Automatic Control Systems Models

POLYNOMIAL APPROACH TO FRACTIONAL DESCRIPTOR ELECTRICAL CIRCUITS

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Abstract: *A new polynomial approach is proposed to analysis of the standard and positive descriptor electrical circuits composed of resistors, coils, capacitors and voltage (current) sources. It is shown that for given descriptor fractional electrical circuit the equivalent standard fractional electrical circuit can be found by premultiplication of the equation of the descriptor electrical circuit by suitable polynomial matrix of elementary row operations. The main result is demonstrated on simple positive fractional electrical circuit.*

Keywords: *Polynomial approach, descriptor, fractional, electrical circuits.*

ACM Classification Keywords: *I.2.8 Computing Methodologies – Control theory*

Introduction

Descriptor (singular) linear systems have been considered in many papers and books [Bru et. all, 2003a, 2003b; Campbell et. all, 1976; Dai, 1989; Guang-Ren, 2010; Kaczorek, 2004, 1992, 2011a, 2011e, 2011f, 2011g, 2013a, 2014a, 2014b; Virnik, 2008]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [Kaczorek, 2004] and the minimum energy control of descriptor linear systems in [Kaczorek, 1992]. In positive systems inputs, state variables and outputs take only non-negative values [Farina et. all, 2000; Kaczorek, 2002]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. The positive fractional linear systems and some of selected problems in theory of fractional systems have been addressed in monograph [Kaczorek, 2011f].

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in Bru et. all, 2003a, 2003b; Campbell et. all, 1976; Kaczorek, 2013a]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [Kaczorek, 2011a]. The stability of positive descriptor systems has been investigated in [Virnik, 2008]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [Kaczorek, 2011e]. Standard and fractional systems and electrical linear circuits have been investigated in [Kaczorek, 2002, 2008, 2010, 2011c, 2011f]. Pointwise completeness and pointwise generacy of standard and positive 1D and 2D systems have been addressed in [Kaczorek, 2009, 2011b].

In this paper a new polynomial approach to analysis of fractional descriptor electrical circuit will be proposed.

The paper is organized as follows. In section 2 basic definitions and theorems concerning the descriptor fractional and positive electrical circuits are recalled. The main result is presented in section 3, where a procedure for reduction of the descriptor fractional electrical circuits to the standard fractional electrical circuits is proposed. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Preliminaries

The following Caputo definition of the fractional derivative will be used [Kaczorek, 2011f]

$$D_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{f^{(p)}(\tau)}{(t-\tau)^{\alpha+1-p}} d\tau, \; p-1 \le \alpha < p \in N = \{1,2,...\}, \tag{1}$$

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(p)}(\tau) = \frac{d^p f(\tau)}{d\tau^p}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Consider the continuous-time fractional linear system described by the state equations

$$D_t^\alpha x(t) = Ax(t) + Bu(t), \;\; 0 < \alpha < 1, \tag{2a}$$

$$y(t) = Cx(t) + Du(t), \tag{2b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Theorem 1. [Kaczorek, 2011f] The solution of equation (2a) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0, \tag{3}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{4}$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}$$

and $E_\alpha(At^\alpha)$ is the Mittag-Leffler matrix function.

Definition 1. [Kaczorek, 2011f] The fractional system (2) is called the internally positive fractional system if and only if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$ for $t \ge 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 2. [Kaczorek, 2011f] The continuous-time fractional system (2) is internally positive if and only if the matrix A is a Metzler matrix and

$$A \in M_n, \;\; B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{6}$$

Let the current $i_C(t)$ in a supercondensator (shortly condensator) with the capacity C be the α order derivative of its charge $q(t)$ [Kaczorek, 2011f]

$$i_C(t) = \frac{d^\alpha q(t)}{dt^\alpha}, \; 0 < \alpha < 1. \tag{7}$$

Using $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha} \tag{8}$$

where $u_C(t)$ is the voltage on the condensator.

Similarly, let the voltage $u_L(t)$ on coil (inductor) with the inductance L be the β order derivative of its magnetic flux $\Psi(t)$ [Kaczorek, 2011f]

$$u_L(t) = \frac{d^\beta \Psi(t)}{dt^\beta}, \quad 0 < \beta < 1. \tag{9}$$

Taking into account that $\Psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta}, \tag{10}$$

where $i_L(t)$ is the current in the coil.

Consider an electrical circuit composed of resistors, n capacitors and m voltage sources. Using the equation (2.8) and the Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{11}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. The components of the state vector $x(t)$ and input vector $u(t)$ are the voltages on the condensators and source voltages respectively. Similarly, using the equation (10) and the Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{d^\beta x(t)}{dt^\beta} = Ax(t) + Bu(t), \quad 0 < \beta < 1, \tag{12}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. In this case the components of the state vector $x(t)$ are the currents in the coils.

Solution of the equation (11) (or (2.12)) satisfying the initial condition $x(0) = x_0$ is given by (3).

Now let us consider electrical circuit composed of resistors, capacitors, coils and voltage (current) source. As the state variables (the components of the state vector $x(t)$) we choose the voltages on the capacitors and the currents in the coils. Using the equations (8), (10) and Kirchhoff's laws we may write for the fractional linear circuits in the transient states the state equation

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u, \quad 0 < \alpha, \; \beta < 1, \tag{13a}$$

where the components $x_C \in \Re^{n_1}$ are voltages on the condensators, the components $x_L \in \Re^{n_2}$ are currents in the coils and the components of $u \in \Re^m$ are the source voltages and

$$A_{ij} \in \Re^{n_i \times n_j},\ B_i \in \Re^{n_i \times m},\quad i, j = 1,2. \tag{13b}$$

Theorem 3. The solution of the equation (13) for $0 < \alpha < 1;\ \ 0 < \beta < 1$ with initial conditions

$$x_C(0) = x_{10} \text{ and } x_L(0) = x_{20} \tag{14}$$

has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t [\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}]u(\tau)d\tau\,, \tag{15a}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},\ B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\ B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

$$T_{kl} = \begin{cases} I_n \ \text{ for } \ k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \ \text{ for } \ k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \ \text{ for } \ k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} \ \text{ for } \ k + l > 0 \end{cases} \tag{15b}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + l\beta}}{\Gamma(k\alpha + l\beta + 1)}\,,$$

$$\Phi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha + l\beta - 1}}{\Gamma[(k+1)\alpha + l\beta]}\,,$$

$$\Phi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + (l+1)\beta - 1}}{\Gamma[k\alpha + (l+1)\beta]}\,.$$

Proof is given in [Kaczorek, 2010, 2011f].

The extension of Theorem 3 to systems consisting of *n* subsystems with different fractional orders is given in [Kaczorek, 2011d].

Reduction of descriptor linear electrical circuits to their standard equivalent forms

The following elementary row (column) operations will be used [Kaczorek, 1992]:

Multiplication of the *i*th row (column) by a real number c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).

Addition to the *i*th row (column) of the *j*th row (column) multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$ ($R[i + j \times c]$).

Interchange of the *i*th and *j*th rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

First the essence of the polynomial approach will be shown on the following simple example.

Example 1. Consider the fractional descriptor electrical circuit shown in Fig. 1 with given resistances R_1, R_2; inductances L_1, L_2 and source current i_z.

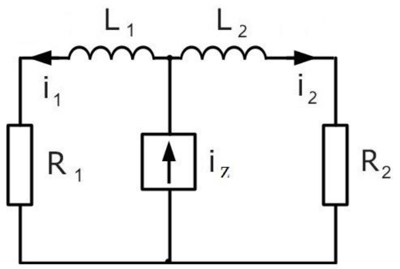

Fig. 1. Fractional electrical circuit

Using Kirchhoff's laws we can write the equations

$$L_1 \frac{d^\alpha i_1}{dt^\alpha} + R_1 i_1 = L_2 \frac{d^\alpha i_2}{dt^\alpha} + R_2 i_2 \tag{16a}$$

$$i_z = i_1 + i_2 \tag{16b}$$

The equations (16) can be written in the form

$$E \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B i_z \tag{17a}$$

where

$$E = \begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix} \; A = \begin{bmatrix} -R_1 & R_2 \\ -1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{17b}$$

Defining

$$E_1 = [L_1 \quad -L_2] \;\; A_1 = [-R_1 \quad R_2], \; A_2 = [-1 \quad -1], B_1 = [0], \; B_2 = [1] \tag{18}$$

we can write the equation (17) in the form

$$E_1 \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_1 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_1 i_z \tag{19a}$$

and

$$0 = A_2 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_2 i_z. \tag{19b}$$

The fractional differentiation of (19b) yields

$$0 = A_2 \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_2 \frac{d^\alpha i_z}{dt^\alpha}. \tag{20}$$

From (19a) and (20) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} i_z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \frac{d^\alpha i_z}{dt^\alpha}. \tag{21}$$

Note that the matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix} \tag{22}$$

is nonsingular and premultiplying (21) by its inverse we obtain

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \overline{A}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \overline{B}_0 i_z + \overline{B}_1 \frac{d^\alpha i_z}{dt^\alpha} \tag{23a}$$

where

$$\begin{aligned}
\overline{A} &= \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1}\begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -R_1 & R_2 \\ 0 & 0 \end{bmatrix} = \frac{1}{L_1 + L_2}\begin{bmatrix} -R_1 & R_2 \\ R_1 & -R_2 \end{bmatrix}, \\
\overline{B}_0 &= \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\overline{B}_1 &= \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} L_1 & -L_2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{L_1 + L_2}\begin{bmatrix} L_2 \\ L_1 \end{bmatrix}.
\end{aligned} \tag{23b}$$

Note that the electrical circuit with (23) is positive since $\overline{A} \in M_2$ and the matrices $\overline{B}_0$ and $\overline{B}_1$ have nonnegative entries.

The standard equation (23a) can be also obtained from the equation (21) by reducing the matrix (22) to the identity matrix I_2 using the elementary row operations

$$L[1 + 2 \times L_2],\ L\left[1 \times \frac{1}{L_1 + L_2}\right],\ L[2 + 1 \times (-1)]. \tag{24}$$

Performing the elementary row operations (324) on the matrix $\begin{bmatrix} 1 & 0 \\ 0 & s^\alpha \end{bmatrix}$ we obtain the polynomial matrix

$$L(s^\alpha) = \frac{1}{L_1 + L_2}\begin{bmatrix} 1 & s^\alpha L_2 \\ -1 & s^\alpha L_1 \end{bmatrix} \tag{25}$$

satisfying the equalities

$$\begin{aligned}
L(s^\alpha)[Es^\alpha - A] &= \frac{1}{L_1 + L_2}\begin{bmatrix} 1 & s^\alpha L_2 \\ -1 & s^\alpha L_1 \end{bmatrix}\begin{bmatrix} s^\alpha L_1 + R_1 & -s^\alpha L_2 - R_2 \\ 1 & 1 \end{bmatrix} \\
&= [I_n s^\alpha - \overline{A}] = \begin{bmatrix} s^\alpha + \frac{R_1}{L_1 + L_2} & -\frac{R_2}{L_1 + L_2} \\ -\frac{R_1}{L_1 + L_2} & s^\alpha + \frac{R_2}{L_1 + L_2} \end{bmatrix}, \\
L(s^\alpha)B &= \frac{1}{L_1 + L_2}\begin{bmatrix} 1 & s^\alpha L_2 \\ -1 & s^\alpha L_1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [\overline{B}_0 + \overline{B}_1 s^\alpha] = \frac{s^\alpha}{L_1 + L_2}\begin{bmatrix} L_2 \\ L_1 \end{bmatrix}.
\end{aligned} \tag{26}$$

Therefore, the reduction of the matrix (22) to identity matrix by the use of elementary row operations (24) is equivalent to premultiplication of the equation

$$[Es^\alpha - A]X = BU \tag{27}$$

by the polynomial matrix of elementary row operations (25), where $X = \mathcal{L}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$, $U = \mathcal{L}[i_z]$ and $\mathcal{L}$ is the Laplace operator.

In general case let consider the descriptor electrical circuit described by the equation

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + Bu \tag{28}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that $\det E = 0$ and the pencil (E,A) is regular.

Applying to (28) the Laplace transform with zero initial conditions we obtain the equation

$$[Es^\alpha - A]X = BU \tag{29}$$

where $X = \mathcal{L}[x(t)]$, $U = \mathcal{L}[u(t)]$.

Theorem 4. There exists a nonsingular polynomial matrix

$$L(s^\alpha) = L_0 + L_1 s^\alpha + ... + L_\mu s^{\alpha\mu} \tag{30}$$

where μ is the nilpotent index of the pair (E,A), such that

$$L(s^\alpha)[Es^\alpha - A] = [I_n s^\alpha - \overline{A}] \tag{31}$$

if and only if the pencil (E,A) is regular, i.e.

$$\det[Es^\alpha - A] \neq 0 \tag{32}$$

for some $s^\alpha \in \mathrm{C}$ where $\mathbf{C}$ is the field of complex numbers.

Proof. The matrix $[I_n s^\alpha - \overline{A}]$ is nonsingular for every matrix $\overline{A} \in \Re^{n\times n}$.

From (31) and (32) it follows that the polynomial matrix (30) is nonsingular. Using elementary row operations the singular matrix E can be always reduced to the form $\begin{bmatrix} E_1 \\ 0 \end{bmatrix}$ where E_1 has the full row rank r_1 and L_1 is the matrix of elementary row operations.

Premultiplying (29) by L_1 we obtain

$$L_1[Es^\alpha - A]X = \begin{bmatrix} E_1 s^\alpha - A_1 \\ -A_2 \end{bmatrix} X = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \tag{33a}$$

where

$$L_1 E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix},\ E_1 \in \Re^{r_1\times n},\ L_1 A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ A_1 \in \Re^{r_1\times n},\ L_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_1 \in \Re^{r_1\times m}. \tag{33b}$$

Using (33) we can write the equation (28) in the form

$$E_1 \frac{d^\alpha x}{dt^\alpha} = A_1 x + B_1 u\,, \tag{34a}$$

$$0 = A_2 x + B_2 u\,. \tag{34b}$$

The fractional differentiation of (34b) yields

$$0 = A_2 \frac{d^\alpha x}{dt^\alpha} + B_2 \frac{d^\alpha u}{dt^\alpha}. \tag{35}$$

From (34a) and (35) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \frac{d^\alpha x}{dt^\alpha} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \frac{d^\alpha u}{dt^\alpha}. \tag{36}$$

If the matrix $\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}$ is nonsingular then from (36) we have

$$\frac{d^\alpha x}{dt^\alpha} = \overline{A}_1 x + \overline{B}_0 u + \overline{B}_1 \frac{d^\alpha u}{dt^\alpha} \tag{37a}$$

where

$$\overline{A}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad \overline{B}_0 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \tag{37b}$$

If the matrix $\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}$ is singular then using elementary row operations we reduced the matrix $\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}$ to the form

$$L_2 \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} E_2 \\ 0 \end{bmatrix} \tag{38}$$

and we repeat the procedure.

It is well known that if the condition (32) is satisfied then after μ steps of the procedure we obtain the nonsingular matrix

$$\begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}. \tag{39}$$

Premultiplying the equation

$$\begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix} \frac{d^\alpha x}{dt^\alpha} = \begin{bmatrix} A_{\mu-1} \\ 0 \end{bmatrix} x + \begin{bmatrix} B_{\mu-1,0} \\ 0 \end{bmatrix} u + \begin{bmatrix} B_{\mu-1,1} \\ B_{\mu-1,0} \end{bmatrix} \frac{d^\alpha u}{dt^\alpha} + \ldots + \begin{bmatrix} 0 \\ B_\mu \end{bmatrix} \frac{d^{\alpha\mu} u}{dt^{\alpha\mu}} \tag{40}$$

by the inverse matrix $\begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}^{-1}$ we obtain the desired equation

$$\frac{d^\alpha x}{dt^\alpha} = \overline{A}_\mu x + \overline{B}_0 u + \overline{B}_1 \frac{d^\alpha u}{dt^\alpha} + \ldots + \overline{B}_\mu \frac{d^{\alpha\mu} u}{dt^{\alpha\mu}} \tag{41a}$$

where

$$\begin{gathered} \overline{A}_\mu = \begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}^{-1} \begin{bmatrix} A_{\mu-1} \\ 0 \end{bmatrix}, \quad \overline{B}_0 = \begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}^{-1} \begin{bmatrix} B_{\mu-1,0} \\ 0 \end{bmatrix}, \\ \overline{B}_1 = \begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}^{-1} \begin{bmatrix} B_{\mu-1,1} \\ B_{\mu-1,0} \end{bmatrix}, \ldots, \quad \overline{B}_\mu = \begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_\mu \end{bmatrix}. \end{gathered} \tag{41b}$$

The standard equation (41a) can be also obtained from the equation (40) by reducing the matrix (39) to the identity matrix I_n using the elementary row operations and this is equivalent to premultiplication of the equation (40) by suitable matrix of elementary row operations.

$$L_\mu \begin{bmatrix} E_\mu \\ -A_\mu \end{bmatrix} = I_n . \tag{42}$$

The desired polynomial matrix of elementary row operations (30) is given by

$$L(s^\alpha) = L_\mu \prod_{i=1}^{\mu} \mathrm{diag}\,[I_{r_i}, I_{n-r_i} s^\alpha]. \tag{43}$$

Note that the matrix $I_{n-r_i} s^\alpha$ corresponds to the fractional differentiation of the algebraic equations.

The considerations can be easily extended to the linear electrical circuits described by the equation (13a).

Example 2. Consider the fractional descriptor electrical circuit shown on Figure 2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 capacitance C and source voltages e_1, e_2.

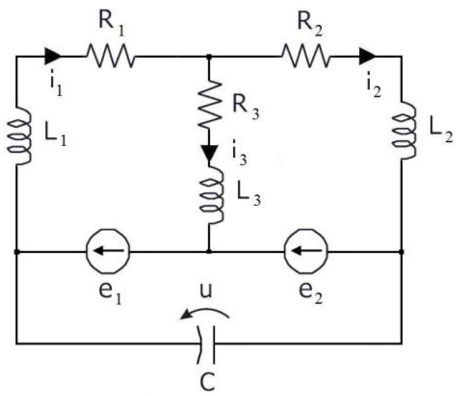

Fig. 2. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = L_1 \frac{d^\beta i_1}{dt^\beta} + R_1 i_1 + L_3 \frac{d^\beta i_3}{dt^\beta} + R_3 i_3, \tag{44a}$$

$$e_2 = L_2 \frac{d^\beta i_2}{dt^\beta} + R_2 i_2 - L_3 \frac{d^\beta i_3}{dt^\beta} - R_3 i_3 \tag{44b}$$

$$i_3 = i_1 - i_2, \tag{44c}$$

$$u = e_1 + e_2. \tag{44d}$$

The equations can be written in the form

$$E \begin{bmatrix} \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \\ \frac{d^\beta i_3}{dt^\beta} \\ \frac{d^\alpha u}{dt^\alpha} \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{45a}$$

where

$$E = \begin{bmatrix} L_1 & 0 & L_3 & 0 \\ 0 & L_2 & -L_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 & 0 \\ 0 & -R_2 & R_3 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{45b}$$

The pencil is regular since

$$\det[Es^\alpha - A] = \begin{vmatrix} s^\alpha L_1 + R_1 & 0 & s^\alpha L_3 + R_3 & 0 \\ 0 & s^\alpha L_2 + R_2 & -s^\alpha L_3 - R_3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= (s^\alpha L_1 + R_1)[s^\alpha (L_2 + L_3) + R_2 + R_3] + (s^\alpha L_3 + R_3)(s^\alpha L_2 + R_2) \neq 0. \tag{46}$$

Defining

$$E_1 = \begin{bmatrix} L_1 & 0 & L_3 & 0 \\ 0 & L_2 & -L_3 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -R_1 & 0 & -R_3 & 0 \\ 0 & -R_2 & R_3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{47}$$

we can write the equation (45a) in the form

$$E_1 \begin{bmatrix} \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \\ \frac{d^\beta i_3}{dt^\beta} \\ \frac{d^\alpha u}{dt^\alpha} \end{bmatrix} = A_1 \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{48a}$$

and

$$0 = A_2 \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + B_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{48b}$$

The α fractional differentiation of (48b) yields

$$0 = A_2 \begin{bmatrix} \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \\ \frac{d^\beta i_3}{dt^\beta} \\ \frac{d^\alpha u}{dt^\alpha} \end{bmatrix} + B_2 \begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \end{bmatrix}. \tag{49}$$

From (48a) and (49) we have

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \begin{bmatrix} \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \\ \frac{d^\beta i_3}{dt^\beta} \\ \frac{d^\alpha u}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \end{bmatrix}. \tag{50}$$

The matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & L_3 & 0 \\ 0 & L_2 & -L_3 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{51}$$

is nonsingular and premultiplying (50) by its inverse we obtain

$$\begin{bmatrix} \frac{d^\beta i_1}{dt^\beta} \\ \frac{d^\beta i_2}{dt^\beta} \\ \frac{d^\beta i_3}{dt^\beta} \\ \frac{d^\alpha u}{dt^\alpha} \end{bmatrix} = \overline{A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u \end{bmatrix} + \overline{B}_0 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \overline{B}_1 \begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \end{bmatrix} \tag{52a}$$

where

$$\overline{A}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \frac{1}{L_1(L_2+L_3)+L_2L_3} \begin{bmatrix} L_2+L_3 & L_3 & L_2L_3 & 0 \\ L_3 & L_1+L_3 & -L_1L_3 & 0 \\ L_2 & -L_1 & -L_1L_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -R_1 & 0 & -R_3 & 0 \\ 0 & -R_2 & R_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_0 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \frac{1}{L_1(L_2+L_3)+L_2L_3} \begin{bmatrix} L_2+L_3 & L_3 & L_2L_3 & 0 \\ L_3 & L_1+L_3 & -L_1L_3 & 0 \\ L_2 & -L_1 & -L_1L_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\overline{B}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \frac{1}{L_1(L_2+L_3)+L_2L_3} \begin{bmatrix} L_2+L_3 & L_3 & L_2L_3 & 0 \\ L_3 & L_1+L_3 & -L_1L_3 & 0 \\ L_2 & -L_1 & -L_1L_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{52b}$$

The standard equation (52a) can be also obtained from the equation (50) by reducing the matrix (51) to the identity matrix I_4 using the elementary row operations

$$L[1+3\times(-L_3)],\ L[2+3\times L_3],\ L\left[1+2\times\left(\frac{L_3}{L_2+L_3}\right)\right],\ L\left[1\times\left(\frac{1}{L_1(L_2+L_3)+L_2L_3}\right)\right],$$

$$L[2+1\times L_3],\ L[3+1\times 1],\ L\left[1\times\left(\frac{1}{L_2+L_3}\right)\right],\ L[3+2\times(-1)], \tag{53}$$

Using the elementary row operations (53) on the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^{\beta} & 0 \\ 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \tag{54}$$

we obtain the polynomial matrix

$$L[s^{\alpha}, s^{\beta}] = \begin{bmatrix} \frac{L_2+L_3}{L} & \frac{L_3}{L} & -\frac{L_2L_3}{L}s^{\beta} & 0 \\ \frac{L_3}{L} & \frac{L_1+L_3}{L} & \frac{L_1L_3}{L}s^{\beta} & 0 \\ \frac{L_2}{L} & -\frac{L_1}{L} & \frac{L_1L_2}{L}s^{\beta} & 0 \\ 0 & 0 & 0 & s^{\alpha} \end{bmatrix} \text{ and } L = L_1(L_2+L_3)+L_2L_3 \tag{55}$$

satisfying the equations

$$L[s^{\alpha}][E\ \mathrm{diag}\,(s^{\beta}, s^{\beta}, s^{\beta}, s^{\alpha}) - A] = \mathrm{diag}\,(s^{\beta}, s^{\beta}, s^{\beta}, s^{\alpha}) - \overline{A}. \tag{56}$$

Conclusion

A new polynomial approach is proposed to analysis of the standard and positive descriptor electrical circuits has been proposed. It has been shown (Theorem 4) that for given descriptor fractional electrical circuit the equivalent standard fractional electrical circuit can be found by premultiplication of the equation of the descriptor electrical circuit by suitable polynomial matrix of elementary row operations. The essence of the proposed method is demonstrated on simple positive fractional descriptor electrical circuit. The considerations can be easily extended to descriptor electrical circuits described by system of linear fractional equations with different orders [Kaczorek, 2010, 2011d]. An open problem is an extension of the approach to two-dimensional continuous-discrete fractional linear systems.

Acknowledgment

This work was supported by National Science Centre in Poland under work S/WE/1/11.

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