DECISION MAKING SUPPORT AND EXPERT SYSTEMS

STUDY RELATIONSHIP BETWEEN UTILITY FUNCTION AND MEMBERSHIP FUNCTION IN THE PROBLEM OF OBJECT RANKING[1]

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***Abstract**: We consider the condition of the same ordering of objects due to using the convolution of utility functions and membership functions. It turns out that the same order takes place when the each utility function of an attribute is constructed as a convolution of fuzzy membership functions of this attribute. To transform membership functions to utility function of an attribute the formula was deduced. The example illustrates that transforming. The paper provides examples of application of the transformation of membership functions in the utility function, and vice versa.*

***Keywords**: utility function, membership function, additive and multiplicative convolution.*

***ACM Classification Keywords**: G. Mathematics of Computing, I.2.1 Applications and Expert Systems.*

Introduction

In monograph [Mikoni, 2004] the author posed the problem of objects ranking based on the results of their classification. Naturally, such task can be solved only if the classes are ranked by quality. This means that each class $h_k, k = \overline{1,m}$ having an intermediate level of quality has only two adjacent class - with the best (h_{k+1}) and worst (h_{k-1}) level of quality. Thus, the best of the two objects x_s and x_t with the same value of membership function of the neighboring classes h_k and h_{k+1}, $\mu_k(x_t)=\mu_{k+1}(x_t)$, will be that one which belongs to the class with best quality level: $x_t \succ x_s$. The quality level of classes is expressed through the coefficients of importance: $p_{k+1} > p_k > p_{k-1}$, $\sum_{k=1}^{m} p_k = 1$. Considering the importance of classes the preference $x_t \succ x_s$ will take place if $\mu_{k+1}(x_t)\cdot p_{k+1} > \mu_k(x_s)\cdot p_k$. It follows that the ratio of estimates of the objects obtained with the membership functions depends on the ratio of the importance of classes. This dependence is taken into account in this paper when searching for the general conditions of matching the results of objects ranking based on the classification results and obtained by the methods of multicriteria utility theory.

[1] The work had been fulfilled under Russian Fundamental Research Fund financial support (project № 10-01-00439)

The objects ranking with the utility functions

The following describes the objects ranking with the methods of multicriteria utility theory. First, for a given j-th criterion, $j=\overline{1,n}$, an utility function $u(y_j)$ is constructed. Its form can be both linear and nonlinear. The linear form is obtained by normalizing values of the criterion with the range of its scale. The usefulness of the j-th criterion, requiring the maximization, is calculated using the following formula:

$$u_{\max}(y_j) = \frac{y_j - y_{j,\min}}{y_{j,\max} - y_{j,\min}}, \qquad j=\overline{1,n}.$$

More complex, piecewise linear and nonlinear utility functions are constructed with expert data.

To convert a vector object evaluation $\mathbf{y}(x_i)=(y_{i1},\ldots,y_{ij},\ldots,y_{in})$ to a scalar evaluation additive or multiplicative convolution is commonly used:

$$u_a^*(x_i) = f(\mathbf{y}) = \sum_{j=1}^{n} w_j u_j(x_i), \tag{1.1}$$

$$u_m^*(x_i) = \prod_{j=1}^{n} u_j(x_i)^{w_j}. \tag{1.2}$$

Based on their scalar estimates $y_a(x_i)$ or $y_m(x_i)$ the objects $x_i \in X$ are assigned ranks in ordinal scale.

The objects ranking with the membership functions

The following describes the objects ranking with the membership functions. First, the membership functions for each criterion are constructed using the experts. For this purpose the scale of the j-th criterion is divided into m ranges according to the number of classes. In the general case there is a nonempty intersection of the ranges allocated to neighboring classes, which is similar to fuzzy boundaries between them:

$[c_{kj,\min}, c_{kj,\max}] \cap [c_{k+1,j,\min}, c_{k+1,j,\max}] \neq \varnothing$.

Then for each object is computed its membership to each of classes on all criteria taking into accounts their importance w_j:

$$\mu_k(x_i) = \sum_{j=1}^{n} w_j \mu_{jk}(x_i), \quad k=\overline{1,m}, \tag{2.1}$$

$$h^*(x_i) = \arg\max_k \mu_k(x_i),$$

In the last expression h^* is the class which the object x_i belongs to stronger than the other ones. Then with the help of experts the importance of classes are assessed p_k, = $\overline{1,m}$, after which the estimate $y^*(x_i)$ of the object x_i is computed from its values of membership functions according to importance of classes:

$$y^*(x_i) = \sum_{k=1}^{m} p_k \mu_k(x_i). \tag{2.2}$$

Objects ranking is simply a sorting with values $y^*(x_i)$.

The condition of matching the results of objects ranking by utility functions and membership functions

The obvious way to achieve identical results of the ordering is to establish correspondence between the utility function and membership functions of each criterion [Mikoni, Garina, 2010]. Using the formula (1.1) (2.1) and (2.2) let us find out the conditions under which such a correspondence can be established:

$$\begin{aligned}
&u_a^*(x_i) = y^*(x_i)\\
&\sum_{j=1}^{n} w_j \cdot u_j(x_i) = \sum_{k=1}^{m} p_k \cdot \mu_k(x_i)\\
&\sum_{j=1}^{n} w_j \cdot u_j(x_i) = \sum_{k=1}^{m} p_k \cdot \sum_{j=1}^{n} w_j \mu_{jk}(x_i) = \sum_{j=1}^{n} w_j \sum_{k=1}^{m} p_k \mu_{jk}(x_i)\\
&u_j(x_i) = \sum_{k=1}^{s} p_k \mu_{jk}(x_i).
\end{aligned} \tag{3.1}$$

Since the domain of the utility function $u(y_j)$ includes the domains of the membership functions of all classes, it is possible to calculate the utility function $u(y_j)$ on the basis of class membership functions on the with (3.1). In this estimates of objects will be identical and, therefore, objects ranking results will be identical too. Fig. 1 shows an example of constructing a utility function $u(y_j)$ of j-th criterion based on three classes of quality with trapezoidal membership functions.

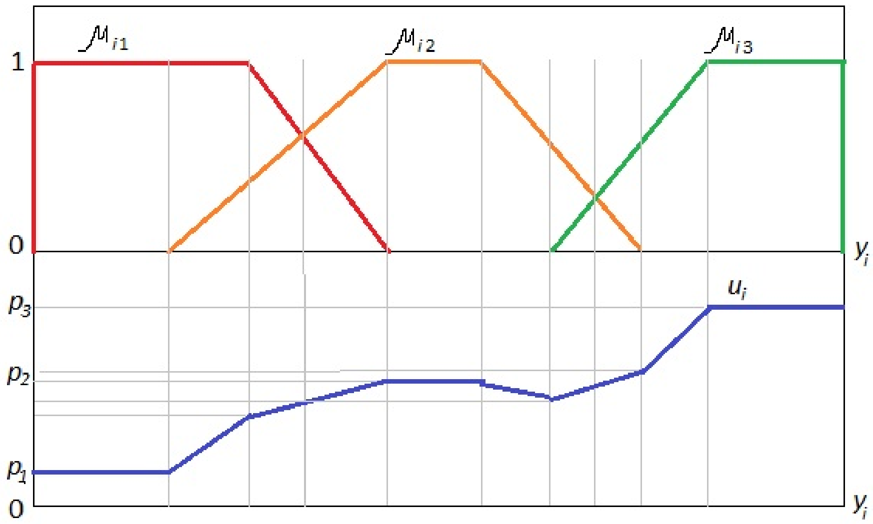

Fig. 1. Utility function constructed with (3.1)

On the border between the 2nd and 3rd classes utility function is non-monotonic, as in this area $\mu_{j2}(x_i)+\mu_{j3}(x_i)<1$, i.e. the requirement of mutual complementarities does not satisfied. Using the multiplicative convolution to compute the global membership is generally impractical, because in the areas of belonging to a class full zero affiliation other classes shall be reset generalized affiliation to them. The condition of application of a multiplicative convolution in classification is a common domain for all classes. In this case the ranking results will match when the utility function $u(y_j)$ is calculated by the formula $u_j = \prod_{k=1}^{s} \mu_{ik}^{\ p_k}$ (proof omitted). Utility functions will then be a piecewise polynomial.

Applications

According to (3.1) the reverse transition from the utility function to membership functions is ambiguity. A unique solution to this problem is possible only for a single membership function under certain other membership functions. Thus, suppose the membership function of *l*-th class is unknown, *l*≠*k*, $k = \overline{1,m}$. Then with the known membership functions and utility function of *j*-th criterion the membership function of *l*-th class is calculated using the following formula:

$$\mu_{j,l}(x_i) = \frac{u_j(x_i) - \sum_{k=1, l\neq k}^{m} p_k \cdot \mu_{j,k}(x_i)}{p_l}.$$

This formula is applied when the additive convolution is using. For multiplicative convolution see the following:

$$\mu_{j,l}(x_i) = \sqrt[p_l]{\frac{u_j(x_i)}{\prod_{k=1, l\neq k}^{m} p_k \cdot \mu_{j,k}(x_i)}}$$

Another application is to restore the importance or weight vector of the classes. Let us represent the utility function, membership functions and the importance of classes in the vector form:

$$\mathbf{U} = \begin{pmatrix} u_1(x_i) \\ \dots \\ u_j(x_i) \\ \dots \\ u_n(x_i) \end{pmatrix}; \quad \mathrm{M} = \begin{pmatrix} \mu_{11}(x_i) & \mu_{1j}(x_i) & \mu_{1n}(x_i) \\ \mu_{k1}(x_i) & \mu_{kj}(x_i) & \mu_{kn}(x_i) \\ \mu_{m1}(x_i) & \mu_{mj}(x_i) & \mu_{mn}(x_i) \end{pmatrix}; \quad \mathrm{P} = \begin{pmatrix} p_1 \\ \dots \\ p_k \\ \dots \\ p_m \end{pmatrix}$$

Let us represent the formula $u_j = \prod_{k=1}^{s} \mu_{ik}^{\ p_k}$ in matrix form: $\mathbf{U} = \mathbf{P}^T \cdot \mathbf{M}$. The solution of system of linear algebraic equations for the non-singular matrix **M** at a fixed point x_i is a vector of weights **P**. Since there are *n* solutions of this system by the number of objects x_i, it is advisable to determine the weight vector for the best (or reference) object *x**.

Conclusion

The condition of matching objects ranking by multi-criteria optimization and classification is to compute the utility functions on the base of given membership functions of classes. The number of classes should be the same for all criterions. The multi-criteria utility functions and functions that calculate the utility based on the membership functions should have the same structure.

The value of weights of classes when ranking on the results of the classification is proportional to the quality of the classes. For complementarily classes the utility function is a monotonic.

The use of a multiplicative convolution in the general case is difficult because there are the different domains of the membership functions of the classes. Zero membership of at least one class leads a zero value of utility function. However, if the using of multiplicative convolution is justified, then to match the results utility functions should also be calculated by a given membership function.

The reverse transition from utility function defined on the entire scale of the criterion to the functions of membership defined on its parts could not be unambiguously. The procedure is unambiguous only for one class recovery when other classes and utility function are certain. Another task deriving from the considered condition is finding the vector of importance of classes by the certain utility function and membership functions.

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