MODIFIED LYAPUNOV'S CONDITIONS FOR HYBRID AUTOMATA STABILITY

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Abstract: *The problem of stability of stationary states of hybrid automata are considered. Sufficient conditions for stability and instability of trivial stationary states of nonlinear and linear hybrid automata with the help of s- and u-hybrid Lyapunov's functions are obtained. Also necessary and sufficient condition for existence of solution of Lyapunov's-like equation (copositive matrix), which can be used to construct quadratic Lyapunov's functions on cones are proved.*

***Keywords**: dynamical systems, discrete-continuous systems, Lyapunov's equations, Lyapunov's functions, hybrid automaton.*

***ACM Classification Keywords**: G.1.7 – Ordinary Differential Equations.*

Introduction

Some dynamical systems combine continuous dynamics and switching between several different discrete states (in certain moments of time or at certain hypersurfaces). Specifically, this property is present in some automatic control systems (for example thermostat, automatic transmission). Difficulties in research of such systems are obvious.

Hybrid automata are used as mathematical descriptions of such systems. Research work on different models of discrete-continuous systems was done by Samoylenko A.M., Martynyuk A.A., Perestyuk N.A., Byslenko N.P., Glushkov V.M., Emel'yanov S.V., DeCarlo R., Branicky M., Pettersson S., Lennartson B. and others [Branicky, 1994]-[Martynyuk, 2002], [Peleties, 1991]- [Ye, 1998].

We will use the variant of method of Lyapunov's functions, based on s- and u- functions [Emel'yanov, 1972] , to investigate stability of stationary states of hybrid automata. Also in the article we propose sufficient conditions for solvability Lyapunov's equations on a some subset of the space of positive define matrices.

For the definition of hybrid automaton we refer the reader to [Bychkov, 2007b], [Alur,1993], [Nicollin, 1993].

We will use the following notation: $|\cdot|$ is Euclidean norm in $\mathbf{R}^n$, $\mathbf{0}$ is the null-vector and $a < b$ ($a \le b$) for vectors is the componentwise comparison.

Definition 1. Trivial stationary state $x = \mathbf{0}$ of hybrid automaton H is called Lyapunov stable, if for any $\varepsilon > 0$ there is $\delta > 0$ such that the inequality $|x(t_0)| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \in \tau$, where τ is a hybrid time.

Main Result

Let's assume, that the hybrid automaton's orbit begins from the first state. Notation $x\,|_{i \to i+1}$ means that automaton switches from the local state i to the state $i+1$ at the point x. Let's build the following sequence:

$$c^0 \in (0, C),\ \ c^1 = \max_{\substack{x^1|_{1\to 2} \\ V^1(x^1) \le c_0}} V^2(x^1),\ \ c^2 = \max_{\substack{x^2|_{2\to 3} \\ V^2(x^2) \le c_1}} V^3(x^2), \dots, c^N = \max_{\substack{x^N|_{N\to 1} \\ V^N(x^N) \le c_{N-1}}} V^1(x^N) \tag{1}$$

Also, let's denote by Ω_{i} the set that describes i-th local state. Let's assume that there exists a set of Lyapunov's functions, defined on sets Ω_{i}.

Definition 2. An indexed family $V(i,\ x) = \{V^i(x)\},\ i = \overline{1,\ N}$ is said to be a hybrid s-function, if $V^i(x)$ are positively defined and $c^N \le c^0$ for any sequence $\{c^i\},\ i = \overline{0,\ N}$, defined as (1).

We will use a hybrid s-functions to investigate stability of a hybrid automaton's trivial stationary state.

Definition 3. The following indexed family is called the derivative of the s-function $V(i, x)$:

$$\dot{V}(i, x) = \left\{ \frac{dV^i(x)}{dx} f_i(x(t)),\ i = \overline{1,\ N} \right\}.$$

Let's introduce the following notations: $B_r = \{x \in \mathbf{R}^n \mid |x| \le r\}$, $S_r = \{x \in \mathbf{R}^n \mid |x| = r\}$.

Theorem 1. Assume that the hybrid automaton H has trivial stationary state and satisfies conditions $|Q| < \infty\ \ i = \overline{1, N-1},\ \ Jump(N, x) = (1, x)$. Also assume that the neighborhood of the origin $D \subset X$ is defined and there exists positively defined hybrid s -function $V(i,\, x)\colon\; Q \times D \to \mathbf{R}$ such that $\frac{dV^i(x)}{dx} f_i(\, x(t)) \le 0$ for all $x \in D \cap \Omega_i$, $i = \overline{1,\, N}$.

Then the trivial stationary state is stable.

Proof. Let's assume that $Q = \{1, 2\}$ and denote $W_r(i) = \{x \in \mathbf{R}^n \mid V^i(x) \le r\}$.

Let's choose arbitrary $\varepsilon > 0$ and show that it is possible to find $\delta > 0$ such that for any orbit $x(t)$, the condition $V^2(x(t)) < a_2(2)$ for all $t \in \tau$, implies $x(t) \in B_\varepsilon$ for all $t \in \tau$.

Let's select $r \in (0, \varepsilon)$ such that $B_{r_2} \subseteq D$ and put $a_2(i) = \min_{x \in S_{r_2}} V^i(x)$. Let's choose $b_2(i) \in (0, a_2(i))$. Then $W_{b_2(i)}(i) \subseteq B_{r_2}$. Let's choose $p_2(i) > 0$ such that $B_{p_2(i)} \subseteq \Omega_{b_2(i)}(i)$ and put $r_1 = \min_{i \in Q} p_2(i)$.

Similarly, the value $b_1(i) \in (0, a_1(i))$ can be defied. Then $W_{b_1(i)}(i) \subseteq B_r$. Let's choose $p_1(i) > 0$ such that $B_{p_1(i)} \subseteq W_{b_1(i)}(i)$ and put $\delta = \min_{i \in Q} p_1(i)$.

Let's assume (for distinctness) that the orbit begins in state 1. If the orbit doesn't move from state 1 to state 2, then the theorem degenerates to Lyapunov's theorem. So assume that it moves form state 1 to state 2 at some moment $\tau_0' = \tau_1$.

Then $|x(\tau_0')| = |x(\tau_1)| < r_1$ for all $t \in [\tau_0, \tau_0']$. If the trajectory doesn't move from state 2 to state 1, the theorem is proved. If there is a jump in point $\tau_1' = \tau_2$, then $|x(\tau_1')| = |x(\tau_2)| < r_2$ for all $t \in [\tau_1, \tau_1']$.

By the theorem's statement, there exists a positively defined hybrid s -function. Let's put $c^0 = V^1(x(\tau_0))$ in s -function's definition. Then $V^1(x(\tau_2)) \le c^2 \le c^0 = V^1(x(\tau_0)) < a_1(1)$, i.e. $|V^1(x(\tau_2))| < a_1(1)$. Continuing by induction we acquire that $|V^1(x(t))| < a_1(1)$ for any $t \ge \tau_0$ in the first state, and $|V^2(x(t))| < a_2(2)$ for any $t \ge \tau_0$ in second state. In both cases $|x(t)| < r_2 < \varepsilon$. Theorem are proved.

Note, that for any i, it is sufficient the check in equalities $c^N \le c^0$ and $\frac{dV^i(x)}{dx} f_i(\, x(t)) \le 0$ only on local state Ω_i.

Now let's consider conditions of instability of trivial stationary state. Let's call a local state unstable if it is not Lyapunov stable. Let's give some definitions.

Definition 4. Hybrid time τ is said to be Zeno, if it consists of infinite number of intervals, but $\lim_{i \to \infty} t_i^* < \infty$ (like in Zeno's paradox about Achilles and a turtle).

Let's build a sequence $\{c^i\}$, $i = \overline{0,\, N}$ as follows:

$$c^0 \in (0,C),\ \ c^1 = \min_{\substack{x^1|_{1\to 2} \\ V^1(x^1)=c^0}} V^2(x^1),\ \ c^2 = \min_{\substack{x^2|_{2\to 3} \\ V^2(x^2)=c^1}} V^3(x^2),\dots,\ c^N = \min_{\substack{x^N|_{N\to 1} \\ V^N(x^N)=c^{N-1}}} V^1(x^N) \tag{2}$$

Definition 5. An indexed family $V(i,\ x)=\left\{V^i(x)\right\},\ i=\overline{1,N}$ is said to be a hybrid u -function, if $V^i(x)$ are positively defined and $c^N \ge c^0$ for any sequence $\left\{c^i\right\}$, $i=\overline{0,\ N}$, defined as (2).

Theorem 2. Let H be a hybrid automata with no Zeno orbits. Assume that H has the trivial stationary state and satisfies conditions $|Q|<\infty,\ Jump(i,\ x)=\{(i+1,\ q_i(x))\}$, $i=\overline{1,N-1}$; $Jump(N,x)=(1,q_N(x))$. Also assume that the neighborhood of the origin $D\subset X$ is defined and there exists a hybrid u -function $V(i,\ x)\colon\ Q\times D\to \mathbf{R}$ such that $\dfrac{dV^i(x)}{dx}f_i(x(t))>0$ for all $x\in D\cap\Omega_i$ and $i=\overline{1,N}$. Then the trivial stationary state is unstable.

Proof. During the live time of a hybrid automaton, there can be two cases:

1) for any $\varepsilon>0$ there exists $x_0\in B_\varepsilon$, such that the orbit, that starts at x_0 performs a finite number of switchings;

2) there exists an $\varepsilon>0$ such that all orbits that start in B_ε perform infinite number of switchings.

In the first case the theorem reduces to Chetaev's theorem about instability.

Let's assume that the second case takes place. For simplicity, let's assume that $Q=\{1,2\}$ and that orbit begins in the first state. Let's assume the contrary: the trivial stationary state is not unstable, i.e. for arbitrarily small $\delta>0$, orbits that begin in B_δ, do not leave the ball B_ε. Then there is some $C>0$, such that $V^i(x(t))<C$ for all $i\in Q$.

From the positive definiteness and continuity of the derivative $\dot V(i,x)=\dfrac{dV^i(x)}{dx}f_i(\,x(t))$ it follows, that there exists some $v>0$ such that $\dot V(i,x(t))>v$ for all $i\in Q$. Let's compute the difference $V^1(x(t))-V^1(x(t_0))$ (we assume that the orbit is in state 1 at time t after M switchings, where M is an even number):

$$V^1(x(t))-V^1(x(t_0))=\sum_{i=2,4,\dots,M}\left(\int_{\tau_{i-1}}^{\tau_i}\dot V^1(s)ds+\left[V^1(x(\tau_{i+1}+0))-V^1(x(\tau_i-0))\right]\right)+\int_{\tau_{M+1}}^{t}\dot V^1(s)ds\ge$$

$$\ge\sum_{i=2,4,\dots,M}\left[V^1(x(\tau_{i+1}+0))-V^1(x(\tau_i-0))\right]+v\left[\sum_{i=2,4,\dots,M}(\tau_i-\tau_{i-1})+(t-\tau_{N+1})\right].$$

Let's assume the series $\sum_{i=2,4,\dots,M}(\tau_i-\tau_{i-1})$ to be divergent. Then the expression $v\left[\sum_{i=2,4,\dots,M}(\tau_i-\tau_{i-1})+(t-\tau_{N+1})\right]$ becomes unbounded with the increase of t. It remains to make sure that the difference $V^1(x(\tau_{i+1}+0))-V^1(x(\tau_i-0))$ is non-negative.

Let's put $c^0=V^1(x(\tau_i))$ in the definition of the u-function. Then $V^1(x(\tau_i))\geq c^1$, and $V^1(x(\tau_{i+1}))\geq c^2\geq c^0$, i.e. inequality $V^1(x(\tau_{i+1}))-V^1(x(\tau_i))\geq 0$ holds.

Thus with the increase of t, the difference $V^1(x(t))-V^1(x(t_0))$ becomes arbitrarily large which contradicts assumption $V^i(x(t))<C$.

If the series $\sum_{i=2,4,\dots,N}(\tau_i-\tau_{i-1})$ converges, then another series $\sum_{i=3,5,\dots,N-1}(\tau_i-\tau_{i-1})$ is divergent and the same arguments can be applied to the second state.

Theorem are proved.

Consider a hybrid automaton with the following properties:

1. for each $k\in Q$, the right-hand side of the equation, describing continuous dynamics in k-th local state has the form $f_k(x)=A_kx$, where A_k is $n\times n$-matrix ;

2. each local state is formed by the convex cone $G_kx\geq\mathbf{0}$, where G_k is $m_k\times n$-matrix;

3. switching is cyclic ($1\to 2\to\dots\to N\to 1$) and occurs on hyper surface $x=B_ky$, where $y\in\mathbf{R}^{n-1}$ and B_k is $n\times(n-1)$-matrix, i.e.

- $Jump(k,x)=\{((k \bmod N)+1,x)\}$, if $x=B_ky$ for some y;
- $Jump(k,x)=\varnothing$, in the other case.

Definition 6. A hybrid automaton, that satisfies properties 1-3 is called linear.

Note, that this definition should not be confused with other definitions of linear hybrid automata which can be found in model checking literature.

Theorem 3. (About piecewise quadratic s-function). Let HA be a linear hybrid automaton. Suppose that there exist positive-definite symmetric $n\times n$-matrices H_k, $k=\overline{1,N}$, such that:

- $a_k=\max\limits_{\substack{G_kx\geq 0\\ x^Tx=1}} x^T(A_k^TH_k+H_kA_k)x<0$;
- the matrix $B_k^T(H_\ell-H_k)B_k$ is negative-semidefinite for each switching $k\to(k \bmod N)+1=\ell$.

Then the trivial stationary state is stable.

Proof. Let's define Lyapunov's functions as follows:

$$V_k(x) = x^T H_k x\,,\ k = \overline{1,N}\,.$$

Let's choose arbitrary $x \neq \mathbf{0}$ such that $G_k x \geq \mathbf{0}$. Then conditions $G_k x \geq \mathbf{0}$ and $a_k < 0$ imply the following inequality

$$\dot{V}(x) = x^T (A_k^T H_k + H_k A_k) x \leq x^T (A_k^T H_k + H_k A_k) x < 0\,.$$

Then the following relation describes transition $k \to (k \bmod N) + 1 = \ell$:

$$c^l - c^k = V_\ell(x) - V_k(x) = x^T (H_\ell - H_k) x = y^T B_k^T (H_\ell - H_k) B_k y \leq 0\,,$$

where c^l , c^k denote Lyapunov's functions $V_\ell(x)$ and $V_k(x)$ respectively.

Thus $\{V_k\}_{k=\overline{1,N}}$ form a hybrid s-function, i.e. $c^N \leq c^{N-1} \leq \ldots \leq c^1 \leq c^0$. So according to the Theorem 1, the trivial stationary state is stable. Theorem are proved.

Theorem 4 (About exponential attenuation coefficients). Suppose that linear hybrid automaton satisfies conditions of the Theorem 3. Then the following inequality holds (where q_0 is the automaton's initial state):

$$|x(t)| \leq \sqrt{\frac{\lambda_{\max}(H_{q_0})}{\min\limits_{k \in Q} \lambda_{\min}(H_k)}} \cdot |x(t_0)| \cdot \exp\left\{ \max_{k \in Q} \frac{a_k}{\lambda_{\min}(H_k)} \cdot \frac{(t - t_0)}{2} \right\}.$$

Proof. The following inequality holds for each $k = \overline{1,N}$:

$$\dot{V}_k(x) = x^T (A_k^T H_k + H_k A_k) x < 0\,.$$

Conditions of the Theorem 3 imply, that $\dot{V}_k(x) \leq a_k |x|^2$. From Rayleigh's inequality it follows that

$$\dot{V}_k(x) \leq \frac{a_k}{\lambda_{\min}(H_k)} V_k(x)\,.$$

Now from the Gronwall-Bellman's inequality implies that

$$V_k(x) \le V_k(x(\tau_i)) \cdot \exp\left\{\frac{a_k(t-\tau_i)}{\lambda_{\min}(H_k)}\right\}.$$

This inequality holds only within local state. Let's write it the current (n-th) segment of hybrid time and for previous segments:

$$V_{q_n}(x(t)) \le V_{q_n}(x(\tau_n)) \cdot \exp\left\{\frac{a_{q_n}(t-\tau_n)}{\lambda_{\min}(H_{q_n})}\right\},$$

$$V_{q_i}(x(\tau_i')) \le V_{q_i}(x(\tau_i)) \cdot \exp\left\{\frac{a_{q_i}(\tau_i'-\tau_i)}{\lambda_{\min}(H_{q_i})}\right\},\ i = 1, \ldots, n-1.$$

Conditions of the Theorem 3 imply that $V_{q_i}(x(\tau_i)) \le V_{q_{i-1}}(x(\tau_{i-1}'))$, so the following inequality holds:

$$V_{q_n}(x(t)) \le V_{q_n}(x(\tau_n)) \cdot \exp\left\{\frac{a_{q_n}(t-\tau_n)}{\lambda_{\min}(H_{q_n})}\right\} \le$$

$$\le V_{q_n}(x(\tau_{n-1}')) \cdot \exp\left\{\frac{a_{q_n}(t-\tau_n)}{\lambda_{\min}(H_{q_n})}\right\} \le V_{q_n}(x(\tau_{n-1})) \cdot \exp\left\{\frac{a_{q_n}(t-\tau_n)}{\lambda_{\min}(H_{q_n})} + \frac{a_{q_{n-1}}(\tau_{n-1}'-\tau_{n-1})}{\lambda_{\min}(H_{q_{n-1}})}\right\} \le \ldots \le$$

$$\le V_{q_0}(x(t_0)) \cdot \exp\left\{\sum_{i=1}^{n-1}\frac{a_{q_i}(\tau_i'-\tau_i)}{\lambda_{\min}(H_{q_i})} + \frac{a_{q_n}(t-\tau_n)}{\lambda_{\min}(H_{q_n})}\right\} \le V_{q_0}(x(t_0)) \cdot \exp\left\{\max_{k\in Q}\frac{a_k}{\lambda_{\min}(H_k)} \cdot (t-t_0)\right\}.$$

Now with the help of Rayleigh's inequality we obtain the following:

$$|x(t)|^2 \le \frac{\lambda_{\max}(H_{q_0})}{\lambda_{\min}(H_{q_n})} \cdot |x(t_0)|^2 \cdot \exp\left\{\max_{k\in Q}\frac{a_k}{\lambda_{\min}(H_k)} \cdot (t-t_0)\right\} \le$$

$$\le \frac{\lambda_{\max}(H_{q_0})}{\min_{k\in Q}\lambda_{\min}(H_k)} \cdot |x(t_0)|^2 \cdot \exp\left\{\max_{k\in Q}\frac{a_k}{\lambda_{\min}(H_k)} \cdot (t-t_0)\right\}.$$

Theorem are proved.

Let's take a look at the following example. Let $|Q| = 2$, $X = (x_1, x_2)$ and $A_1 = \begin{pmatrix} -0,1 & 1 \\ -2 & -0,1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -0,1 & 2 \\ -1 & -0,1 \end{pmatrix}$.

Let $Inv_1 = \{x_1 > 0\}$, $Inv_2 = \{x_1 < 0\}$, $G_1 = (1,0)$, $G_2 = (-1,0)$, $B_1 = B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let's put $H_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0,5 & 0 \\ 0 & 1 \end{pmatrix}$ and check conditions of the theorem 4:

- $a_1 = \max\limits_{\substack{G_1 x \geq 0 \\ x^T x = 1}} x^T (A_1^T H_1 + H_1 A_1) x = -0,2$; $a_2 = -0,1$;
- $B_1^T (H_2 - H_1) B_1 = 0$, $B_2^T (H_1 - H_2) B_2 = 0$.

Thus trivial stationary state is stable. If we assume, that the first state is initial, then:

$$\lambda_{\max}(H_1) = 2 \text{ , } \lambda_{\min}(H_1) = 1 \text{ , } \lambda_{\min}(H_2) = 0,5 \text{ , } \frac{a_1}{\lambda_{\min}(H_1)} = \frac{a_2}{\lambda_{\min}(H_2)} = -0,2 \text{ ,}$$

i.e. we obtain the following exponential inequality:

$$|x(t)| \leq 2|x(t_0)| \cdot e^{-0,1(t-t_0)} .$$

Let's investigate conditions that imply existence of a H that satisfies condition 1 of the theorem 3. Let's introduction the following notations:

- $A > 0$ ($A \geq 0$) – matrix A has positive (non-negative) elements;
- $\mathbf{S}^{n \times n}$ – a set of real symmetric $n \times n$-matrices;
- E_n – identity $n \times n$-matrix;
- $(a_1, a_2, .., a_n)$ – horizontal concatenation of column vectors;
- $cl(X)$ – closure of the set $X \subseteq \mathbf{R}^n$ (in usual topology on $\mathbf{R}^n$);
- $\mathrm{Im}_H(X) = \{Hx \mid x \in X\}$, $H \in \mathbf{R}^{n \times n}$, $X \subseteq \mathbf{R}^n$.

The following lemma can be found in convex analysis literature:

Lemma 1. Let $X, Y \subseteq \mathbf{R}^n$ be open convex cones and $cl(X) \cap cl(Y) = \{\mathbf{0}\}$. Then there exists $p \in \mathbf{R}^n$ for which $\forall x \in X, y \in Y : p^T x < 0 < p^T y$.

We obtain the following corollary from this lemma.

Corollary. Let $X, Y \subseteq \mathbf{R}^n$ be convex cones such that $cl(X) \cap cl(Y) = \{\mathbf{0}\}$. Then there exists a nonsingular matrix $C \in \mathbf{R}^{n \times n}$ such that

$$\forall x \in \mathrm{cl}(X) \setminus \{\mathbf{0}\}, y \in \mathrm{cl}(Y) \setminus \{\mathbf{0}\} \;\; Cx < \mathbf{0} < Cy .$$

Proof. Let's assume that $X \neq \{\mathbf{0}\}$, $Y \neq \{\mathbf{0}\}$, because in the other case the corollary is trivial. Let's apply lemma 1 to X and Y. Then

$$\forall x \in \mathrm{cl}(X) \setminus \{\mathbf{0}\}, y \in \mathrm{cl}(Y) \setminus \{\mathbf{0}\} \;\; p^T x < 0 < p^T y$$

for some $p \in \mathbf{R}^n$. Let's denote

$$C(\varepsilon) = \underbrace{(p, p, .., p)}_{n}^T + \varepsilon E_n , \varepsilon \in \mathbf{R} .$$

Then

$$\forall x \in \mathrm{cl}\, X \cap S_1, y \in \mathrm{cl}\, Y \cap S_1 \;\; C(0)x < \mathbf{0} < C(0)y ,$$

where $S_1 = \{x \in \mathbf{R}^n \mid |x| = 1\}$ – unit sphere.

Nonempty sets $\mathrm{cl}\, X \cap S_1$ and $\mathrm{cl}\, Y \cap S_1$ are compact, so there are $a_0, b_0 \in \mathbf{R}$, such that

$$\max_{x \in \mathrm{cl}\, X \cap S_1} \max_{i=1..N} e_i^T C(0)x < a_0 < 0 < b_0 < \min_{y \in \mathrm{cl}\, Y \cap S_1} \min_{i=1..N} e_i^T C(0)y .$$

The function $\max\limits_{i=1..N} e_i^T C(\varepsilon)x$ is continuous in ε, x, so there exists $\varepsilon^* > 0$, such that

$$\forall x \in \mathrm{cl}\, X \cap S_1, y \in \mathrm{cl}\, Y \cap S_1 .$$

Then

$$\max_{i=1..N} e_i^T C(\varepsilon^*)x \le a_0 < 0 < b_0 \le \min_{i=1..N} e_i^T C(\varepsilon^*)y ,$$

whence

$$\forall x \in \mathrm{cl}\, X \setminus \{\mathbf{0}\}, \; y \in \mathrm{cl}\, Y \setminus \{\mathbf{0}\} \;\; C(\varepsilon^*)x < \mathbf{0} < C(\varepsilon^*)y .$$

So we can put $C = C(\varepsilon^*)$. Corollary are proved.

Lemma 2. Let $X \subseteq \mathbf{R}^n$ be a convex cone and $A \in \mathbf{R}^{n \times n}$ be a matrix, such that $\mathrm{cl}\, X \cap \mathrm{Im}_A \mathrm{cl}\, X = \{\mathbf{0}\}$. Then there exists nonsingular matrix $H \in \mathbf{S}^{n \times n}$, such that $\forall x \in \mathrm{cl}\, X \setminus \{\mathbf{0}\}$ $x^T H x > 0$ and $x^T (A^T H + HA)x \le 0$.

Proof. Let's apply the corollary from the previous lemma to the cones X and $\mathrm{Im}_A X$. Then

$$\forall x \in \mathrm{cl}(X) \setminus \{\mathbf{0}\}, y \in \mathrm{Im}_A \mathrm{cl}(X) \setminus \{\mathbf{0}\} \;\; Cx < \mathbf{0} < Cy$$

for some nonsingular matrix C. Let's put $H = C^T C$. Then for any $x \in \mathrm{cl}\, X \setminus \{\mathbf{0}\}$ the following inequalities hold

$$x^T H x = (Cx)^T Cx > 0,$$

$$x^T (A^T H + HA) x = 2x^T C^T CAx = 2(Cx)^T (CAx) \le 0,$$

because $Ax \in \mathrm{clIm}_A X$ and $Cx < \mathbf{0} \le C(Ax)$.

Lemma are proved.

Lemma 2 can be used to prove a condition for existence of positive quadratic Lyapunov's function on the cone $\{x \in \mathbf{R}^n \mid x \ge \mathbf{0}\}$.

Theorem 5. Condition $\forall x \ge \mathbf{0}, x \ne \mathbf{0} \;\; Ax \not\ge 0$ is necessary and sufficient for existence of nonsingular matrix $H \in \mathbf{S}^{n \times n}$ such that $H > \mathbf{0}$ and $A^T H + HA < \mathbf{0}$.

Proof. Sufficiency follows from lemma 2, so let's prove necessity. Let's assume that there is matrix $H \in \mathbf{S}^{n \times n}$, such that $H > \mathbf{0}$ and $A^T H + HA < \mathbf{0}$, and prove that $\forall x \ge \mathbf{0}, x \ne \mathbf{0}$, $Ax \not\ge 0$. Let's suppose contrary: there exists vector $x_0 \ge \mathbf{0}, x_0 \ne \mathbf{0}$, such that $Ax_0 \ge \mathbf{0}$. Then $x_0^T (A^T H + HA) x_0 = 2(Hx_0)^T Ax_0 > 0$ because $Hx_0 \ge \mathbf{0}$, $Ax_0 \ge \mathbf{0}$ and $Hx_0 \ne \mathbf{0}$, $Ax_0 \ne \mathbf{0}$. The inequality $x_0^T (A^T H + HA) x_0 > 0$ contradicts assumption $A^T H + HA < \mathbf{0}$. So $\forall x \ge \mathbf{0}, x \ne \mathbf{0}$, $Ax \not\ge 0$.

Theorem are proved.

Let's show how to use theorem 5. Let's consider the system that describes local dynamics of linear hybrid automaton:

$$\dot{x} = Ax, \; Gx \ge \mathbf{0}, \tag{3}$$

where $A, G \in \mathbf{R}^{n \times n}$, and A is nonsingular. Let's apply a nonsingular change of variables $y = (y_1, .. y_n)^T = Tx$, such that for some indices i, j, $0 \le i < j \le n+1$, the condition $Gx \ge \mathbf{0}$ is equivalent to $y_1, .. y_i \ge 0, y_j, .. y_n = 0$.

Without loss of generality we can assume that equations $y_j,..y_n = 0$ are not present. Then system (3) will be equivalent to the system like

$$\dot{y} = TAT^{-1}y = \overline{A}y \ , \ y_1,..y_n \geq 0 \, , \tag{4}$$

where all variables y_i are non-negative. Then we can apply theorem 5: if $\forall y \geq \mathbf{0}, y \neq \mathbf{0} \ \ \overline{A}y \ngeq \mathbf{0}$, then there exists quadratic form $y^T Hy, H \in \mathbf{S}^{n\times n}$, such that $y^T Hy > 0$ and $y^T(\overline{A}^T H + H\overline{A})y < 0$ for any $y \geq \mathbf{0}, y \neq \mathbf{0}$. Then the following conditions are satisfied for any x, such that $Gx \geq \mathbf{0}$:

$$x^T T^T HTx > 0 \, ;$$

$$x^T T^T ((TA^T T^{-1})^T H + HTAT^{-1})Tx = x^T (A^T T^T HT + T^T HTA)x < 0 \, .$$

Then the quadratic form $x^T T^T HTx$ is the Lyapunov's function for the system (3). So it can be used to construct hybrid s-function.

Conclusion

In the article we considered the problem of stability of stationary states of hybrid automata. We obtained sufficient conditions for stability and instability of trivial stationary states with the help of s- and u-hybrid Lyapunov's functions. Also in the article we obtained sufficient condition for stability of stationary states of linear hybrid automata, and necessary and sufficient condition for existence of solution of Lyapunov's-like equation, which can be used to construct quadratic Lyapunov's functions on cones (theorem 5).

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