $\overline{B}_k \in \Re_+^{n\times m}$ if and only if $B_k \in \Re_+^{n\times m}$ and $\overline{C}_k \in \Re_+^{p\times n}$ if and only if $C_k \in \Re_+^{p\times n}$.

Using (2.10) we obtain

$$\begin{aligned}\overline{T}(s) &= \overline{C}_k[I_n s - \overline{A}_k]^{-1}\overline{B}_k + \overline{D}_k = C_kP^{-1}[I_n s - PA_kP^{-1}]^{-1}PB_k + D_k \\ &= C_kP^{-1}\{P[I_n s - A_k]P^{-1}\}^{-1}PB_k + D_k = C_kP^{-1}P[I_n s - A_k]^{-1}P^{-1}PB_k + D_k \\ &= C_k[I_n s - A_k]^{-1}B_k + D_k = T(s).\end{aligned} \tag{2.12}$$

Therefore, the matrices (2.10) are a positive asymptotically stable realization of $T(s)$ if and only if the matrices (2.11) are also its positive asymptotically stable realization.

The problem under considerations can be stated as follows: Given a rational proper matrix $T(s) \subset \Re^{p\times m}(s)$, find a set of its positive asymptotically stable realizations (2.11).

In this paper necessary and sufficient conditions for existence of the set of the positive asymptotically stable realizations for a given $T(s)$ will be established and a procedure for computation of the set of realizations will be proposed.

Systems with real negative poles

In this section the single-input single-output linear continuous-time linear systems with the proper transfer function

$$T(s)=\frac{b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0} \tag{3.1}$$

having only real negative poles (not necessarily distinct) $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$, i.e.

$$p_n(s)=(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)=s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0,$$
$$a_{n-1}=\alpha_1+\alpha_2+\ldots+\alpha_n,\;\; a_{n-2}=\alpha_1(\alpha_2+\alpha_3+\ldots+\alpha_n)+\alpha_2(\alpha_3+\alpha_4+\ldots+\alpha_n)+\ldots+\alpha_{n-1}\alpha_n,\ldots,$$
$$a_0=\alpha_1\alpha_2\ldots\alpha_n \tag{3.2}$$

will be considered.

First we shall address the problem for $n = 1$ with the transfer function

$$T(s)=\frac{b_1 s+b_0}{s+a},\; a>0. \tag{3.3}$$

Theorem 3.1. There exists the set of positive asymptotically stable realizations

$$\overline{A}_k = PA_kP^{-1},\; \overline{B}_k = PB_k,\; \overline{C}_k = C_kP^{-1},\; \overline{D}_k = D_k,\; k = 1,2 \tag{3.4}$$

for any positive parameter $P > 0$ and A_k, B_k, C_k, D_k having one of the forms

$$A_1=[-a],\;\; B_1=[1],\;\; C_1=[b_0-ab_1],\;\; D_1=[b_1] \tag{3.5}$$

or

$$A_2=[-a],\;\; B_2=[b_0-ab_1],\;\; C_2=[1],\;\; D_2=[b_1] \tag{3.6}$$

of the transfer function (3.3) if and only if

$$a>0,\;\; b_1\ge 0,\;\; b_0-ab_1\ge 0. \tag{3.7}$$

Proof. It is easy to check that the matrices (3.5) are a realization of (3.3). The matrix $A_1 \in M_{1s}$ and $C_1 \in \Re_+^{1\times 1}$, $D_1 \in \Re_+^{1\times 1}$ if and only if the conditions (3.7) are satisfied. By Lemma 2.3 the matrices (3.4) are a positive asymptotically stable realization of (3.3) for any $P > 0$ if and only if the matrices (3.5) are its positive asymptotically stable realization. Proof for matrices (3.6) is similar.

Theorem 3.2. There exists the set of positive asymptotically stable realizations

$$\overline{A}_{Mk} = PA_{Mk}P^{-1} \in M_{2s},\; \overline{B}_k = PB_k \in \Re_+^{2\times 1},\; \overline{C}_k = C_kP^{-1} \in \Re_+^{1\times 2},\; \overline{D}_k = D_k \in \Re_+^{1\times 1},\; k = 1,2 \tag{3.8}$$

for any monomial matrix $P \in \Re_{+}^{2\times 2}$ and A_{Mk}, B_k, C_k, D_k having one of the forms

$$A_{M1} = \begin{bmatrix} -a & a_1 a - a^2 - a_0 \\ 1 & a - a_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_0 - ab_1 + (aa_1 - a_0)b_2 \\ b_1 - a_1 b_2 \end{bmatrix}, \quad C_1^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = [b_2],$$
$$0 < a < a_1, \quad a_1 a - a^2 - a_0 \geq 0 \tag{3.9a}$$

$$A_{M2} = A_{M1}^T = \begin{bmatrix} -a & 1 \\ a_1 a - a^2 - a_0 & a - a_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2^T = \begin{bmatrix} b_0 - ab_1 + (aa_1 - a_0)b_2 \\ b_1 - a_1 b_2 \end{bmatrix}, \quad D_2 = [b_2],$$
$$0 < a < a_1, \quad a_1 a - a^2 - a_0 \geq 0 \tag{3.9b}$$

of the transfer function

$$T(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{3.10}$$

if and only if

$$a_1^2 - 4a_0 \geq 0 \tag{3.11}$$

and

$$b_2 \geq 0, \quad b_0 - ab_1 + (aa_1 - a_0)b_2 \geq 0, \quad b_1 - a_1 b_2 \geq 0 \text{ for } 0 < a < a_1. \tag{3.12}$$

Proof. The matrix $A_{M1} \in M_{2s}$ if and only if its characteristic polynomial

$$\det[I_2 s - A_{M1}] = \begin{vmatrix} s + a & a^2 + a_0 - a_1 a \\ -1 & s + a_1 - a \end{vmatrix} = s^2 + a_1 s + a_0$$

has negative real zeros and this is the case if and only if the condition (3.11) is met and $0 < a < a_1$. The matrix

$$D_1 = \lim_{s \to \infty} T(s) = [b_2] \in \Re_{+}^{1\times 1}$$

if and only if $b_2 \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D_1 = \frac{\bar{b}_1 s + \bar{b}_0}{s^2 + a_1 s + a_0} \tag{3.13}$$

where $\bar{b}_1 = b_1 - a_1 b_2$, $\bar{b}_0 = b_0 - a_0 b_2$. Assuming $C_1 = [0 \ 1]$ we obtain

$$T_{sp}(s) = C_1[I_2 s - A_{M1}]^{-1}B_1 = [0 \quad 1]\begin{bmatrix} s+a & a^2+a_0-a_1 a \\ -1 & s+a_1-a \end{bmatrix}^{-1}\begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} = \frac{b_{12}s+b_{11}+ab_{12}}{s^2+a_1 s+a_0}. \quad (3.14)$$

From comparison of (3.13) and (3.14) we have

$$b_{12} = \bar{b}_1 = b_1 - a_1 b_2,$$
$$b_{11} = \bar{b}_0 - ab_{12} = b_0 - a_0 b_2 - a(b_1 - a_1 b_2) = b_0 - ab_1 + (aa_1 - a_0)b_2. \quad (3.15)$$

From (3.15) it follows that $B_1 \in \Re_+^{2\times 1}$ if and only if the conditions (3.12) are satisfied. The proof for (3.9b) is similar. By Lemma 2.3 the matrices (3.8) are a positive asymptotically stable realization for any monomial matrix $P \in \Re_+^{2\times 2}$ if and only if the matrices (3.9) are its positive asymptotically stable realization.

Example 3.1. Compute the set of positive asymptotically stable realizations (3.8) of the transfer function

$$T(s) = \frac{2s^2+12s+26}{s^2+5s+6}. \quad (3.16)$$

The transfer function (3.16) satisfies the conditions (3.11) and (3.12) since

$a_1^2 - 4a_0 = 1 > 0$ and

$b_2 = 2,\ b_0 - ab_1 + (aa_1 - a_0)b_2 = 14 - 2a \ge 0,\ b_1 - a_1 b_2 = 2 > 0$ for $0 \le a \le 7$.

Using (3.9) we obtain

$$A_{M1} = \begin{bmatrix} -a & 5a-a^2-6 \\ 1 & a-5 \end{bmatrix},\ B_1 = \begin{bmatrix} 14-2a \\ 2 \end{bmatrix},\ C_1 = [0 \quad 1],\ D_1 = [2] \quad (3.17a)$$

and

$$A_{M2} = \begin{bmatrix} -a & 1 \\ 5a-a^2-6 & a-5 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_2 = [14-2a \quad 2],\ D_2 = [2] \quad (3.17b)$$

for the parameter a satisfying $2 \le a \le 3$. The desired set of positive asymptotically stable realizations of (3.16) is given by

$$\bar{A}_{M1} = P\begin{bmatrix} -a & 5a-a^2-6 \\ 1 & a-5 \end{bmatrix}P^{-1},\ \bar{B}_1 = P\begin{bmatrix} 14-2a \\ 2 \end{bmatrix},\ \bar{C}_1 = [0 \quad 1]P^{-1},\ \bar{D}_1 = [2] \quad (3.18a)$$

and

$$\overline{A}_{M2} = P\begin{bmatrix} -a & 1 \\ 5a-a^2-6 & a-5 \end{bmatrix}P^{-1},\ \ \overline{B}_2 = P\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \ \overline{C}_2 = [14-2a \quad 2]P^{-1},\ \ \overline{D}_2 = [2] \tag{3.18b}$$

where $P \in \Re_+^{2\times 2}$ is any monomial matrix.

Theorem 3.3. Let the transfer function

$$T(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{3.19}$$

have only real negative poles $-\alpha_1, -\alpha_2, -\alpha_3$, i.e.

$$d_3(s) = (s+\alpha_1)(s+\alpha_2)(s+\alpha_3) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{3.20a}$$

where

$$a_2 = \alpha_1 + \alpha_2 + \alpha_3,\ \ a_1 = \alpha_1(\alpha_2+\alpha_3) + \alpha_2\alpha_3,\ \ a_0 = \alpha_1\alpha_2\alpha_3. \tag{3.20b}$$

There exists the set of positive asymptotically stable realizations

$$\overline{A}_{Mk} = PA_{Mk}P^{-1} \in M_{3s},\ \overline{B}_k = PB_k \in \Re_+^{3\times 1},\ \overline{C}_k = C_k P^{-1} \in \Re_+^{1\times 3},$$
$$\overline{D}_k = D_k = [b_3] \in \Re_+^{1\times 1},\ k = 1,2 \tag{3.21}$$

for any monomial matrix $P \in \Re_+^{3\times 3}$ and A_{Mk}, B_k, C_k, D_k having one of the forms

$$A_{M1} = \begin{bmatrix} -\alpha_1 & 1 & 0 \\ 0 & -\alpha_2 & 1 \\ 0 & 0 & -\alpha_3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C_1^T = \begin{bmatrix} b_0 - \alpha_1 b_1 + \alpha_1^2 b_2 + (a_1\alpha_1 - a_0 + a_2\alpha_1^2)b_3 \\ b_1 - (\alpha_1+\alpha_2)b_2 + [a_2(\alpha_1+\alpha_2) - a_1]b_3 \\ b_2 - a_2 b_3 \end{bmatrix},\ D_1 = [b_3] \tag{3.22a}$$

or

$$A_{M2} = A_{M1}^T,\ \ B_2 = C_1^T,\ \ C_2 = B_1^T,\ \ D_2 = D_1 \tag{3.22b}$$

of the transfer function (3.19) if and only if the conditions

$$b_0 - \alpha_1 b_1 + \alpha_1^2 b_2 + (a_1\alpha_1 - a_0 + a_2\alpha_1^2)b_3 \ge 0 \tag{3.23a}$$

$$b_1 - (\alpha_1+\alpha_2)b_2 + [a_2(\alpha_1+\alpha_2) - a_1]b_3 \ge 0 \tag{3.23b}$$

$$b_2 - a_2 b_3 \ge 0 \tag{3.23c}$$

are met.

Proof. The matrix $A_{M1} \in M_{3s}$ if and only if $\alpha_k > 0$ for $k = 1,2,3$. The matrix

$$D_1 = \lim_{s\to\infty} T(s) = [b_3] \in \Re_+^{1\times 1}$$

if and only if $b_3 \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D_1 = \frac{\bar{b}_2 s^2 + \bar{b}_1 s + \bar{b}_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{3.24}$$

where $\bar{b}_2 = b_2 - a_2 b_3$, $\bar{b}_1 = b_1 - a_1 b_3$, $\bar{b}_0 = b_0 - a_0 b_3$.

Assuming $B_1^T = [0 \quad 0 \quad 1]$ we obtain

$$T_{sp}(s) = C_1[I_3 s - A_{M1}]^{-1} B_1 = [c_1 \quad c_2 \quad c_3] \begin{bmatrix} s+\alpha_1 & -1 & 0 \\ 0 & s+\alpha_2 & -1 \\ 0 & 0 & s+\alpha_3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{[c_1 \quad c_2 \quad c_3]}{s^3 + a_2 s^2 + a_1 s + a_0} \begin{bmatrix} 1 \\ s+\alpha_1 \\ (s+\alpha_1)(s+\alpha_2) \end{bmatrix} = \frac{c_3 s^2 + [c_2 + c_3(\alpha_1 + \alpha_2)]s + c_1 + \alpha_1 c_2 + \alpha_1 \alpha_2 c_3}{s^3 + a_2 s^2 + a_1 s + a_0}$$

(3.25)

From comparison of (3.24) and (3.25) we have

$$c_3 = \bar{b}_2 = b_2 - a_2 b_3,$$
$$c_2 = \bar{b}_1 - c_3(\alpha_1 + \alpha_2) = b_1 - a_1 b_3 - c_3(\alpha_1 + \alpha_2) = b_1 - (\alpha_1 + \alpha_2) b_2 + [a_2(\alpha_1 + \alpha_2) - a_1] b_3,$$
$$c_1 = \bar{b}_0 = b_0 - a_0 b_3 - c_2 \alpha_1 - c_3 \alpha_1 \alpha_2 = b_0 - \alpha_1 b_1 + \alpha_1^2 b_2 + (a_1 \alpha_1 - a_0 + a_2 \alpha_1^2) b_3.$$

(3.26)

From (3.26) it follows that $C_1 \in \Re_+^{1\times 3}$ if and only if the conditions (3.23) are met. The proof for (3.22b) follows immediately from the equality that

$$\begin{aligned} T(s) = T^T(s) &= [C_1[I_3 s - A_{M1}]^{-1} B_1 + D_1]^T = B_1^T [I_3 s - A_{M1}^T]^{-1} C_1^T + D_1 \\ &= C_2[I_3 s - A_{M2}]^{-1} B_2 + D_2. \end{aligned} \tag{2.27}$$

By Lemma 2.3 the matrices (3.21) are a positive asymptotically stable realization of (3.19) for any monomial matrix $P \in \Re_+^{3\times 3}$ if and only if the matrices (3.22) are its positive asymptotically stable realization.

Theorem 3.4. There exists the set of positive asymptotically stable realizations

$$\bar{A}_{Mk} = P A_{Mk} P^{-1} \in M_{ns}, \ \bar{B}_k = P B_k \in \Re_+^{n\times 1}, \ \bar{C}_k = C_k P^{-1} \in \Re_+^{1\times n}, \ \bar{D}_k = D_k \in \Re_+^{1\times 1},$$
$$k = 1,2 \tag{3.28}$$

for any monomial matrix $P \in \Re_+^{n\times n}$ and A_{Mk}, B_k, C_k, D_k having one of the forms

$$A_{M1} = \begin{bmatrix} -\alpha_1 & 1 & 0 & \dots & 0 \\ 0 & -\alpha_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_n \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_1^T = \begin{bmatrix} \bar{b}_0 - \bar{a}_{10}c_2 - \bar{a}_{20}c_3 - \dots - \bar{a}_{n-1,0}c_n \\ \vdots \\ \bar{b}_{n-2} - \bar{a}_{n-1,n-2}c_n \\ \bar{b}_{n-1} \end{bmatrix}, \quad D_1 = [b_n]$$

(3.29a)

or

$$A_{M2} = A_{M1}^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T, \quad D_2 = D_1 \qquad (3.29b)$$

of the transfer function (3.1) with only real negative poles $-\alpha_1, -\alpha_2, \dots, -\alpha_n$ if and only if the conditions

$$\begin{aligned} c_n &= b_{n-1} - a_{n-1}b_n \geq 0 \\ c_{n-1} &= b_{n-2} - a_{n-2}b_n - \bar{a}_{n-1,n-2}c_n \geq 0 \\ &\vdots \\ c_1 &= b_0 - a_0 b_n - \bar{a}_{10}c_2 - \bar{a}_{20}c_3 - \dots - \bar{a}_{n-1,0}c_n \geq 0 \end{aligned} \qquad (3.30a)$$

where

$$\bar{a}_{10} = \alpha_1, \ \bar{a}_{20} = \alpha_1\alpha_2, \ \bar{a}_{21} = \alpha_1 + \alpha_2, \ \bar{a}_{30} = \alpha_1\alpha_2\alpha_3, \ \bar{a}_{31} = \alpha_1(\alpha_2 + \alpha_3) + \alpha_2\alpha_3, \ \bar{a}_{32} = \alpha_1 + \alpha_2 + \alpha_3,$$
$$\vdots$$
$$\bar{a}_{n-1,0} = \alpha_1\alpha_2 \dots \alpha_n, \ \bar{a}_{n-1,1} = \alpha_1(\alpha_2 + \alpha_3 + \dots + \alpha_n) + \alpha_2(\alpha_3 + \alpha_4 + \dots + \alpha_n) + \dots + \alpha_{n-1}\alpha_n, \dots,$$
$$\bar{a}_{n-1,n-2} = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

(3.30b)

are met.

Proof. The matrix $A_{M1} \in M_{ns}$ if and only $\alpha_k > 0$ for $k = 1,2,\dots,n$. The matrix

$$D_1 = \lim_{s \to \infty} T(s) = [b_n] \in \Re_+^{1\times 1} \qquad (3.31)$$

if and only if $b_n \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D_1 = \frac{\bar{b}_{n-1}s^{n-1} + \dots + \bar{b}_1 s + \bar{b}_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \qquad (3.32a)$$

where

$$\bar{b}_k = b_k - a_k b_n \text{ for } k = 0,1,\dots,n-1. \qquad (3.32b)$$

Assuming $B_1^T = [0 \quad \dots \quad 0 \quad 1] \in \Re_+^{n\times 1}$ we obtain

$$T_{sp}(s)=C_1[I_n s-A_{M1}]^{-1}B_1=[c_1 \quad \dots \quad c_n]\begin{bmatrix} s+\alpha_1 & -1 & 0 & \dots & 0 \\ 0 & s+\alpha_2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & s+\alpha_n \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$=\frac{[c_1 \quad \dots \quad c_n]}{d_n(s)}\begin{bmatrix} 1 \\ p_1(s) \\ \vdots \\ p_{n-1}(s) \end{bmatrix}=\frac{c_1+c_2p_1(s)+\dots+c_np_{n-1}(s)}{d_n(s)}$$

(3.33a)

where

$$\begin{aligned} &d_n(s)=s^n+a_{n-1}s^{n-1}+\dots+a_1s+a_0, \\ &p_1(s)=s+\alpha_1=s+\bar{a}_{10}, \quad \bar{a}_{10}=\alpha_1, \\ &p_2(s)=(s+\alpha_1)(s+\alpha_2)=s^2+\bar{a}_{21}s+\bar{a}_{20}, \quad \bar{a}_{21}=\alpha_1+\alpha_2, \quad \bar{a}_{20}=\alpha_1\alpha_2, \\ &\vdots \\ &p_{n-1}(s)=(s+\alpha_1)(s+\alpha_2)\dots(s+\alpha_{n-1})=s^{n-1}+\bar{a}_{n-1,n-2}s^{n-2}+\dots+\bar{a}_{n-1,1}s+\bar{a}_{n-1,0}, \\ &\quad \bar{a}_{n-1,n-2}=\alpha_1+\alpha_2+\dots+\alpha_{n-1},\dots, \quad \bar{a}_{n-1,0}=\alpha_1\alpha_2\dots,\alpha_{n-1}. \end{aligned}$$

(3.33b)

From comparison of (3.33a) and (3.32a) we have

$$\begin{aligned} &c_n=\bar{b}_{n-1}=b_{n-1}-a_{n-1}b_n, \\ &c_{n-1}=\bar{b}_{n-2}-\bar{a}_{n-1,n-2}c_n=b_{n-2}-a_{n-2}b_n-\bar{a}_{n-1,n-2}c_n, \\ &\vdots \\ &c_1=\bar{b}_0-\bar{a}_{10}c_2-\bar{a}_{20}c_3-\dots-\bar{a}_{n-1,0}c_n. \end{aligned} \quad (3.34)$$

From (3.34) it follows that $C_1 \in \Re_+^{1\times n}$ if and only if the conditions (3.30) are met. The proof for (3.29b) follows immediately from (2.27). By Lemma 2.3 the matrices (3.28) are a positive asymptotically stable realization of (3.1) for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices (3.29) are its positive asymptotically stable realization.

From above considerations we have the following procedure for computation of the set of positive asymptotically stable realizations (3.28) of the transfer function (3.1) with real negative poles.

Procedure 3.1.

Step 1. Check the conditions (3.30). If the conditions are met, go to Step 2, if not then does not exist the set of realizations.

Step 2. Using (3.29) compute the matrices A_{Mk}, B_k, C_k, D_k for example for $k = 1$ or $k = 2$.

Step 3. Using (3.28) compute the desired set of realizations.

Example 3.2. Compute the set of positive asymptotically table realizations of the transfer function

$$T(s) = \frac{0.2s^4 + 2.2s^3 + 8.6s^2 + 12.4s + 7.8}{s^4 + 6s^3 + 13s^2 + 12s + 4}. \tag{3.35}$$

The transfer function (3.35) has two real double poles $-\alpha_1 = -\alpha_2 = -1$, $-\alpha_3 = -\alpha_4 = -2$. Using Procedure 3.1 we obtain the following.

Step 1. The conditions (3.30) are satisfied since

$$\begin{aligned}
c_4 &= b_3 - a_3 b_4 = 1 > 0, \\
c_3 &= b_2 - a_2 b_4 - \bar{a}_{32} c_4 = 2 > 0, \\
c_2 &= b_1 - a_1 b_4 - \bar{a}_{21} c_3 - \bar{a}_{31} c_4 = 1 > 0, \\
c_1 &= b_0 - a_0 b_4 - \bar{a}_{10} c_2 - \bar{a}_{20} c_3 - \bar{a}_{30} c_4 = 2 > 0.
\end{aligned} \tag{3.36}$$

Step 2. In this case the matrices (3.29a) have the forms

$$A_{M1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_1 = [2 \quad 1 \quad 2 \quad 1], \quad D_1 = [0.2]. \tag{3.37}$$

Step 3. The desired set of realizations of (3.35) is given by

$$\bar{A}_{M1} = P \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} P^{-1}, \quad \bar{B}_1 = P \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{C}_1 = [2 \quad 1 \quad 2 \quad 1] P^{-1}, \quad \bar{D}_1 = [0.2] \tag{3.38}$$

for any monomial matrix $P \in \Re_+^{4\times 4}$.

Systems with complex conjugate poles

In this section the single-input single-output linear continuous-time system with the transfer function (3.1) having at least one pair of complex conjugate poles will be considered.

Theorem 4.1. There exists the set of positive asymptotically stable realizations

$$\bar{A}_{Mk} = PA_{Mk}P^{-1} \in M_{3s}\,,\ \bar{B}_k = PB_k \in \Re_+^{3\times 1}\,,\ \bar{C}_k = C_kP^{-1} \in \Re_+^{1\times 3}\,,$$
$$\bar{D}_k = D_k = [b_3] \in \Re_+^{1\times 1}\,,\ k = 1,2 \qquad (4.1)$$

for any monomial matrix $P \in \Re_+^{3\times 3}$ and the matrices A_{Mk}, B_k, C_k, D_k having one of the forms

$$A_{M1} = \begin{bmatrix} p_1 + p_2 - a_2 & 1 & a_{13} \\ 0 & -p_1 & a_{23} \\ 1 & 0 & -p_2 \end{bmatrix},\ B_1 = \begin{bmatrix} b_1 + (p_2 - a_2)b_2 + (a_2^2 - a_1 - a_2p_2)b_3 \\ b_0 - p_1b_1 + p_1^2b_2 + (a_1p_1 - a_0 - a_2p_1^2)b_3 \\ b_2 - a_2b_3 \end{bmatrix},$$
$$C_1 = [0 \quad 0 \quad 1],\ D_1 = [b_3],$$
$$a_{13} = (a_2 - p_1 - p_2)(p_1 + p_2) + p_1p_2 - a_1,\ \ a_{23} = (a_2 - p_1 - p_2)p_1p_2 - a_{13}p_1 - a_0 \qquad (4.2a)$$

or

$$A_{M2} = A_{M1}^T,\ \ B_2 = C_1^T,\ \ C_2 = B_1^T,\ \ D_2 = D_1 \qquad (4.2b)$$

of the transfer function

$$T(s) = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0} \qquad (4.3)$$

if and only if the coefficients of the polynomial

$$d_3(s) = s^3 + a_2s^2 + a_1s + a_0 \qquad (4.4)$$

satisfies the conditions

$$a_2^2 - 3a_1 \ge 0,\ \ -2a_2^3 + 9a_1a_2 - 27a_0 \ge 0 \qquad (4.5)$$

and

$$\begin{aligned} &b_1 + (p_2 - a_2)b_2 + (a_2^2 - a_1 - a_2p_2)b_3 \ge 0, \\ &b_0 - p_1b_1 + p_1^2b_2 + (a_1p_1 - a_0 - a_2p_1^2)b_3 \ge 0, \\ &b_2 - a_2b_3 \ge 0 \end{aligned} \qquad (4.6)$$

are where p_1, p_2 are positive parameters satisfying $0 < p_1 + p_2 < a_2$.

Proof. If the matrix $A_{M1} \in M_{3s}$ if and only if its characteristic polynomial

$$d_3(s) = \det[I_3 s - A_{M1}] = \begin{vmatrix} s + a_2 - p_1 - p_2 & -1 & -a_{13} \\ 0 & s + p_1 & -a_{23} \\ -1 & 0 & s + p_2 \end{vmatrix} = s^3 + a_2 s^2 + a_1 s + a_0 \tag{4.7}$$

has the coefficients satisfying the conditions (4.5) [19] and $0 < p_1 + p_2 < a_2$.

The matrix

$$D_1 = \lim_{s \to \infty} T(s) = [b_3] \in \Re_+^{1\times 1} \tag{4.8}$$

if and only if $b_3 \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D_1 = \frac{\bar{b}_2 s^2 + \bar{b}_1 s + \bar{b}_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{4.9a}$$

where

$$\bar{b}_2 = b_2 - a_2 b_3,\ \bar{b}_1 = b_1 - a_1 b_3,\ \bar{b}_0 = b_0 - a_0 b_3. \tag{4.9b}$$

Assuming $C_1 = [0 \quad 0 \quad 1]$ we obtain

$$\begin{aligned} T_{sp}(s) &= C_1 [I_3 s - A_{M1}]^{-1} B_1 = [0 \quad 0 \quad 1] \begin{bmatrix} s + a_2 - p_1 - p_2 & -1 & -a_{13} \\ 0 & s + p_1 & -a_{23} \\ -1 & 0 & s + p_2 \end{bmatrix}^{-1} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} \\ &= \frac{[s + p_1 \quad 1 \quad (s + a_2 - p_1 - p_2)(s + p_1)]}{s^3 + a_2 s^2 + a_1 s + a_0} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} \\ &= \frac{b_{13} s^2 + [b_{11} + (a_2 - p_2) b_{13}] s + b_{12} + p_1 b_{11} + (a_2 - p_1 - p_2) p_1 b_{13}}{s^3 + a_2 s^2 + a_1 s + a_0}. \end{aligned} \tag{4.10}$$

From comparison of (4.9a) and (4.10) we have

$$\begin{aligned} b_{13} &= \bar{b}_2 = b_2 - a_2 b_3, \\ b_{11} &= \bar{b}_1 - (a_2 - p_2) b_{13} = b_1 + (p_2 - a_2) b_2 + (a_2^2 - a_1 - a_2 p_2) b_3, \\ b_{12} &= \bar{b}_0 - p_1 b_{11} - (a_2 - p_1 - p_2) p_1 b_{13} = b_0 - p_1 b_1 + p_1^2 b_2 + (a_1 p_1 - a_0 - a_2 p_1^2) b_3. \end{aligned} \tag{4.11}$$

From (4.11) it follows that $B_1 \in \Re_+^{3\times 1}$ if and only if the conditions (4.6) are met. The proof for (4.2b) is similar. By Lemma 2.3 the matrices (4.1) are a positive asymptotically stable realization for any monomial matrix $P \in \Re_+^{3\times 3}$ of (4.3) if and only if the matrices (4.2) are its positive asymptotically stable realization.

Remark 4.1. The matrix A_{M1} in Theorem 4.1 can be replaced by the matrices [19]

$$A_{M3} = \begin{bmatrix} p_1 + p_2 - a_2 & 0 & 1 \\ a_{21} & -p_1 & 0 \\ a_{31} & 1 & -p_2 \end{bmatrix}, \quad A_{M4} = \begin{bmatrix} p_1 + p_2 - a_2 & a_{12} & 0 \\ 0 & -p_1 & 1 \\ 1 & a_{32} & -p_2 \end{bmatrix} \tag{4.12a}$$

and the matrix A_{M2} by A_{M3}^T, A_{M4}^T. For A_{M3} the matrices B_3 and C_3 have the forms

$$B_3 = \begin{bmatrix} b_2 - a_2 b_3 \\ b_0 - p_1 b_1 + p_1^2 b_2 + (a_1 p_1 - a_0 - a_2 p_1^2) b_3 \\ b_1 - (p_1 + p_2) b_2 + [a_2(p_1 + p_2) - a_1] b_3 \end{bmatrix}, \quad C_3^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{4.12b}$$

and for A_{M4} the matrices B_4 and C_4 have the forms

$$B_4 = \begin{bmatrix} b_0 + (p_1 + p_2 - a_2) b_1 + [(p_1 + p_2)^2 - a_2^2] b_2 + (p_1 + p_2 - a_2)[a_2^2 - a_1 - a_2(p_1 + p_2)] b_3 \\ b_2 - a_2 b_3 \\ b_1 + (p_2 + a_2) b_2 + (a_2^2 - a_1 - a_2 p_2) b_3 \end{bmatrix}, \quad C_4^T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.12c}$$

From above considerations we have the following procedure for computation of the set of positive asymptotically stable realizations.

Procedure 4.1.

Step 1. Check the conditions (4.5) and (4.6). If the conditions are met, go to Step 2, if not then does not exist the set of realizations.

Step 2. Using (4.2) compute the matrices A_{Mk}, B_k, C_k, D_k for example for $k = 1$ or $k = 2$.

Step 3. Using (4.1) compute the desired set of realizations.

Example 4.1. Compute the set of positive asymptotically table realizations of the transfer function

$$T(s) = \frac{0.1s^3 + s^2 + 4s + 12}{s^3 + 9s^2 + 25s + 17}. \tag{4.13}$$

Using Procedure 4.1 we obtain the following.

Step 1. The transfer function (4.13) satisfies the conditions (4.5) and (4.6) since

$$\begin{aligned} & a_2^2 - 3a_1 = 6 > 0, \\ & -2a_2^3 + 9a_1 a_2 - 27a_0 = 108 > 0 \end{aligned} \tag{4.14a}$$

and

$$
\begin{aligned}
&b_1 + (p_2 - a_2)b_2 + (a_2^2 - a_1 - a_2 p_2)b_3 = 0.6 + 0.1p_2 > 0,\\
&b_0 - p_1 b_1 + p_1^2 b_2 + (a_1 p_1 - a_0 - a_2 p_1^2)b_3 = 10.3 + p_1(0.1p_1 - 1.5) > 0,\\
&b_2 - a_2 b_3 = 0.1 > 0
\end{aligned} \tag{4.14b}
$$

for $0 < p_1 + p_2 < 9$.

Step 2. Using (4.2a), (4.13) and (4.14b) we obtain

$$
A_{M1} = \begin{bmatrix} p_1 + p_2 - 9 & 1 & a_{13} \\ 0 & -p_1 & a_{23} \\ 1 & 0 & -p_2 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.6 + 0.1p_2 \\ 10.3 + p_1(0.1p_1 - 1.5) \\ 0.1 \end{bmatrix},\ C_1^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ D_1 = [0.1] \tag{4.15}
$$

where

$$
\begin{aligned}
&a_{13} = [9 - (p_1 + p_2)](p_1 + p_2) + p_1 p_2 - 25,\\
&a_{23} = [8 - (p_1 + p_2)]p_1 p_2 - 9(p_1 + p_2) + (p_1 + p_2)^2 + 8
\end{aligned}
$$

and p_1, p_2 are arbitrary parameters satisfying $0 < p_1 + p_2 < 9$.

Step 3. The desired set of positive stable realizations is given by

$$
\overline{A}_{M1} = PA_{M1}P^{-1},\ \overline{B}_1 = PB_1,\ \overline{C}_1 = C_1 P^{-1},\ \overline{D}_1 = D_1 \tag{4.16}
$$

for any monomial matrix $P \in \Re_+^{3\times 3}$.

Theorem 4.2. There exists the set of positive asymptotically stable realizations

$$
\overline{A}_{Mk} = PA_{Mk}P^{-1} \in M_{4s},\ \overline{B}_k = PB_k \in \Re_+^{4\times 1},\ \overline{C}_k = C_k P^{-1} \in \Re_+^{1\times 4},\ \overline{D}_k = D_k \in \Re_+^{1\times 1} \tag{4.17}
$$

for any monomial matrix $P \in \Re_+^{4\times 4}$ and the matrices A_{Mk}, B_k, C_k, D_k having one of the forms

$$
A_{M1} = \begin{bmatrix} -p_1 & 1 & 0 & a_{14} \\ 0 & -p_2 & 1 & a_{24} \\ 0 & 0 & -p_3 & a_{34} \\ 1 & 0 & 0 & p_1 + p_2 + p_3 - a_3 \end{bmatrix},\ B_1 = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{bmatrix},\ C_1 = [0\ \ 0\ \ 0\ \ 1],\ D_1 = [b_4],
$$

$$
\begin{aligned}
&\tilde{b}_1 = b_2 - (p_1 + p_2 + p_3)b_3 + [a_3(p_1 + p_2 + p_3) - a_2]b_4\\
&\tilde{b}_2 = b_1 - (p_2 + p_3)b_2 + (p_2^2 + p_3^2 + p_2 p_3)b_3 + [a_3(p_2 + p_3) - a_1 - a_3(p_2^2 + p_3^2 + p_2 p_3)]b_4\\
&\tilde{b}_3 = b_0 - p_3 b_1 + p_3^2 b_2 - p_3^2 b_3 + [a_1 p_3 - a_0 + a_2 p_2 p_3 - a_3 p_2 p_3]b_4\\
&\tilde{b}_4 = b_3 - a_3 b_4
\end{aligned} \tag{4.18a}
$$

and

$$a_{14} = p_1(a_3 - p_1) + p_2(a_3 - p_1 - p_2) + p_3(a_3 - p_1 - p_2 - p_3) - a_2 \geq 0,$$
$$a_{24} = (p_1 + p_2)p_3(a_3 - p_1 - p_2 - p_3) + p_1 p_2(a_3 - p_1 - p_2) - a_{14}(p_2 + p_3) - a_1 \geq 0,$$
$$a_{34} = p_1 p_2 p_3(a_3 - p_1 - p_2 - p_3) - a_{14} p_2 p_3 - a_{24} p_3 - a_0 \geq 0$$

(4.18b)

or

$$A_{M2} = A_{M1}^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T, \quad D_2 = D_1 \qquad (4.18c)$$

of the transfer function

$$T(s) = \frac{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \qquad (4.19)$$

if and only if the coefficients of the polynomial

$$d_4(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \qquad (4.20)$$

satisfies the conditions

$$3a_3^2 - 8a_2 \geq 0, \quad -a_3^3 + 4a_2 a_3 - 8a_1 \geq 0, \quad 3a_3^4 - 16a_2 a_3^2 + 64a_1 a_3 - 256a_0 \geq 0 \qquad (4.21)$$

and

$$b_2 - (p_1 + p_2 + p_3)b_3 + [a_3(p_1 + p_2 + p_3) - a_2]b_4 \geq 0$$
$$b_1 - (p_2 + p_3)b_2 + (p_2^2 + p_3^2 + p_2 p_3)b_3 + [a_3(p_2 + p_3) - a_1 - a_3(p_2^2 + p_3^2 + p_2 p_3)]b_4 \geq 0$$
$$b_0 - p_3 b_1 + p_3^2 b_2 - p_3^2 b_3 + [a_1 p_3 - a_0 + a_2 p_2 p_3 - a_3 p_2 p_3]b_4 \geq 0$$
$$b_3 - a_3 b_4 \geq 0$$

(4.22)

are where p_1, p_2, p_3 are positive parameters satisfying $0 < p_1 + p_2 + p_3 < a_3$.

The proof is similar to the proof of Theorem 4.1.

Remark 4.2. The matrix A_{M1} in Theorem 4.2 can be replaced by the matrices

$$A_{M3} = \begin{bmatrix} -p_1 & 1 & a_{13} & 0 \\ 0 & -p_2 & a_{23} & 1 \\ 1 & 0 & -p_3 & 0 \\ 0 & 0 & a_{43} & p_1 + p_2 + p_3 - a_3 \end{bmatrix}, \quad A_{M4} = \begin{bmatrix} -p_1 & a_{12} & 1 & 0 \\ 1 & -p_2 & 0 & 0 \\ 0 & a_{32} & -p_3 & 1 \\ 0 & a_{42} & 0 & p_1 + p_2 + p_3 - a_3 \end{bmatrix},$$

$$A_{M5} = \begin{bmatrix} -p_1 & 0 & 0 & 1 \\ a_{21} & -p_2 & 0 & 0 \\ a_{31} & 1 & -p_3 & 0 \\ a_{41} & 0 & 1 & p_1 + p_2 + p_3 - a_3 \end{bmatrix}, \quad 0 < p_1 + p_2 + p_3 < a_3$$

(4.23)

and the matrix A_{M2} by the matrices A_{M3}^T, A_{M4}^T, A_{M5}^T.

In general case let us consider the transfer function

$$T(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \quad (4.24)$$

with at least one pair of complex conjugate poles.

Theorem 4.3. There exists the set of positive asymptotically stable realizations

$$\overline{A}_{Mk} = PA_{Mk}P^{-1} \in M_{ns}, \; \overline{B}_k = PB_k \in \Re_+^{n\times 1}, \; \overline{C}_k = C_k P^{-1} \in \Re_+^{1\times n}, \; \overline{D}_k = D_k \in \Re_+^{1\times 1} \quad (4.25)$$

for any monomial matrix $P \in \Re_+^{n\times n}$ and A_{Mk}, B_k, C_k, D_k having one of the forms

$$A_{M1} = \begin{bmatrix} -p_1 & 1 & 0 & \ldots & 0 & a_{1,n} \\ 0 & -p_2 & 1 & \ldots & 0 & a_{2,n} \\ 0 & 0 & -p_3 & \ldots & 0 & a_{3,n} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_{n-2,n} \\ 0 & 0 & 0 & \ldots & -p_{n-1} & a_{n-1,n} \\ 1 & 0 & 0 & \ldots & 0 & p_1 + \ldots + p_{n-1} - a_{n-1} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_{n-2} - a_{n-2}b_n - \hat{a}_{n,n-2}b_{1,n} \\ \vdots \\ b_0 - a_0 b_n - \hat{a}_{n,0}b_{1,n} - \hat{a}_{10}b_{1,1} - \ldots - \hat{a}_{n-2,0}b_{1,n-2} \\ b_{n-1} - a_{n-1}b_n \end{bmatrix}, \; C_1^T = [0 \;\; \ldots \;\; 0 \;\; 1], \; D_1 = [b_n] \quad (4.26a)$$

where $p_1, p_2, \ldots, p_{n-1}$ are positive parameters satisfying $0 < p_1 + p_2 + \ldots + p_{n-1} < a_{n-1}$ or

$$A_{M2} = A_{M1}^T, \; B_2 = C_1^T, \; C_2 = B_1^T, \; D_2 = D_1 \quad (4.26b)$$

and

$$\begin{aligned} a_{1,n} &= p_1(a_{n-1} - p_1) + p_2(a_{n-1} - p_1 - p_2) + \ldots + p_{n-1}(a_{n-1} - p_1 - \ldots - p_{n-1}) - a_{n-2} \\ &\vdots \\ a_{n-1,n} &= p_1 \ldots p_{n-1}(a_{n-1} - p_1 - \ldots - p_{n-1}) - \hat{a}_{10}a_{1,n} - \ldots - \hat{a}_{n-2,0}a_{n-2,n} \end{aligned} \quad (4.26c)$$

of the transfer function (4.24) if and only if the coefficients of the polynomial

$$d_n(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (4.27)$$

satisfies the conditions

$$
\begin{aligned}
&C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2} \geq 0,\\
&C_3^n\left(\frac{a_{n-1}}{n}\right)^3 - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right]C_1^{n-2}\left(\frac{a_{n-1}}{n}\right) - a_{n-3} \geq 0,\\
&\vdots\\
&C_n^n\left(\frac{a_{n-1}}{n}\right)^n - \left[C_2^n\left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right]C_1^{n-2}\left(\frac{a_{n-1}}{n}\right)^{n-2} - \ldots - C_1^1\left(\frac{a_{n-1}}{n}\right) - a_0 \geq 0\\
&C_k^n = \frac{n!}{k!(n-k)!}
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
&b_{n-2} - a_{n-2}b_n - \hat{a}_{n,n-2}b_{1,n} \geq 0\\
&\vdots\\
&b_0 - a_0 b_n - \hat{a}_{n,0}b_{1,n} - \hat{a}_{10}b_{11} - \ldots - \hat{a}_{n-2,0}b_{1,n-2} \geq 0\\
&b_{n-1} - a_{n-1}b_n \geq 0
\end{aligned}
\tag{4.29a}
$$

where

$$
\begin{aligned}
&\hat{a}_{10} = p_2 p_3 \ldots p_{n-1},\ \ \hat{a}_{20} = p_3 p_4 \ldots p_{n-1}, \ldots, \hat{a}_{n,0} = p_1 p_2 \ldots p_{n-1},\\
&\vdots\\
&\hat{a}_{1,n-3} = p_2 + p_3 + \ldots + p_{n-1},\ \ \hat{a}_{2,n-4} = p_3 + p_4 + \ldots + p_{n-1}, \ldots, \hat{a}_{n,n-2} = p_1 + p_2 + \ldots + p_{n-1}.
\end{aligned}
\tag{4.29b}
$$

Proof. It is well-known [Kaczorek 2012] that there exists $A_{M1} \in M_{ns}$ if and only if the coefficients of the polynomial (4.27) are positive and satisfy the conditions (4.28). The matrix

$$D_1 = \lim_{s\to\infty} T(s) = [b_n] \in \Re_+^{1\times 1} \tag{4.30}$$

if and only if $b_n \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D_1 = \frac{\bar{b}_{n-1}s^{n-1} + \ldots + \bar{b}_1 s + \bar{b}_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \tag{4.31a}$$

where

$$\overline{b}_k = b_k - a_k b_n \text{ for } k = 1,2,...,n-1. \tag{4.31b}$$

Assuming $C_1 = [0 \quad ... \quad 0 \quad 1] \in \Re_+^{1\times n}$ we obtain

$$T_{sp}(s) = C_1[I_n s - A_{M1}]^{-1} B_1$$

$$= [0 \quad ... \quad 0 \quad 1]\begin{bmatrix} s+p_1 & -1 & 0 & ... & 0 & -a_{1,n} \\ 0 & s+p_2 & -1 & ... & 0 & -a_{2,n} \\ 0 & 0 & s+p_3 & ... & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & -1 & -a_{n-2,n} \\ 0 & 0 & 0 & ... & s+p_{n-1} & -a_{n-1,n} \\ -1 & 0 & 0 & ... & 0 & s+a_{n-1}-p_1-...-p_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} \\ \vdots \\ b_{1,n-1} \\ b_{1,n} \end{bmatrix} \tag{4.32a}$$

$$= \frac{[p_1(s) \quad ... \quad p_n(s)]}{d_n(s)} \begin{bmatrix} b_{11} \\ \vdots \\ b_{1,n-1} \\ b_{1,n} \end{bmatrix} = \frac{p_1(s)b_{11} + p_2(s)b_{12} + ... + p_n(s)b_{1,n}}{d_n(s)}$$

where

$$\begin{aligned}
&p_1(s) = (s+p_2)(s+p_3)...(s+p_{n-1}) = s^{n-2} + \hat{a}_{1,n-3}s^{n-3} + ... + \hat{a}_{11}s + \hat{a}_{10}, \\
&\quad \hat{a}_{1,n-3} = p_2 + p_3 + ... + p_{n-1}, ..., \hat{a}_{10} = p_2 p_3 ... p_{n-1}, \\
&p_2(s) = (s+p_3)(s+p_4)...(s+p_{n-1}) = s^{n-3} + \hat{a}_{2,n-4}s^{n-4} + ... + \hat{a}_{21}s + \hat{a}_{20}, \\
&\quad \hat{a}_{2,n-4} = p_3 + p_4 + ... + p_{n-1}, ..., \hat{a}_{20} = p_3 p_4 ... p_{n-1}, \\
&\vdots \\
&p_{n-2}(s) = s + p_{n-1} = s + \hat{a}_{n-2,0}, \quad \hat{a}_{n-2,0} = p_{n-1}, \\
&p_{n-1}(s) = 1 \\
&d_n(s) = (s+p_1)(s+p_2)...(s+p_{n-1}) = s^{n-1} + \hat{a}_{n,n-2}s^{n-2} + ... + \hat{a}_{n,1}s + \hat{a}_{n,0}, \\
&\quad \hat{a}_{n,n-2} = p_1 + p_2 + ... + p_{n-1}, ..., \hat{a}_{n,0} = p_1 p_2 ... p_{n-1}.
\end{aligned} \tag{4.32b}$$

From comparison of (4.31a) and (4.32a) we have

$$\begin{aligned}
&b_{1,n} = \overline{b}_{n-1} = b_{n-1} - a_{n-1}b_n, \\
&b_{11} = \overline{b}_{n-2} - \hat{a}_{n,n-2}b_{1,n} = b_{n-2} - a_{n-2}b_n - \hat{a}_{n,n-2}b_{1,n}, \\
&\vdots \\
&b_{1,n-1} = \overline{b}_0 - \hat{a}_{n,0}b_{1,n} - \hat{a}_{10}b_{11} - ... - \hat{a}_{n-2,0}b_{1,n-2} = b_0 - a_0 b_n - \hat{a}_{n,0}b_{1,n} - \hat{a}_{10}b_{11} - ... - \hat{a}_{n-2,0}b_{1,n-2}.
\end{aligned} \tag{4.33}$$

From (4.33) it follows that $B_1 \in \Re_+^{n\times 1}$ if and only if the conditions (4.29a) are met. The proof for (4.26b) follows immediately from (4.27). By Lemma 2.3 the matrices (4.25) are a positive asymptotically stable realization of (4.24) for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices (4.26) are its positive asymptotically stable realization.

Remark 4.2. The matrix A_{M1} in Theorem 4.2 can be replaced by the matrices

$$A_{M3} = \begin{bmatrix} -p_1 & 1 & 0 & \dots & a_{1,n-1} & 0 \\ 0 & -p_2 & 1 & \dots & a_{2,n-1} & 1 \\ 0 & 0 & -p_3 & \dots & a_{3,n-1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-2,n-1} & 0 \\ 1 & 0 & 0 & \dots & -p_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{n,n-1} & p_1+\dots+p_{n-1}-a_{n-1} \end{bmatrix},\dots,$$

$$A_{Mn+1} = \begin{bmatrix} -p_1 & 0 & 0 & \dots & 0 & 1 \\ a_{21} & -p_2 & 0 & \dots & 0 & 0 \\ a_{31} & 1 & -p_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-2,1} & 0 & 0 & \dots & 0 & 0 \\ a_{n-1,1} & 0 & 0 & \dots & -p_{n-1} & 0 \\ a_{n,1} & 0 & 0 & \dots & 1 & p_1+\dots+p_{n-1}-a_{n-1} \end{bmatrix} \quad (4.23)$$

where $p_1, p_2, \dots, p_{n-1}$ are positive parameters satisfying $0 < p_1 + p_2 + \dots + p_{n-1} < a_{n-1}$ and the matrix A_{M2} by the matrices $A_{M3}^T,\dots,A_{Mn+2}^T$.

To compute the desired set of positive asymptotically stable realizations (4.25) of (4.24) Procedure 4.1 with slight modifications can be used.

Conclusion

The problem of existence and computation of the set of positive asymptotically stable realizations of a proper transfer function of linear continuous-time systems has been formulated and solved. Necessary and sufficient conditions for existence of the set of realizations have been established (Theorems 3.1 – 3.4 and 4.1 – 4.3). Procedure for computation of the set of realizations for transfer functions with only real negative poles and with at least one pair of complex conjugate poles have been proposed

(Procedures 3.1 and 4.1). The effectiveness of the procedures have been demonstrated on numerical examples. The presented methods can be extended to positive asymptotically stable discrete-time linear systems and also to multi-input multi-output continuous-time and discrete-time linear systems. An open problem is an existence of these considerations to fractional linear systems [Kaczorek 2011c].

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