FUZZY MODELS AND METHODS OF RATIONAL RESOURCE ALLOCATION

Olexii Voloshyn, Vasyl Laver

Abstract*: We consider the application of fuzzy sets theory to classical rationing problem. Fuzzy generalizations of classical methods are proposed, the theory is illustrated by a numeric example. The fuzzy methods are compared with their crisp analogs using the inequality measures.*

Keywords*: rationing problem, rational distribution, fuzzy models and methods, inequality measures.*

ACM Classification Keywords*: H.4.2 Information Systems Applications - Types of Systems - Decision Support*

Introduction

Rational ("fair", "optimal") distribution of joint costs (or collectively produced goods) between agents with different contributions to the production or with different types of input (or output) resources is an important social and economic problem. So it can be considered as a central theme of transferable utility games theory [Moulin, 2001]. Various interesting axiomatic results of modern microeconomic theory have been obtained in this field.

In particular, a simple problem of distributing one resource according to a profile of claims (that describe individual preferences or needs) is also a rich field for axiomatic analysis. This model of resource allocation is often referred to as "bankruptcy problem". Among the first papers, dedicated to this model were [O'Neill, 1982], [Aumann, Maschler, 1985], [Young, 1988]. Recent reviews on the topic are given in [Thomson, 2013] and [Moulin, 2001]. However, the classical models, as a rule, do not take into account the uncertainty of input data, that is common for real processes.

The model

A rationing problem is defined by a triple (*N*,*c*,*b*), where *N* is a finite set of agents, the nonnegative real number *c* represents the amount of resources to be divided, the vector $b=(b_i)_{i\in N}$ specifies for each agent *i* a claim b_i, and these numbers are such that $\sum_{i=1}^{n} b_i > 0$ and

$$0 \le b_i, \forall i \in N : 0 \le c \le \sum_{i=1}^{n} b_i . \tag{1}$$

A solution to the rationing problem is a vector $x = (x_i)_{i \in N}$, specifying a share x_i for each agent *i*, such that

$$0 \le x_i \le b_i, \tag{2}$$

$$\forall i \in N : \sum_{i \in N} x_i = c. \tag{3}$$

A rationing method (rule) is a function *r* that associates with each triple (*N*,*c*,*b*) a unique vector $x = (x_i)_{i \in N}$, $x = r(N, c, b)$. There are different ways of interpreting the rationing problem. Without reducing the generality, let us consider the rationing problem as a cost sharing problem. Thus, *c* is interpreted as the production cost of an indivisible public good, b_i are interpreted as initial amount of money of agent *i.* Both *c* and b_i are non-negative real numbers.

In general, it is not correct to talk about the “optimality” (“reasonableness”, “justice”) of allocation. It seems more adequate to use the concept of “rationality’, which was proposed by a Nobel-prize winner Herbert A. Simon. In [Simon, 1947] the concept of “administrative man” who (in contrary to “economic man” who makes “optimal” decisions) makes “rational” decisions using simplified view of reality and heuristic approach is proposed.

Fuzzy approach

In general, there can be a situation, where some of the agents will not be satisfied with their shares, which can lead to the collapse of maximal coalition.

To prevent such situations, it may be reasonable to allow certain flexibility in determining the agents’ shares. Thus, we should allow agents to deviate from their shares to some degree. We propose to present the set of agents *N* as two subsets – the subset of “poor” agents N_1 and subset of “rich” agents N_2, where $N_1 \cup N_2 = N$, $N_1 \cap N_2 = \emptyset$ [Voloshyn, Laver, 2013]. We can set the threshold values *α* and *β*, where (1-*α*) $\hat{x}_i$ is the most desired share by agent $i \in N_1$ and $(1+\beta)\hat{x}_j$ is the maximal share that agent $j \in N_2$ can afford. The way in which N is divided into subsets N_1 and N_2 is determined by the person, who makes the decisions.

Agents’ shares can be presented as right-sided triangular fuzzy numbers $((1-\alpha)\hat{x}_i; (1-\alpha)\hat{x}_i; \hat{x}_i)$ for each $i \in N_1$ and $(\hat{x}_j; \hat{x}_j; (1+\beta)\hat{x}_j)$ for each $j \in N_2$.

Denote $x_i^L = \begin{cases} (1-\alpha)\hat{x}_i, i \in N_1; \\ \hat{x}_i, i \in N_2, \end{cases}$ i $x_i^R = \begin{cases} \hat{x}_i, i \in N_1; \\ (1+\beta)\hat{x}_i, i \in N_2, \end{cases}$ for each $i \in N$. In these notations, the agents’ shares can be presented as fuzzy numbers (x_i^L, x_i^L, x_i^R).

The process of finding the optimal distribution can be reduced to the following linear programming problem:

$$\lambda \to \max, \tag{4}$$

$$\mu_i(x_i) \geq \lambda, \ \forall i \in N, \tag{5}$$

$$\sum_{i=1}^{n} x_i = c, \tag{6}$$

$$0 \leq x_i \leq b_i, \forall i \in N, 0 \leq \lambda \leq 1. \tag{7}$$

Denote

$$\lambda_0 = \mu_\Sigma(c) = \frac{\sum_{i=1}^{n} x_i^{\ R} - c}{\sum_{i=1}^{n} x_i^{\ R} - \sum_{i=1}^{n} x_i^{\ L}} \tag{8}$$

In [Voloshyn, Laver, 2014] it is shown that $\lambda = \lambda_0$ is the unique solution of (4)-(7). The agents' shares can be found using the formula

$$x_k = x_k^R - \lambda_0 \left(x_k^R - x_k^L\right), \ \forall k \in N . \tag{9}$$

Speaking about the presenting *N* as two subsets, we can propose this general approach: we compute the average agent money amount $\frac{\sum_{i \in N} b_i}{n}$ and include in N_1 all the agents whose initial money amount is less or equal to this number, all other agents we include in N_2. However, there can be other approaches to dividing N into subsets.

We propose the following algorithm for finding the agents' shares for fuzzy generalization of rationing method *r*:

1. We find the crisp shares $\hat{x}_i = r(N,c,b)$ using *r.*
2. Break the set N into two subsets N_1 and N_2 (using some rule, defined by the decision-maker).
3. Set the threshold values for each subset α $(i \in N_1)$ і другої β $(j \in N_2 = N \setminus N_1)$.
4. Using (8)-(9) find $(x_1, x_2,..., x_n; \lambda_0)$.
5. If λ_0 doesn't satisfy the decision-maker, then go to 3. Otherwise, stop the process.

Case of Fuzzy Costs

Let the cost value be a triangular fuzzy number $C = (\underline{c}, \hat{c}, \overline{c})$. One of the approaches to the solution of this problem is dividing it in two subproblems – "optimistic" scenario (cost are low, i.e. $c \in [\underline{c}, \hat{c}]$) and "pessimistic" scenario (costs are high, i.e. $c \in [\hat{c}, \overline{c}]$). For optimistic case [Voloshyn, Laver, 2015]

$$c_0 = \frac{\hat{c}\sum_{i=1}^{n} x_i^{\,R} - \underline{c}\sum_{i=1}^{n} x_i^{\,L}}{\sum_{i=1}^{n} x_i^{\,R} - \sum_{i=1}^{n} x_i^{\,L} + \hat{c} - \underline{c}}. \tag{10}$$

λ_0 and x_k, $\forall k \in N$ are computed using (8) and (9), taking into account that $c = c_0$.

Pessimistic case can be reduced to finding optimal distribution for $c_0 = \hat{c}$ with agents' shares membership functions computed for $\bar{c}$.

As a solution of the initial problem we take the distribution with higher rate of λ_0.

Case of Fuzzy Initial Money Amounts

Let the agents' money amounts be a triangular fuzzy numbers $B_i = \left(\underline{b}_i, \hat{b}_i, \bar{b}_i\right), \forall i \in N$ [Voloshyn, Laver, 2015].

Denote

$$\hat{x}_i^{L} = \min\left\{r\left(N, c, \underline{b}_i\right), r\left(N, c, \bar{b}_i\right)\right\}, \forall i \in N;$$

$$\hat{x}_i^{C} = r\left(N, c, \hat{b}_i\right), \forall i \in N;$$

$$\hat{x}_i^{R} = \max\left\{r\left(N, c, \underline{b}_i\right), r\left(N, c, \bar{b}_i\right)\right\}, \forall i \in N.$$

Agents' shares can be approximately presented as triangular fuzzy numbers $\hat{X}_i = \left(\hat{x}_i^{L}, \hat{x}_i^{C}, \hat{x}_i^{R}\right)$. At the same time the desired shares are right-sided triangular fuzzy numbers $X_i = \left(\underline{x}_i, \hat{x}_i^{C}\right), \forall i \in N_1$, $X_i = \left(\hat{x}_i^{C}, \bar{x}_i\right), \forall i \in N_2$, where $\underline{x}_i = (1-\alpha)\hat{x}_i^{C}$ i $\bar{x}_i = (1+\beta)\hat{x}_i^{C}$.

Resulting membership function of agents' shares will be $\mu_i(\tilde{x}_i) = \min\{\mu_i(\hat{x}_i), \mu_i(x_i)\}, \forall i \in N_1$.

Taking into account the shape of corresponding membership functions, we obtain that $\tilde{X}_i = \left(\tilde{x}_i^{L}, \tilde{x}_i^{C}, \tilde{x}_i^{R}\right) = \left(\hat{x}_i^{L}, \tilde{x}_i^{C}, \hat{x}_i^{C}\right), \forall i \in N_1$. We can find $\tilde{x}_i^{C}$. In this point the following equality holds:

$$\frac{\hat{x}_i^{C} - \tilde{x}_i^{C}}{\hat{x}_i^{C} - \underline{x}_i} = \frac{\tilde{x}_i^{C} - \hat{x}_i^{L}}{\hat{x}_i^{C} - \hat{x}_i^{L}},$$

Thus, $\tilde{x}_i^{C} = \hat{x}_i^{C}\,\dfrac{\alpha\hat{x}_i^{L} + \hat{x}_i^{C} - \hat{x}_i^{L}}{\alpha\hat{x}_i^{C} + \hat{x}_i^{C} - \hat{x}_i^{L}}$.

As the highest rate of λ is when $\tilde{x}_i^{C}$, put $x_i = \tilde{x}_i^{C}$ for $\forall i \in N_1$ and $\tilde{c} = c - \sum_{i \in N_1} x_i$.

Shares of the agents from second group are right-sided triangular fuzzy numbers $\left(\hat{x}_i^{C}, \hat{x}_i^{C}, x_i^{\max}\right), \forall i \in N_2$, where $x_i^{\max} = \max\left\{\hat{x}_i^{R}, \bar{x}_i\right\}$.

Denote $x_i^L = \hat{x}_i^C$ and $x_i^R = x_i^{\max}$. We obtain linear programming problem:

$$\lambda \to \max,\ \frac{x_i^R - x_i}{x_i^R - x_i^L} \ge \lambda,\ \forall i \in N_2,\ \sum_{i \in N_2} x_i = \widetilde{c},\ 0 \le \lambda \le 1.$$

The solution of this problem can be found using (8) (9).

Fully Fuzzy Case

Let the costs and initial money amounts be triangular fuzzy numbers $C = (\underline{c}, \hat{c}, \overline{c})$, $B_i = (\underline{b}_i, \hat{b}_i, \overline{b}_i), \forall i \in N$.

Agents' shares are triangular fuzzy numbers $\hat{X}_i = (\hat{x}_i^L, \hat{x}_i^C, \hat{x}_i^R)$ where $\hat{x}_i^L$ are computed for cost equal to $\underline{c}$ and initial money amounts $\underline{b}_i$ (analogically we find $\hat{x}_i^C$ and $\hat{x}_i^R$). When the shares for $\underline{c}$ are bigger than the shares for $\overline{c}$, we take for $\hat{x}_i^L$ the shares computed for $\overline{c}$, and for $\hat{x}_i^R$ shares computed for $\overline{c}$.

Desired agents' money amounts are right-sided triangular fuzzy numbers $X_i = (\underline{x}_i, \underline{x}_i, \hat{x}_i^C), \forall i \in N_1$, $X_i = (\hat{x}_i^C, \hat{x}_i^C, \overline{x}_i), \forall i \in N_2$, where $\underline{x}_i = (1-\alpha)\hat{x}_i^C$ i $\overline{x}_i = (1+\beta)\hat{x}_i^C$. Resulting shares are found as in the previous case

As the highest level of λ is reached on $\widetilde{x}_i^C$, put $x_i = \widetilde{x}_i^C$ for $\forall i \in N_1$. Then $\widetilde{C} = \left(\underline{c} - \sum_{i \in N_1} x_i ; \hat{c} - \sum_{i \in N_1} x_i ; \overline{c} - \sum_{i \in N_1} x_i \right)$.

For the agents from the second group the problem is reduced to the problem with the fuzzy costs and can be solved using the abovementioned techniques.

Fuzzy Rationing Methods and Inequality

Important question is "How to compare obtained distributions of resources?". One of the ways to do so is to use inequality measures (Gini, Theil, Atkinson e.t.c. [De Maio, 2007]) for corresponding vectors of shares. In this case the "better" distribution is one that has lower inequality rate [Voloshyn, Laver, 2015].

As the abovementioned methods are based on the redistribution of resources between agents, the following statement holds:

Theorem. Let us consider some rationing method *r(N,c,b)* (with exception of uniform gains method) and its fuzzy generalization. Suppose that for corresponding shares vectors $x \in R^N$ and $x' \in R^N$ is true that $b_i - x_i \ge 1$ and $b_i - x_i' \ge 1$ for each $i \in N_2$. Then $I \ge I'$, where *I* is some inequality measure

(Gini, Theil, Atkinson), computed for agents' "free money" (which is left after the distribution) ($b_i - x_i$) using *r* and I' - the corresponding inequality measure for $x' \in R^N$.

The proof of this theorem is given in [Voloshyn, Laver, 2015].

Generalizations, Based on the Use of Fuzzy Arithmetic

Other approach to generalization of rationing methods is based on the use of fuzzy arithmetic. In this approach, we use fuzzy analogs of arithmetic operations and for comparison of fuzzy numbers we use their expected values [Laver. 2015].

Let the agents' shares and cost value be trapezoidal fuzzy numbers $\widetilde{b}_i = \left(b_i^{L_1}; b_i^{L_2}; b_i^{R_1}; b_i^{R_2}\right),\ \forall i \in N$, and $\widetilde{c}_i = \left(c^{L_1}; c^{L_2}; c^{R_1}; c^{R_2}\right)$. In this approach, the use of triangular fuzzy numbers and crisp numbers can be viewed as a particular case. For triangular fuzzy numbers we have $b_i^{L_2} = b_i^{R_1}$ or $c^{L_2} = c^{R_1}$, in the case of crisp numbers we have to deal with trapezoidal fuzzy numbers where all the four components are equal.

The easiest for computation in this type of generalization is the proportional method. We can get the corresponding formula for this case:

$$\widetilde{x}_i = \left(\frac{b_i^{L_1}}{\sum_{i=1}^{n} b_i^{R_2}} c^{L_1}; \frac{b_i^{L_2}}{\sum_{i=1}^{n} b_i^{R_1}} c^{L_2}; \frac{b_i^{R_1}}{\sum_{i=1}^{n} b_i^{L_2}} c^{R_1}; \frac{b_i^{R_2}}{\sum_{i=1}^{n} b_i^{L_1}} c^{R_2} \right) \tag{11}$$

In the case of other rationing methods, we use the classical algorithms, using fuzzy arithmetic and using expected values of fuzzy numbers for comparison.

Let us consider an illustrative example.

There are five agents. We have to distribute *c*=(29,30,31) among them. Agents' initial money amounts are (3,4,5), (11,12,13), (19,20,21), (23,24,25), (29,30,31).

Consider the proportional method. As triangular fuzzy numbers are a particular case of trapezoidal fuzzy numbers, we can use (11). We get following shares: (0,92;1,33;1,82); (3,36;4;4,74); (5,8;6,67; 7,66); (7,02; 8; 9,12); (8,85; 10; 11,31).

Consider the uniform losses method. If we divide the costs equally, the share of each agent will be (5,8;6;6;6,2).

The expected value of this fuzzy number is $\frac{1}{4}(5{,}8 + 2 \cdot 6 + 6{,}2) = 6$. Let us compare it with the expected value of the first agent's initial money amount: $\frac{1}{4}(3 + 2 \cdot 4 + 5) = 4 < 6$. So, the first agents share will be his money amount, thus (3;4;4;5).

We have to distribute (29,30,30,31) - (3,4,4,5)= (26,26,26,26) cost units. This number has to be divided between four agents. The share of each agent will be (6,5;6,5;6,5;6,5), which is equivalent to the crisp number 6,5 (which is the expected value of this fuzzy number). Expected value of the second agent is $\frac{1}{4}(11+2\cdot 12+13)=12>6{,}5$, so the second agent's share is (6,5;6,5;6,5;6,5). As other agent's initial money amounts are also bigger (it's easy to check this by computing the expected values of these fuzzy numbers), the shares of all the remaining agents also will be equal to (6,5;6,5;6,5;6,5). So we can present the shares as triangular fuzzy numbers (3,4,5); (6.5;6,5;6,5); (6.5;6,5;6,5); (6.5;6,5;6,5); (6.5;6,5;6,5).

Consider the uniform gains method. Let us find $\frac{\sum_{i=1}^{n}\tilde{b}_i-\tilde{c}}{n}$ ("free money" amount of the maximal coalition). The result is trapezoidal fuzzy number (10,8; 12; 12; 13,2). The expected value of this number is 12. As 12>4 (the expected value of the first agent's initial money amount), the first agent's share will be (0;0;0;0). So we have to find the shares of four more agents. Let us compute the amount of "free money" left:

$$\frac{\sum_{i=2}^{5}\tilde{b}_i-\tilde{c}}{4}=(12{,}75;14;14;15{,}25).$$

The expected value of this fuzzy number is 56/4 = 14. As the expected value of the second agent's initial money amount is 12<14, the second agent's share will also be (0;0;0;0). Exclude the second agent from the distribution and compute the new value of "free money". We have:

$$\frac{\sum_{i=3}^{5}\tilde{b}_i-\tilde{c}}{3}=(13\tfrac{1}{3};14\tfrac{2}{3};14\tfrac{2}{3};16).$$

The expected value of this fuzzy number will be $14\frac{2}{3}$. This is less than 20 (expected value of the third agent's money). So the third agent's share will be a fuzzy number

$$\tilde{b}_3-\frac{\sum_{i=3}^{5}\tilde{b}_i-\tilde{c}}{3}=(3;5\tfrac{1}{3};5\tfrac{1}{3};6\tfrac{2}{3}).$$

There are $\tilde{c}-\tilde{x}_3=(22\frac{1}{3};24\frac{2}{3};24\frac{2}{3};28)$ cost units left to distribute. As the expected values of the fourth and fifth agent's initial money amounts are also larger than $14\frac{2}{3}$, we can find their shares - $(7;9\frac{1}{3};9\frac{1}{3};11\frac{2}{3})$ and $(13;15\frac{1}{3};15\frac{1}{3};17\frac{2}{3})$. So the shares are (0,0,0); (0,0,0); $(3;5\frac{1}{3};6\frac{2}{3})$; $(7;9\frac{1}{3};11\frac{2}{3})$ i $(13;15\frac{1}{3};17\frac{2}{3})$. We can find the agents' shares using Talmudic, reverse Talmudic and Piniles' methods [Thomson, 2013]. As these methods are based on uniform gains and uniform losses methods, we can omit the calculations. Resulting shares are (1.5,2,2.5); (5.5,6,6.5); (6.67,7.33,8); (6.67,7.33,8);

(6.67,7.33,8) for Talmudic method, (0,0,0); (1.25,2.75,4.25); (5.25,6.75,8.25); (7.25,8.75,10.25); (10.25, 11.75,13.25) for reverse Talmudic method, (1.5,2,2.5); (5.5,6,6.5); (6.67,7.33,8); (6.67,7.33,8); (6.67,7.33,8) for Piniles' method.

New Approach to the Fuzzy Gemeralizations

The abovementioned generalizations allow to take into account the fuzziness of input data, but it is hard to build an axiomatic characterization for them. To avoid these difficulties, we propose to construct fuzzy generalizations using the arithmetic operations on fuzzy numbers proposed in [Gani, Assarudeen, 2012].

Consider the proportional method.

Let the agents' money amounts and the cost value be triangular fuzzy numbers: $B_i = \left(b_i^L; b_i^C; b_i^R\right)$, $C_i = \left(c^L; c^C; c^R\right)$. To find the agents' shares we will use operations of addition and multiplication of fuzzy numbers and generalized division operation.

Suppose that the following condition holds:

$$\left|\frac{\left(b_i^R c^R - b_i^L c^L\right)/2}{\left(b_i^R c^R + b_i^L c^L\right)/2}\right| \geq \left|\frac{\left(\sum_{i=1}^{n} b_i^R - \sum_{i=1}^{n} b_i^L\right)/2}{\left(\sum_{i=1}^{n} b_i^R + \sum_{i=1}^{n} b_i^L\right)/2}\right|, \text{ thus } \frac{b_i^R c^R - b_i^L c^L}{b_i^R c^R + b_i^L c^L} \geq \frac{\sum_{i=1}^{n} b_i^R - \sum_{i=1}^{n} b_i^L}{\sum_{i=1}^{n} b_i^R + \sum_{i=1}^{n} b_i^L}, \ \forall i \in N. \tag{12}$$

Then we can use the division operation, proposed in [Gani, Assarudeen, 2012]:

$$PR_i = \left(\frac{b_i^L}{\sum_{i=1}^{n} b_i^L} c^L; \frac{b_i^C}{\sum_{i=1}^{n} b_i^C} c^C; \frac{b_i^R}{\sum_{i=1}^{n} b_i^R} c^R\right), \tag{13}$$

where PR_i - fuzzy share of i-th agent, found by using the proportional method.

Let us consider the numerical example.

There are three agents, their initial money amounts are triangular fuzzy numbers: (2,3,4), (4,5,6), (11,12,13). We have to distribute (9,10,11) cost units among them. To use (13) we have to check (12) on each step.

As, $\sum_{i=1}^{n} B_i = (17;20;23)$, we have $\frac{\sum_{i=1}^{n} b_i^R - \sum_{i=1}^{n} b_i^L}{\sum_{i=1}^{n} b_i^R + \sum_{i=1}^{n} b_i^L} = \frac{23-17}{23+17} = \frac{6}{40} \approx 0{,}15 \cdot$

So: $\frac{b_1^R c^R - b_1^L c^L}{b_1^R c^R + b_1^L c^L} = \frac{4 \cdot 11 - 2 \cdot 9}{4 \cdot 11 + 2 \cdot 9} = \frac{26}{62} \approx 0{,}4194 > 0{,}15;\ \frac{b_2^R c^R - b_2^L c^L}{b_2^R c^R + b_2^L c^L} = \frac{30}{102} \approx 0{,}2941 > 0{,}15;$

$$\frac{b_3^R c^R - b_3^L c^L}{b_3^R c^R + b_3^L c^L} = \frac{44}{242} \approx 0{,}18 > 0{,}15$$
.

That means that we can find the shares using (13): $\left(\frac{18}{17};1.5;\frac{44}{23}\right)$; $\left(\frac{36}{17};2.5;\frac{66}{23}\right)$; $\left(\frac{99}{17};2.5;\frac{143}{23}\right)$.

The advantage of this approach is that finding the agents' shares in this case is reduced to finding the solution of three separate problems – left, central and right (for corresponding parts of triangular fuzzy numbers). So the characterization of the crisp proportional method can be used for its fuzzy generalization.

But we should check (12) on every step and this makes the use of this generalization rather difficult. On the other hand, this type of generalization gives us an opportunity to build the characterization of fuzzy rationing methods. Therefore, this type of generalization is worth of further research.

Conclusion

There are many ways of how we can obtain fuzzy generalizations of classical rationing methods. In this article we considered three of them. Every approach has its benefits and disadvantages. The first approach allows to lower the inequality between agents, but for this approach (as well as for the second one) it is difficult to build the characterization. For the third approach it is easy to build the characterization, but it can be used for a limited set of problems.

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