DIVERGENT AND MULTIPLE-VALUED SEQUENCES AND FUNCTIONS

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Abstract*: On the basis of the theory of hyper-random phenomena, the calculus approaches for divergent and multiple-valued sequences and functions have been developed. For divergent sequences and functions, a number of new concepts, in particular concepts of generalized limit, spectrum of limit points, convergence to spectrum of limit points, distribution function and density function of limit points, bounds of a distribution function etc. are introduced. Concepts known for single-valued functions, in particular concepts of convergence, continuity, derivative, differentiability, indefinite and definite integrals are generalized on multiple-valued functions. A link between multiple-valued and divergent functions is found. Transformation particularities of single-valued functions into multiple-valued functions and multiple-valued functions into single-valued ones are researched.*

Keywords*: divergent function, partial limit, multiple-valued function, divergent integral, theory of hyper-random phenomena.*

ACM Classification Keywords: *G.3 Probability and Statistics*

1. Introduction

The limit and convergence concepts are considered as basics of modern mathematics. Main mathematical results have been obtained on the basis of them. Concepts of uniform convergence, continuous function, derivative, integral, and so on are introduced by them.

Important request to classic definitions as of limit of the function determined in a set of real numbers, as of convergence of the numerical sequence to a limit, is obligatory to the existence of just a single limit. If there is no just a single limit, it is said that the function or the sequence has no limit or that there is a divergence of it.

Not all sequences and functions have limits. Interesting, in the real physical world a lot of processes are divergent ones. Such processes, for instance, are flicker noises, statistically instable processes [Gorban, 2012-2], chaotic processes with strange attractors, and others ones.

The lack of convergence is a serious problem related to many mathematical objects. However, it is not researched well yet. Mainly it is discussed in the limit theory and in connection with convergence disturbances of series and integrals [Ильин, 1985, Корн, 1973, Фихтенгольц, 1958, Харди, 1951].

A divergent numerical sequence $\{x_n\}$, terms x_n of which are alternately increasing and decreasing, when number n tends to infinity and also a divergent function $x(t)$, whose value is fluctuated in some bounds, when t tends to definite value t_0 are of especial interest.

The lack of convergence does not mean that it is impossible to say anything about the behavior of the sequence $\{x_n\}$, when $n \to \infty$ or about the behavior of the function $x(t)$ when $t \to t_0$. We should emphasize that the limit is only one parameter from the set of them characterizing a sequence or a function under the limit passage.

Research of different physical processes in large observation intervals showed [Gorban, 2007, 2010, 2011-1, 2011-2, 2011-3, 2012-2] that in the overwhelming majority of cases their sample averages were not convergent. Search of effective methods for description of such processes led to the new physical-mathematical theory of hyper-random phenomena [Gorban, 2007, 2011-1] oriented on learning the physical phenomena that could not be characterized by single-valued probability characteristics. Researching physical objects of this theory are statistically unstable physical phenomena, in particular physical magnitudes, the variance of sample average of which does not tend to zero, when the sample size tends to infinity. In this theory, to describe physical phenomena instead of concrete probability parameters and characteristics, a set of their possible alternatives are used. Main abstract mathematical objects of the theory are hyper-random variable, which is the set of random variables and hyper-random function (the latter being the set of random functions). Probability measure of these random variables and functions depends on a parameter which value belongs to a finite, countable, or non-countable set.

Research shows [Gorban, 2012-1, 2012-3] that the methods of this theory may be effectively used for solving many tasks including those being far away from the statistic tasks, in particular for the description of divergent deterministic sequences and functions, as well as of closely coupled with them multiple-valued variables, sequences, and functions.

The purpose of the article is to present new scientific results in these two directions.

2. Generalized limits of single-valued sequences and functions

In [Gorban, 2012-1], the new concept of convergence for an unlimited single-valued numerical sequence $\{x^n\}^{n\to\infty} = x^1, x^2, \ldots, x^n, \ldots$ has been introduced.

According to the classic view, the sequence $\{x^n\}^{n\to\infty}$ is the convergent one if just a single limit $a = \lim\limits_{n\to\infty} x^n$ exists. The sequence that does not have just a single limit is the divergent one.

Divergent sequences may be of different types.

It is known, that from any infinite sequence a set of partial sequences obtaining from the initial sequence by deletion of some its terms can be formed.

It has been proven that if the sequence is a converged one, then all its partial sequences are converged ones too. If the sequence is a divergent one, then not necessarily all its partial sequences are divergent ones. Some of them can converge to the definite limit points a_m (accumulation points or partial limits). The set of all limit points of the sequence has been called the spectrum $\tilde{S}_x$ [Gorban, 2012-1].

The spectrum of a divergent sequence is an analog of the limit of a convergent sequence. Analytically it can be written by the following expression:

$$\tilde{S}_x = \operatorname*{LIM}_{n\to\infty} x^n , \tag{1}$$

where in contrast to the classic limit $\lim\limits_{n\to\infty}$ the multiple-valued generalized limit $\operatorname*{LIM}\limits_{n\to\infty}$ [Gorban, 2012-1] is used.

The spectrum $\tilde{S}_x$ is characterized by the distribution function of limit points [Gorban, 2012-1]

$$\tilde{F}(x) = \operatorname*{LIM}_{n\to\infty} \frac{n(x)}{n} , \tag{2}$$

where $n(x)$ is a number of terms of the sequence $\{x^n\} = x^1, x^2, \ldots, x^n$ that are less than x.

The generalized limit (2) can be converged to a number (fig. 1a), converged to a set of numbers (fig. 1b), or can be a divergent one (fig. 1c). In the first two cases, the distribution function $\tilde{F}(x)$ is a single-valued one ($\tilde{F}(x) = F(x)$) and in the third case – a multiple-valued one.

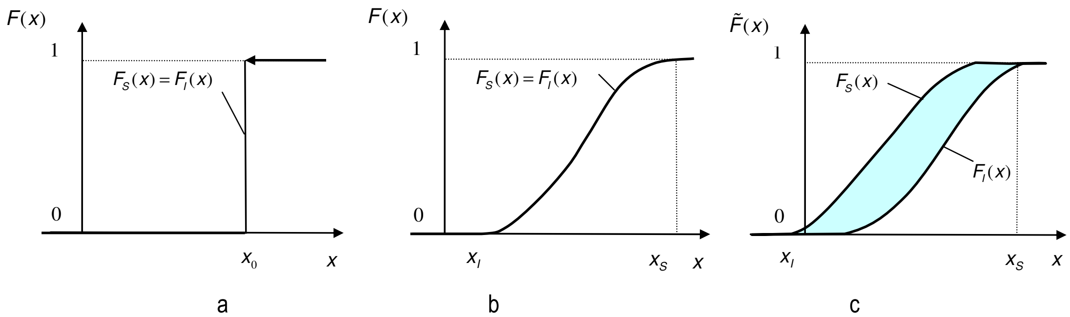

Fig. 1. The distribution function of limit points $\tilde{F}(x)$ and its bounds $F_I(x)$, $F_S(x)$ for the single-valued sequence $\{x^n\}^{n\to\infty}$ converging to the number x_0 (a), converging to the set of numbers (b), and for the divergent sequence (c)

For description of the distribution function $\tilde{F}(x)$, its low $F_I(x)$ and upper $F_S(x)$ bounds (fig. 1) can be used as well as density functions of bounds $f_I(x)=\frac{dF_I(x)}{dx}$, $f_S(x)=\frac{dF_S(x)}{dx}$, moments of bounds: expectations of bounds m_I, m_S, variances of bounds D_I, D_S, and other characteristics and parameters of the theory of hyper-random phenomena.

Similarly, it has been introduced [Gorban, 2012-1] the concept of convergent of a single-volume function $x(t)$ from the left-hand side (when $t\to t_0-0$) and the right-hand side (when $t\to t_0+0$) to sets (spectrums) of limit points $\tilde{S}_x^-(t_0)=\mathop{\mathrm{LIM}}\limits_{t\to t_0-0} x(t)$ and $\tilde{S}_x^+(t_0)=\mathop{\mathrm{LIM}}\limits_{t\to t_0+0} x(t)$ respectively.

Sets $\tilde{S}_x^{\pm}(t)$ can be characterized by multiple (in a general case) distribution functions of limit points $\tilde{F}^{\pm}(x;t)$.

For description of the distribution functions $\tilde{F}^{\pm}(x;t)$ it can be used their low and upper bounds $F_I^{\pm}(x;t)$, $F_S^{\pm}(x;t)$, distribution functions of bounds $f_I^{\pm}(x;t)=\frac{dF_I^{\pm}(x;t)}{dx}$, $f_S^{\pm}(x;t)=\frac{dF_S^{\pm}(x;t)}{dx}$, moments of bounds: expectations of bounds $m_I^{\pm}(t)$, $m_S^{\pm}(t)$, variances of bounds $D_I^{\pm}(t)$, $D_S^{\pm}(t)$, and other characteristics.

3. Multiple-valued variables and functions

In mathematics, single-valued and multiple-valued variables and functions are recognized. A single-valued variable has a concrete value and a multiple-valued – a set of values. A single-valued function sets a single meaning point-to-point correspondence and a multiple-valued function – a multiple meanings correspondence.

It will be supposed that all values of a multiple-valued variable, an argument, and values of multiple-valued functions are scalar real values.

A particular case of a multiple-valued function is the multiple-valued number sequence that is the multiple-valued function of an integer argument. Another particular case of a multiple-valued function is a multiple-valued variable that is the multiple-valued function, the applicable domain of which is a number.

We will mark a multiple-valued particularity by the tilde above a letter identifying a multiple-valued variable, sequence, or function.

Different approaches for description of multiple-valued variables and functions are known. One of them widely used in the trigonometry, in the theory of functions of complex variable, and others mathematical directions is based on the branch concept.

A branch of the function is defined [Корн, 1973] as a single-valued continues function in its applicable domain.

Multiple meanings of the function is interpreted as a heightened dimension of its actual range or applicable domain.

In the first case, the multiple-valued function $\tilde{x}(t)$ of an argument t is regarded as a parametrically specified single-valued continues function $x_g(t)$, the parameter of which $g \in G$ (where G is a finite, countable, or non-countable set) characterizes the branch of the function $\tilde{x}(t)$.

In the second case, the multiple-valued function $\tilde{x}(t)$ is presented as the single-valued continues function $x(t,g)$ of two arguments: t and g. When the argument g is fixed, the dependence from the argument t is a continues function, which is presented as the g-th branch of the function $\tilde{x}(t)$.

So in both cases, the function $\tilde{x}(t)$ is regarded as a finite, countable, or non-countable set of branches. Notice that branches of a multiple-valued function can have common points, applicable domains of branches can be different, and there are a lot of branch decomposition variants.

Description of multiple-valued functions by branches is very comfortable and visual, especially when a quantity of branches is a finite or countable one. When a branch number does not a countable, an ability of visualization is lost and an extraction of branches becomes a problem.

An original approach for description of multiple-valued variables and functions proposes the probability theory. This theory is oriented on the study of statistical stable physical objects, in particular physical magnitudes that have sample averages trending to definite values, when a sample number tends to infinity. Ones of the main mathematical objects of the probability theory are a random variable and a random function. A random variable can be regarded as a multiple-valued variable, for which there is a probability measure (distribution function). A random function is interpreted or as a set of random variables depended from the argument of the function, or as a set of single-valued realizations of the multiple-valued function, for which there are a probability measure.

The theory of hyper-random phenomena [Gorban, 2011-1] opens new possibilities for description of multiple-valued variables and functions.

4. Description of multiple-valued variables and functions on the base of the theory of hyper-random phenomena

Let a variable $x(p)$ depends from a parameter $p \in P$ where P is a neighbourhood of the point p_0. We shall propose that for all fixed $p \neq p_0$ the variable has a single value. Mark, in the point p_0 the variable may be a multiple-volume one.

Examine the single-valued sequence $\{x^n\} = x^1, x^2, \ldots, x^n$ obtained from the variable $x(p)$ when $p \to p_0$. The generalized limit $\underset{n\to\infty}{\mathrm{LIM}}\, x^n = \underset{p\to p_0}{\mathrm{LIM}}\, x(p)$ of the sequence can tend to a number, to a set of numbers, or be the divergent limit.

Definition 1. A multiple-valued determine variable $\tilde{x}$ can be regarded as a generalized limit $\underset{n\to\infty}{\mathrm{LIM}}\, x^n$ presented by the spectrum $\tilde{S}_{\tilde{x}}$ and the distribution function $\tilde{F}(x)$.

Mark, the spectrum of the multiple-valued variable $\tilde{x}$ does not necessary coincide with the values of the variable $x(p)$ in the point p_0.

The distribution function $\tilde{F}(x)$ can be as a single-valued as a multiple-valued one. If it is a single-valued ($\tilde{F}(x) = F(x)$), the density function $f(x) = dF(x)/dx$ exists.

More strictly, the multiple-valued variable can be defined by an assistance of the space that is the analog of the hyper-random space described by the tetrad $(\Omega, \Im, G, P_g)$ [Gorban, 2011-1], where Ω is a set of simple events (for instance, values x of the multiple-valued variable $\tilde{x}$), $\Im$ is a sigma field, G is a set of conditions $g \in G$, and P_g is a measure of event's subset that depends from conditions g.

Definition 1a. A multiple-valued determine variable $\tilde{x}$ can be regarded as a mathematical object specifying by the spectrum of values $\tilde{S}_{\tilde{x}}$ and the distribution function $\tilde{F}(x)$.

In a particular case, when there is single conditions, the multiple-valued variable can be defined by assistance of the space similar to the probability space described by the triad $(\Omega, \Im, P)$ [Колмогоров, 1974], where P is a measure of event's subset. Then a multiple-valued variable can be presented by the spectrum $\tilde{S}_{\tilde{x}}$ and the single-valued distribution function $F(x)$.

A multiple-valued determine function $\tilde{x}(t)$ can be specified as a set of multiple-valued variables according to fixed values of the argument t. Therefore, a function $\tilde{x}(t)$ can be characterized by the spectrum $\tilde{S}_{\tilde{x}}(t)$ and the distribution function $\tilde{F}(x;t)$.

Mark, presented single-valued distribution functions $F(x)$, $F(x;t)$ and according density functions $f(x)$, $f(x;t)$ are similar to correspondently distribution probability functions and probability density functions of the random variable and the random function. The properties of the researched characteristics are the same as the properties of probability analogues.

5. Spectrums of multiple-valued sequences and functions

Definition 2. A single-valued sequence formed from the multiple-valued finite $\{\tilde{x}_i\} = \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_i$ or infinite $\{\tilde{x}_i\}_{i\to\infty}$ sequence by casting-out a part of terms and retaining in the rest terms only one value in every of them will be called the subsequence (partial sequence) of the sequence.

Definition 3. A limit a_m (a number) of a single-valued m-th partial sequence formed from the multiple-valued sequence $\{\tilde{x}_i\}_{i\to\infty}$ will be called the m-th partial limit (accumulation point) of the sequence.

Definition 4. A limit (a number) of a single-valued m-th partial sequence formed from the multiple-valued function $\tilde{x}(t)$ with finite volumes, when $t \to t_0 - 0$ or $t \to t_0 + 0$ will be called the m-th partial limit (accumulation point) of the function.

Note, not all single-volume partial sequences are converged (have single limits). Therefore not all single-volume partial sequences of a multiple-valued sequence or a multiple-valued function have single limit points.

For multiple-valued sequences and functions the analogue of usual limit is the set (spectrum) of limit points.

Let $\tilde{S}_{\tilde{x}}$ is the set (spectrum) of limit points of a multiple-valued sequence $\{\tilde{x}_i\}_{i\to\infty}$ and $\tilde{S}^{-}_{\tilde{x}}(t_0)$, $\tilde{S}^{+}_{\tilde{x}}(t_0)$ are the left- and right-hand sides sets (spectrums) of limit points of a multiple-valued function $\tilde{x}(t)$, when $t \to t_0 - 0$ and $t \to t_0 + 0$ respectively.

Analytically, the convergence of the multiple-valued sequence to the set of limit points can be written by the expression $\tilde{S}_{\tilde{x}} = \operatorname*{LIM}_{i\to\infty} \tilde{x}_i$ and the convergence of the multiple-valued function to the left- and right-hand sides sets of limit points – by the expressions $\tilde{S}^{-}_{\tilde{x}}(t_0) = \operatorname*{LIM}_{t\to t_0-0} \tilde{x}(t)$ and $\tilde{S}^{+}_{\tilde{x}}(t_0) = \operatorname*{LIM}_{t\to t_0+0} \tilde{x}(t)$.

Limit points of a multiple-valued sequence $\{\tilde{x}_i\}_{i\to\infty}$ are bounded by the interval $[x_I, x_S]$, where x_I, x_S are low and upper bounds of limit points of the sequence. Limit points of the multiple-valued function $\tilde{x}(t)$, when $t \to t_0 - 0$ or $t \to t_0 + 0$ are bounded by correspondently intervals $[x_I^-(t_0), x_S^-(t_0)]$ and $[x_I^+(t_0), x_S^+(t_0)]$, where $x_I^-(t_0)$, $x_S^-(t_0)$ are low and upper bounds of the left-hand side limit points of the function and $x_I^+(t_0)$, $x_S^+(t_0)$ are low and upper bounds of the right-hand side limit points of it.

Mark, the request to finite volumes of the function and to volumes of the argument t_0 figured in Definition 4 is not essential one. By the same manner, the concepts of a set of limit points for an unbounded multiple-valued function and for a multiple-valued function $\tilde{x}(t)$ when $t \to +\infty$ or $t \to -\infty$ can be defined too.

6. The distribution function of the multiple-valued sequence

Every term $\tilde{x}_j$ of a multiple-valued finite sequence $\{\tilde{x}_i\}$ ($j = \overline{1,i}$) can be presented as the generalized limit $\operatorname*{LIM}_{n\to\infty} x_j^n$ of the generated sequence $\{x_j^n\} = x_j^1, x_j^2, \ldots, x_j^n$ and described by the distribution function

$$\tilde{F}_j(x) = \operatorname*{LIM}_{n\to\infty} \frac{n_j(x)}{n},$$

where $n_j(x)$ is a number of terms of the sequence $\{x_j^n\}$ that are less than x.

The spectrum $\tilde{S}_{\tilde{x}_i}$ of the sequence $\{\tilde{x}_i\}$ can be described by the distribution function

$$\tilde{F}^i(x) = \operatorname*{LIM}_{n\to\infty} \frac{\sum_{j=1}^{i} n_j(x)}{ni} = \frac{1}{i}\sum_{j=1}^{i} \tilde{F}_j(x).$$

Definition 5. The distribution function of limit points of a sequence $\{\tilde{x}_i\}_{i\to\infty}$ will call the function

$$\tilde{F}(x) = \operatorname*{LIM}_{i\to\infty} \tilde{F}^i(x) = \operatorname*{LIM}_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} \tilde{F}_j(x) = \operatorname*{LIM}_{i\to\infty}\operatorname*{LIM}_{n\to\infty} \frac{\sum_{j=1}^{i} n_j(x)}{ni}.$$

Pay attention, that the function $\tilde{F}(x)$ may be a multiple-volume one. If for all $x \in (-\infty, +\infty)$ it is a single-volume one then $\tilde{F}(x) = F(x)$.

Examined distribution function of limit points $\tilde{F}(x)$ is similar to the distribution function of limit points (2) of a single-volume sequence.

We will recognize multiple-volume sequences $\{\tilde{x}_i\}_{i\to\infty}$ converging to a number, to a set of numbers and divergent multiple-volume sequences.

Definition 6. A multiple-valued sequence $\{\tilde{x}_i\}_{i\to\infty}$ will be called the converging sequence to the number x_0 if the distribution function $\tilde{F}(x)$ of limit points is described by the single jump function $\mathrm{sign}[x - x_0] = \begin{cases} 0, & \text{when} \quad x < x_0, \\ 1, & \text{when} \quad x \geq x_0 \end{cases}$ in the point x_0, the converging sequence to the set of numbers (in particular to the interval) if the distribution function is described by the single-volume function in the interval $(-\infty, +\infty)$, and the divergent sequence if distribution function has a multiple meanings even in a single point x.

So not only single-volume but multiple-volume sequences can be converging to a number, to a set of numbers, and can be divergent sequences.

As in a single-value sequence case, a distribution function $\tilde{F}(x)$ for a multiple-volume sequence can be characterized by the single-value low $F_I(x)$ and upper $F_S(x)$ bounds. If for all x the distribution function $\tilde{F}(x)$ has concrete volumes (the sequence converges to a number or to a set of numbers), then the bounds are coincided. In the contrary case we have $F_I(x) \neq F_S(x)$.

7. The distribution function of the multiple-valued function

Similar to the distribution function $\tilde{F}(x)$ of a multiple-valued sequence it can be introduced the concept of the left-hand side $\tilde{F}^-(x;t)$ and the right-hand side $\tilde{F}^+(x;t)$ distribution functions of limit points of the multiple-valued function $\tilde{x}(t)$ (fig. 2) that characterize the volume frequency of the function, when the argument tends to t from the left- and right-hand side respectively.

Mark, the distribution functions $\tilde{F}^\pm(x;t)$ may be as single-volume as multiple-volume ones.

Definition 7. A multiple-volume function $\tilde{x}(t)$ will be called a converging function from the left (when t is fixed) to the definite number $x^-(t)$ if its left-hand side distribution function of limit point $\tilde{F}^-(x;t)$ is described by the single jump function $\mathrm{sign}[x - x^-(t)]$ (fig. 2a), will be called a converging function from the left to the set of numbers if its left-hand side distribution function is described by a single-volume function for all $x \in (-\infty, +\infty)$ (fig. 2b), and will be called a divergent function from the left if according distribution function has a multiple meanings even in a single point x (fig. 2 c).

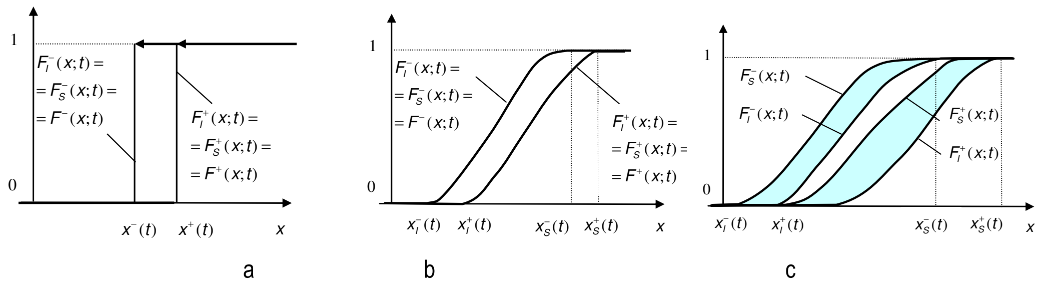

Fig. 2. Bounds of the distribution function of limit points of the multiple-valued function $\tilde{x}(t)$ that converges from the left- and right-hand side to the number (a), converges from the left- and right-hand side to the set of numbers (b) and bounds of the distribution function of the divergent function $\tilde{x}(t)$ (c).

Examining the right distribution function $\tilde{F}^+(x;t)$ according to the right-hand side limit of the function $\tilde{x}(t)$ we will recognize the multiple-volume function $\tilde{x}(t)$ converging from the right to the definite number $x^+(t)$ (fig. 2a), converging from the right to the set of numbers (fig. 2b) and the divergent function from the right (fig. 2 c).

The left- $\tilde{F}^-(x;t)$ and right-hand $\tilde{F}^+(x;t)$ side distribution functions can be characterized by single-volume bounds $F_I^-(x;t)$, $F_S^-(x;t)$ and $F_I^+(x;t)$, $F_I^+(x;t)$ respectively. If under $x \in (-\infty, +\infty)$ they have concrete volumes (the function $\tilde{x}(t)$ is converged to a number or to a set of numbers), then according bounds are coincided: $F^-(x;t) = F_I^-(x;t) = F_S^-(x;t)$, $F^+(x;t) = F_I^+(x;t) = F_S^+(x;t)$ (fig. 2a, 2b). In the contrary case they are different: $F_I^-(x;t) \neq F_S^-(x;t)$, $F_I^+(x;t) \neq F_S^+(x;t)$ (fig. 2c).

8. A continuous multiple-volume function

Definition 8. A multiple-volume function $\tilde{x}(t)$ will be called the continuous function in the point t from the left (right) if

1) the function is determined as in the point t as in the left-hand (right-hand) side area of this point,

2) the function is converged in the point t from the left-hand (right-hand) side to a number or a set of numbers (according distribution function is a single-volume one),

3) in the point t the left-hand $F^-(x;t)$ (the right-hand $F^+(x;t)$) side distribution function equals to the distribution function $F(x;t)$ in the point t: $F^-(x;t) = F(x;t)$ ($F^+(x;t) = F(x;t)$) (the distribution function $F(x;t)$ is continuous along t from the left (right)).

In any contrary case, the function will be called the discontinuous one in the point t from the left (right).

Definition 9. A multiple-volume function $\tilde{x}(t)$ will be called the continuous function in the interval (t_1, t_2), if it is a continuous in all points of this interval from the left and right.

For a continuous function $\tilde{x}(t)$, it can be written the following equality: $x_I^-(t) = x_I^+(t) = x_I(t)$, $x_S^-(t) = x_S^+(t) = x_S(t)$, where $x_I(t)$, $x_S(t)$ are low and upper bounds of the function $\tilde{x}(t)$ (fig. 3a, 3c).

For continuous functions new branch concept can be defined.

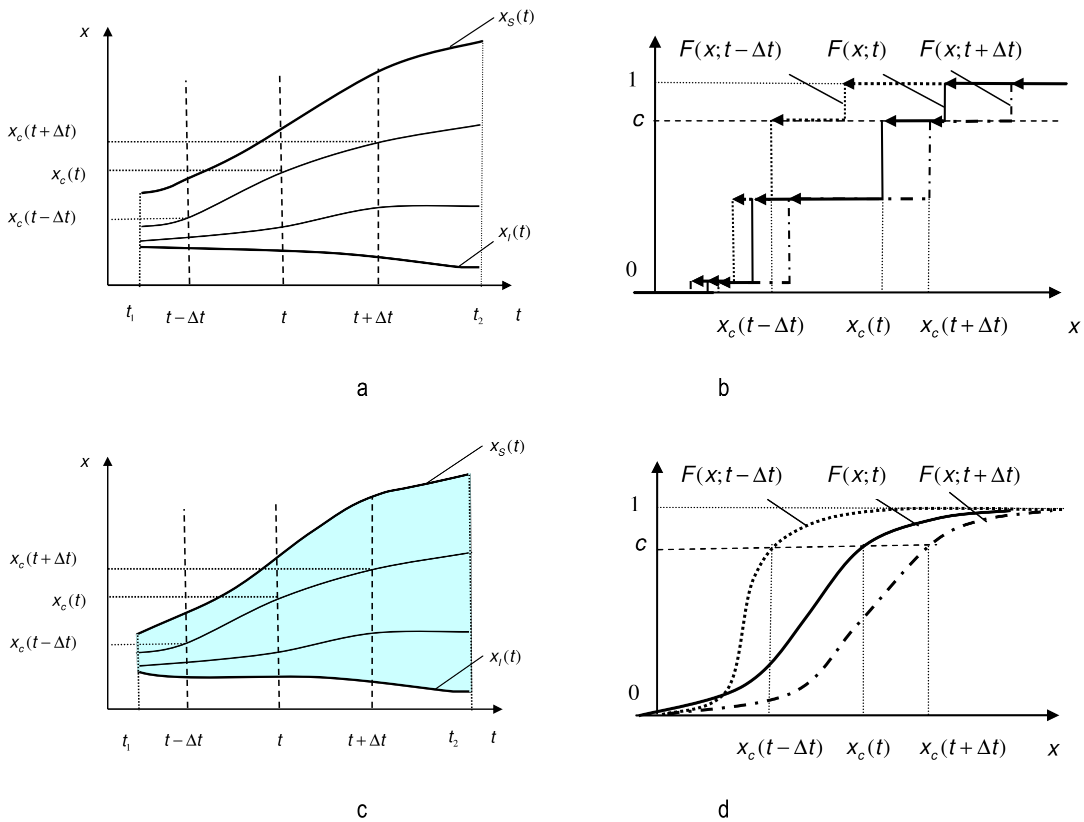

Fig. 3. Multiple-volume continuous functions $\tilde{x}(t)$ (a, c) and according sections of their distribution functions $F(x;t-\Delta t)$, $F(x;t)$, $F(x;t+\Delta t)$ in the points $t-\Delta t$, t, and $t+\Delta t$ (b, d). Thin continuous lines in the fig. 3a, 3c correspond to branches of the function $\tilde{x}(t)$ and bold lines – to its bounds.

Definition 10. The c-th branch ($c \in (0,1]$) of the function $\tilde{x}(t)$, determined in the interval $t \in (t_1, t_2)$ and describing by the distribution function $F(x;t)$ will call the single-volume function $x_c(t) = \inf\limits_x \arg\limits_x (F(x;t) = c)$ (fig. 3a, 3c).

For existence of the c-th branch in the interval $t \in (t_1, t_2)$, it is necessary and sufficient that the equation $F(x;t) = c$ has a solution (or solutions) for all $t \in (t_1, t_2)$ (fig. 3b, 3d).

The number of branches may be a finite, countable, or non-countable one.

If the number of branches is a finite (fig. 3a) or countable, then under $t = \mathrm{const}$ the distribution function $F(x;t)$ is a stepped function of the argument x (fig. 3b). If the number of branches is a non-countable one and for all $t \in (t_1, t_2)$ function volumes closely fill the interval $(x_I(t),\ x_S(t))$ (fig. 3c), then under $t = \mathrm{const}$ the distribution function $F(x;t)$ is a strictly increasing function of x (fig. 3d).

Theorem 1. The branches of a multiple-volume continuous function are continuous functions that have not common points.

The continuity of the branches of the multiple-volume continuous function follows from the continuity along t of its distribution function $F(x;t)$.

The absence of common points of branches can be proofed by ex adverso. Let the multiple-volume continuous function $\tilde{x}(t)$ described by the single-volume distribution function $F(x;t)$ has branches $x_{c_1}(t)$ and $x_{c_2}(t)$ ($c_2 \neq c_1$) that have a common point when $t = t_0$: $x_{c_1}(t_0) = x_{c_1}(t_0) = x_0$. It follows from this that in the point (x_0, t_0) the distribution function $F(x;t)$ has two different volumes: c_1 and c_2. This fact contradicts to the request of one-valuedness of the distribution function $F(x;t)$.

Definition 11. A multiple-volume function $\tilde{x}(t)$ that is continuous in the interval $t \in (t_1, t_2)$ will be called the decomposable one on branches, if it can be presented by a set of branches: $\tilde{x}(t) = \{x_c(t), \quad c \in C\}$, where C is a set of branches.

Notice, not all multiple-volume continuous functions can be decomposed on branches. If a function $\tilde{x}(t)$ is decomposed on branches, then it can be described by the set of branches C and the distribution function $F_c(x)$ of branches.

9. A derivative of the multiple-volume function

Definition 12. The left derivative $\tilde{x}'^{-}(t)$ of the multiple-volume continuous function $\tilde{x}(t)$ that can be decomposed on branches will call the set of the left derivatives

$$\tilde{x}_c'^{-}(t) = \operatorname*{LIM}_{\Delta t \to +0} \frac{x_c(t) - x_c(t - \Delta t)}{\Delta t}, \tag{3}$$

and the right derivative $\tilde{x}'^{+}(t)$ – the set of the right derivatives

$$\tilde{x}_c'^{+}(t) = \operatorname*{LIM}_{\Delta t \to +0} \frac{x_c(t + \Delta t) - x_c(t)}{\Delta t}, \tag{4}$$

calculated in the point t for all branches of $c \in C$.

Generalized limits in expressions (3) and (4) are not, in general, single-volume ones. They can converge to a set of numbers or be divergent limits.

If in expressions (3) and (4) the limits are single-volume ones ($\mathrm{LIM} = \lim$) for all $c \in C$, then $\tilde{x}_c'^{\pm}(t) = x_c'^{\pm}(t)$ and the derivative $x'^{-}(t)$ (derivative $x'^{+}(t)$) is the set of velocities, with that branches $x_c(t)$ are changing when the argument is tending to t from the left (right).

Definition 13. A multiple-volume continuous function $\tilde{x}(t)$ that can be decomposed on branches will be called the differentiable one in the point t, if all its derivatives on branches are single-volume ones and for all branches the left derivative equals to the right derivative: $x_c'^{-}(t) = x_c'^{+}(t) = x_c'(t)$.

Definition 14. A multiple-volume continuous function $\tilde{x}(t)$ that can be decomposed on branches will be called the differentiable one, if it is differentiated in all points of its applicable domain.

For a continuous derivative $\tilde{x}'(t)$ that can be decomposed on branches the second derivative $\tilde{x}''^{\pm}(t)$ can be defined. By iteration, for a continuous r-th derivative $\tilde{x}^{(r)\pm}(t)$ that can be decomposed on branches the $r+1$-th derivative $\tilde{x}^{(r+1)\pm}(t)$ can be defined too.

For the differentiable function $\tilde{x}(t)$ with the differentiable derivative $\tilde{x}'(t)$, the second derivative $\tilde{x}''^{-}(t)$ (derivative $\tilde{x}''^{+}(t)$) in the point t is the set of accelerations with that branches $x_c(t)$ are changing, when the argument is tending to t from the left (right).

A multiple-volume differentiable function $\tilde{x}(t)$ that in the point t_0 has single-volume derivatives $\tilde{x}^{(r)}(t_0)$ of any order r can be described by the set of branches $x_c(t)$, every of which is decomposed in the point t_0 in Taylor series. Therefore, the function $\tilde{x}(t)$ can be described by the set $\{x(t_0)\}$ of volumes of function in the point t_0, the set $\{x^{(r)}(t_0)\}$ ($r=1,2,\dots$) of volumes of its derivatives, and the set of according distribution functions $F(x;t_0)$, $F(x^{(r)};t_0)$ ($r=1,2,\dots$).

10. Examples of multiple-volume functions

Multiple-volume functions that are single-volume in all applicable domains except some intervals are especially interesting. Examples of such functions are presented in fig. 4a-4d.

These functions are single-volume ones in the intervals $t<t_1$, $t>t_2$ and multiple-volume ones in the interval $t_1 \le t \le t_2$. Multiple meanings of the function in fig. 4a appear spontaneously and spontaneously disappear. In other functions (fig. 4b-4d), the conversion to multiple meanings and then to single meaning is accompanied by the branch processes marked in the figures by dash lines. Part limits are formed in these zones.

Functions presented in fig. 4a and 4b are discontinues ones. If for all points of the applicable domain of functions presented in fig. 4c and 4d the conditions of the Definition 8 are satisfied, then the functions are continues ones. The functions in fig. 4a–4c are not differentiable ones and the function in fig.4d is differentiable one (if the last is a continuous function).

Examine the function

$$\tilde{x}(t) = \begin{cases} \sin\left(\dfrac{1}{\omega_1(t-t_1)}\right), & \text{when} \quad t<t_1, \\ [-1,1], & \text{when} \quad t_1 \le t \le t_2, \\ \sin\left(\dfrac{1}{\omega_2(t-t_2)}\right), & \text{when} \quad t>t_2, \end{cases} \tag{5}$$

that is a single-volume one in intervals $(-\infty,t_1)$, $(t_2,+\infty)$ and a multiple-volume one in the interval $[t_1,t_2]$, where $\omega_1 \ne 0$, $\omega_2 \ne 0$ (fig. 5a).

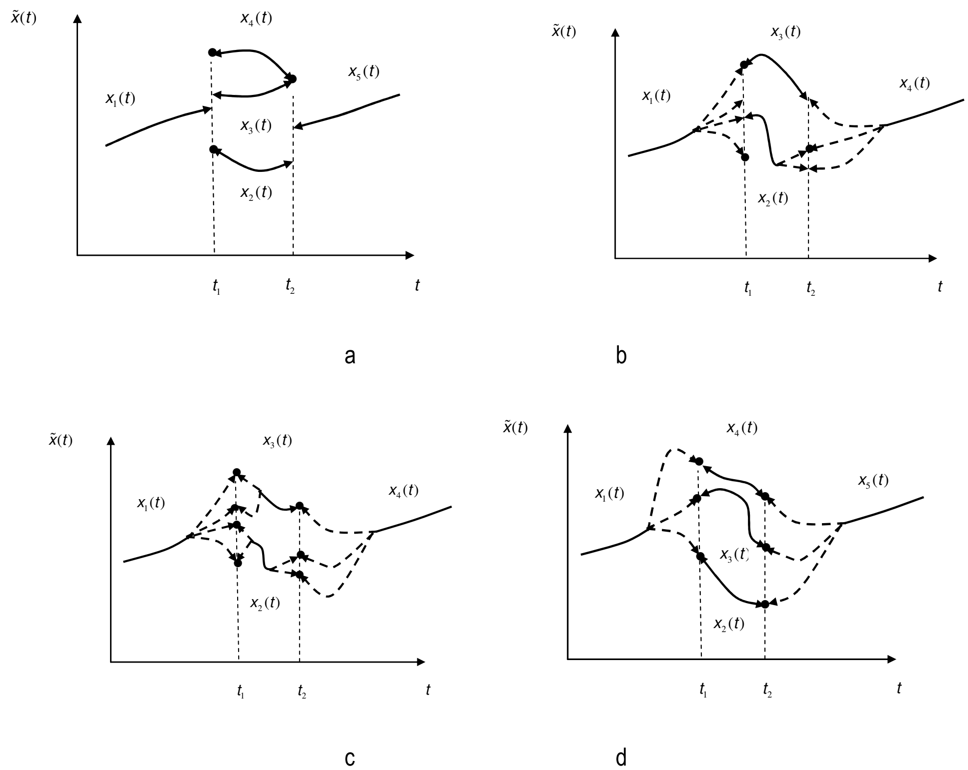

Fig 4. Multiple-volume functions: discontinuous (a, b) ones and continuous ones (if in all points of applicable domain the conditions of the Definition 8 are satisfied) (c, d)

When t tends to t_1 from the left-hand side and t tends to t_2 from the right-hand side, the single-volume parts of the function (5) are delaminated. In this case, left $F^-(x;t_1)$ and right $F^+(x;t_2)$ distribution functions are described [Gorban, 2012-1] by the expression

$$F^-(x;t_1) = F^+(x;t_2) = \frac{1}{2} + \frac{1}{\pi}\arcsin x \,. \tag{6}$$

Therefore, if in the interval $[t_1, t_2]$ the distribution function $F(x;t)$ is described by the same expression (6), then the function (5) is a continuous and differentiable one.

In this case, the derivative of the function (5) is a single-volume function for $t \in (-\infty, +\infty)$:

$$x'(t) = \begin{cases} -\dfrac{1}{\omega_1 (t-t_1)^2}\cos\left(\dfrac{1}{\omega_1 (t-t_1)}\right), & \text{when} \quad t < t_1, \\ 0, \quad \text{when} \quad t_1 \le t \le t_2, & \\ -\dfrac{1}{\omega_2 (t-t_2)^2}\cos\left(\dfrac{1}{\omega_2 (t-t_2)}\right), & \text{when} \quad t > t_2. \end{cases}$$

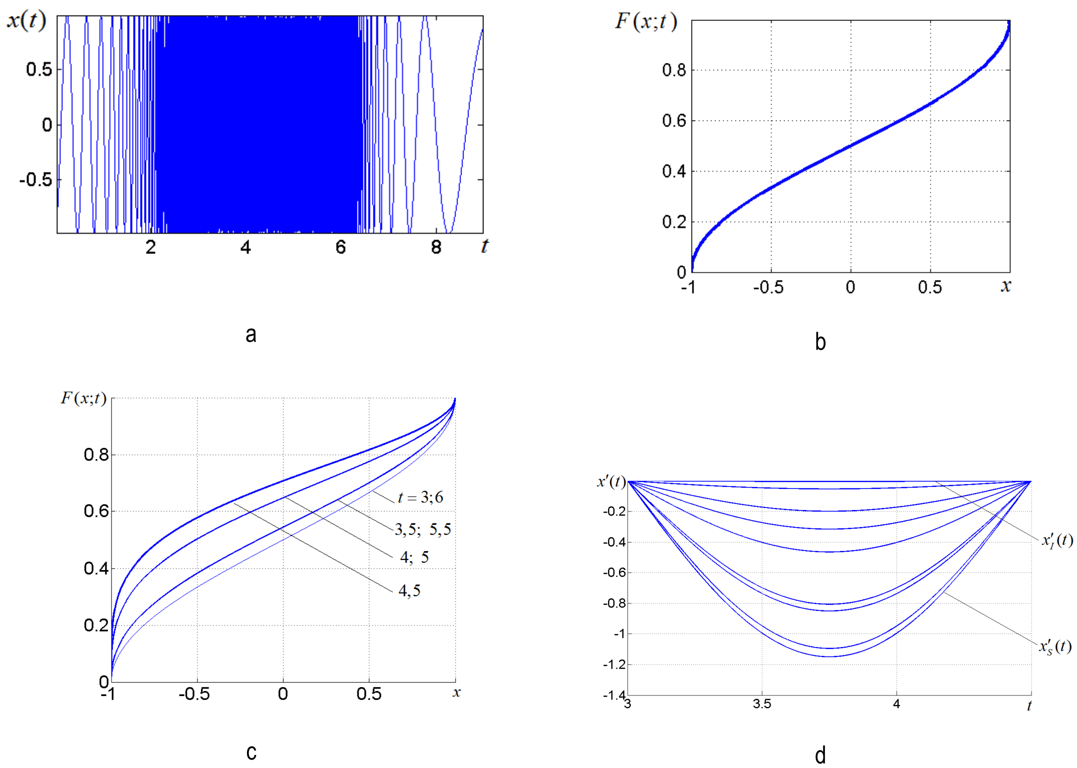

Fig. 5. The multiple-volume function (5) (a), its distribution functions (6) (b), (7) (c) in the interval $[t_1,t_2]$, and its derivative (8) (d) ($c=\overline{0,1}$ with the step 0,1): $t_1=3$, $t_2=6$, $\omega_1=10^{-2}$, $\omega_2=4\cdot 10^{-2}$

The distribution function of the derivative is the function $F(x';t)=\mathrm{sign}[x'-x'(t)]$.

If the distribution function $F(x;t)$ for $t\in[t_1,t_2]$ is described by the expression

$$F(x;t)=\left(\frac{1}{2}+\frac{1}{\pi}\arcsin x\right)^{a(t)}, \tag{7}$$

where $a(t)=\frac{3}{4}+\frac{1}{4}\cos\frac{2\pi(t-t_1)}{t_2-t_1}$ (fig. 5c), then the c-th branch $x_c(t)$ of the function $\tilde{x}(t)$ is described by the equation

$$\left(\frac{1}{2}+\frac{1}{\pi}\arcsin x_c(t)\right)^{a(t)}=c.$$

The solution of this equation is the function $x_c(t)=-\cos\pi\sqrt[a(t)]{c}$. Then, the derivative can be obtained in the following form:

$$x_c'(t)=\frac{\pi^2 c^{1/a(t)}\ln c}{2a^2(t)(t_2-t_1)}\sin\left(\pi c^{1/a(t)}\right)\sin\left(\frac{2\pi(t-t_1)}{t_2-t_1}\right).$$

So in this case, the function (5) is a continuous and differentiable one for any t. In the interval $[t_1,t_2]$ its derivative is multiple-volume one and described by the expression

$$\tilde{x}'(t) = \left\{ \frac{\pi^2 c^{1/a(t)} \ln c}{2a^2(t)(t_2 - t_1)} \sin\left(\pi c^{1/a(t)}\right) \sin\left(\frac{2\pi(t - t_1)}{t_2 - t_1}\right), \quad c \in (0,1] \right\}. \tag{8}$$

The distribution function of the derivative $\tilde{F}(x';t)$ can be as a single-volume one (then $\tilde{F}(x';t) = F(x';t)$) as a multiple-volume one.

11. The integral from the multiple-volume function

Definition 15. The multiple-volume differentiable function $\tilde{y}(t)$ determined in the interval $[a,b]$ will be called the primitive of the multiple-volume function $\tilde{x}(t)$ determined in the same interval, if in all its points the derivative $\tilde{y}'(t)$ equals to the function $\tilde{x}(t)$.

As any multiple-volume differentiable function (and therefore continuous and decomposable on branches), the primitive $\tilde{y}(t)$ is described by the set of volumes $\tilde{S}_{\tilde{y}}(t)$ and the distribution function $F(y;t)$ of volumes.

Definition 16. The indefinite integral from the multiple-volume function $\tilde{x}(t)$ determined in the interval $[a,b]$ will call the multiple-volume differentiable function $\int \tilde{x}(t)\,\mathrm{d}t = \tilde{y}(t) + C_0$, where C_0 is any constant.

Definition 17. The definite integral from the multiple-volume bounded continuous function $\tilde{x}(t)$ determined in the interval $[a,b]$ and decomposable on branches will call the set of limit points

$$\tilde{S}_{\tilde{y}} = \int_a^b \tilde{x}(t)dt = \left\{ \underset{\max \Delta t_i \to 0}{\mathrm{LIM}} \sum_{i=1}^{I} x_c(\xi_i)\Delta t_i, \quad c \in C \right\}, \tag{9}$$

where $a = t_0 < t_1 < \ldots < t_I = b$, $\Delta t_i = t_i - t_{i-1}$, $\tilde{x}(\xi_i)$ is the volume in any point $\xi_i \in [t_{i-1}, t_i]$, and low y_I and upper y_S bounds of the integral will call accordingly low and upper boundaries of this set.

The definite integral $\int_a^b \tilde{x}(t)dt$, as any set of limit points is described not only by the set of its volumes $\tilde{S}_{\tilde{y}}$ but also by the distribution function $\tilde{F}(y)$ that, in a general is a multiple-volume one.

In the particular case, when limits in the expression (9) are single-volume ones, the set of limit points

$$\tilde{S}_{\tilde{y}} = \int_a^b \tilde{x}(t)dt = \left\{ \int_a^b x_c(t)dt, \quad c \in C \right\}.$$

Mark, the Definition 17 can be generalized to nonintrinsic integrals too.

12. The generalized main volume of the definite integral

Introduced concept of the definite integral from the multiple-volume function can be useful for estimation of divergent integrals. Such tasks are occurred [Gorban, 2012-2], for instance, in the case of functional transformations.

Examine the continuous single-volume bounded function $x(t,\lambda)$ of the scalar argument t determined in the interval $[a(\lambda), b(\lambda)]$, where $\lambda \in \Lambda$ is a parameter. Let the integral of this function converges, when $\lambda \neq \lambda_0$ and diverges, when $\lambda = \lambda_0$.

Definition 18. The generalized main volume of the definite integral of the function $x(t,\lambda)$, when $\lambda = \lambda_0$ will call the multiple-volume (in general) set of limit points

$$\tilde{S}^{\circ}_{\tilde{y}} = \underset{\lambda \to \lambda_0}{\mathrm{LIM}} \int_{a(\lambda)}^{b(\lambda)} x(t,\lambda)dt,$$

and the distribution function of generalized main volume – the function

$$\tilde{F}(y) = \underset{\lambda \to \lambda_0}{\mathrm{LIM}} \frac{m_\lambda(y)}{m_\lambda},$$

where $m_\lambda(y)$ is the number of volumes of the integral $\int_{a(\lambda)}^{b(\lambda)} x(t,\lambda)dt$ that are less than y and m_λ is the common number of volumes of this integral.

The researched generalized main volume of the integral, in contrast to the main volume of a usual integral has a set of meanings. Bounds y_I, y_S of this integral are low and upper boundaries of the set of limit points $\tilde{S}^{\circ}_{\tilde{y}}$.

13. Conclusion

On the basis of the theory of hyper-random phenomena, the calculus approaches for divergent and multiple-valued sequences and functions have been developed.

For divergent sequences and functions, a number of new concepts, in particular concepts of generalized limit, spectrum of limit points, convergence to spectrum of limit points, distribution function and density function of limit points, bounds of a distribution function etc. are introduced.

Concepts known for single-valued functions, in particular concepts of convergence, continuity, derivative, differentiability, indefinite and definite integrals are generalized on multiple-valued functions.

A link between multiple-valued and divergent functions is found.

Transformation particularities of single-valued functions into multiple-valued functions and multiple-valued functions into single-valued ones are researched.

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