PROBLEM OF RESTORING THE FUNCTIONS-SIGNALS BY FINITE SET OF DATA WITH ERRORS

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***Abstract**: This paper researches the problem of restoring the functions-signals which allow a spectral representation by a finite set of data which is corrupted by errors. The authors consider the cases when the errors are random values and which certain statistical information is either known or not (the errors are considered as fuzzy values). The pseudoinverse solution, solution in the form of an expansion by eigenfunctions as well as Pareto-optimal and the maximum plausible estimations are constructed. A connection between them is established.*

***Keywords**: function-signal, pseudoinversion, maximum plausible estimations, Pareto-optimization, fuzzy values.*

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Introduction

Let's consider the class of signals that can be represented by the following structural formula:

$$f(z)=\int_Q g(z,t)u(t)dt\ ,\ z\in D, \tag{1}$$

where $u(t)$ and $g(z,t)$ (complex-valued functions) are a spectrum and a kernel (transfer function) of a signal; $Q\subset R^{n_1}$, $D\subset R^{n_2}$; $u(t)\in L_2[Q]$, $g(z,t)$ is a continuous function at $D\times Q$. The function $f(z)$ admitting the integral representation should be named a function-signal.

We understand the problem of restoring the function-signal (1) by the finite set of data as the following interpolation problem: it is necessary to restore (to estimate) the value $f(z)$ at the arbitrary point $z\in D$ by the values $f(z)$ at the points $z_1,...,z_m\in D$, known with random errors ν_k , $k=1,...,m$ In other words, it is necessary to estimate the linear functional (1) at the arbitrary point $z\in D$ by a set of values m of the linear functionals:

$$y_k=\int_Q g_k(t)u(t)dt+\nu_k\ ,\ k=1,...,m, \tag{2}$$

We assume that statistical characteristics (mathematical expectation and correlation matrix) of random errors are known, and $M(\nu)=0$, $M(\nu\nu^*)=R$, $\det R\neq 0$, $\nu=(\nu_1,...,\nu_m)^T$, $*-$ is a symbol of Hermitian conjugation. It obviously follows from this that the problem of restoring the function-signal by values of derivatives.

This problem has many important applications in the theory of communication, spectral estimation, control theory, radio- and hydrolocation, radioastronomy, optics, radio astronomy, medicine and other applied spheres.

The stated problem is incorrect in the mathematical sense because of incomplete data (the finite set of values $f(z_k)$) and the presence of errors in them, and therefore requires special approaches to its solving.

Traditional approaches, similar to Tikhonov regularization which are based on minimizing the residual norm and on narrowing the set of possible solutions up to a compact, don't take into consideration the statistical information on errors at the data. Furthermore, it isn't always possible to estimate the error of the regularized solution. Therefore, the authors have set themselves the task to consider other possible approaches to solving the defined problem, both in terms of the availability of the statistical information on errors and in its absence (in this case the errors were considered as the fuzzy variables) and to establish a connection between the obtained results and to offer an optimal solution of this problem.

Pseudoinverse solution

As the first approach, let's consider the solution of the system of integral equations (2) which is obtained by means of pseudoinversion tools [Альберт, 1977].

Now we rewrite the system (2) in the operator form:

$$y = Gu + \nu \tag{3}$$

$$y = (y_1,...,y_m)^T,\ G = \int_Q g(t)dt : L_2[Q] \to C^m,\ g(t) = (g_1(t),...,g_m(t))^T .$$

Let's denote the unknown vector by f : $f = Gu = (f_1,...,f_m)^T$.

By definition [Альберт, 1977], the operator of the following form is named as the pseudoinverse to the operator G :

$$G^{-} = \lim_{\alpha \to 0} G^{*}(GG^{*} + \alpha E)^{-1},\ \alpha > 0 .$$

G^{*} , presented here is an operator conjugated with G relative to scalar products:

$$(G^{*}r,u)_{L_2[Q]} = (r,Gu)_{C^m},\ r = (r_1,...,r_m)^T ,$$

$u = u(t)$, E is an identity matrix.

In this condition, G^{*} is m -dimensional vector-function of the following form: $G^{*} = g^{*}(t) = (\overline{g}_1(t),...,\overline{g}_m(t))$.

The operator GG^{*} is a covariance matrix of kernels of the system (2):

$$K = (k_{ij})_{i,j=1}^{m} = (\int_Q g_i(t)\overline{g}_j(t)dt)_{i,j=1}^{m} ,$$

which characterizes the “quality” of the points z_k for restoring (the matrix K , depending on the choice of $z_k, k = 1,...,m$ may be better or worse conditioned). Then the estimation of the spectrum $u(t)$ will be as follows $(\alpha > 0)$:

$$\hat{u}(t) = G^{-}y = \lim_{\alpha \to 0} g^{*}(t)(K + \alpha E)^{-1} y \tag{4}$$

and

$$\hat{f}_0(z) = \lim_{\alpha \to 0} \int_Q g(z,t)g^{*}(t)dt(K + \alpha E)^{-1} y \tag{5}$$

The estimation (4) can be written as:

$$\hat{u}(t) = \lim_{\alpha \to 0}(-\frac{1}{\Delta_\alpha}) \begin{vmatrix} 0 & \overline{g}_1(t) & ... & \overline{g}_m(t) \\ y_1 & k_{11} + \alpha & ... & k_{1m} \\ ... & ... & ... & ... \\ y_m & k_{m1} & ... & k_{mm} + \alpha \end{vmatrix},\ \alpha > 0 \tag{6}$$

$$\Delta_\alpha = \begin{vmatrix} k_{11}+\alpha & ... & k_{1m} \\ ... & ... & ... \\ k_{m1} & ... & k_{mm}+\alpha \end{vmatrix}, \ \alpha > 0.$$

Relying on the pseudoinversion properties at Hilbert spaces, there's known that for the linear operator equation $Aw = v$ the element $w = A^{-}v$ minimizes the residual norm $\|v - Aw\|$, and among all the elements that have this property $w = A^{-}v$ has a minimum norm. Therefore, for this problem one can interpret $\hat{u}(t) = G^{-}y$ as the energy spline (or spline that smoothes the values y_k, $k = 1,...,m$) (depending on whether $\min\|y - G\hat{u}(t)\|_{C^m}$ is reduced to zero or not), because:

$$\|u(t)\|^2_{L_2[Q]} = \int_Q |u(t)|^2 dt$$

is physically interpreted as the energy of spectrum of the signal (1).

The signal restoration by the energy spline is optimal in the following sense. As shown in [Белов, 1986], the set of values of the function $f(z)$ of the form (1) at the point $z \in D$, where $u(t)$ ranges over the class of functions $L_2[Q]$ satisfying the following inequality:

$$\int_Q |u(t)|^2 dt \le l^2,$$

is a circle $\left|f(z) - \hat{f}_0(z)\right| \le S(z)$ centered at the point $\hat{f}_0(z)$. It turns out, $\hat{f}_0(z)$ is the same as the energy spline (5). The radius $S(z)$ is determined by the dissipation of energy $l^2 - \int_Q |\hat{u}(t)|^2 dt$ and the choice of the points z_k. The energy spline is optimal in the sense of:

$$\inf_{\hat{f}(z)} \sup_{f(z)} \left|f(z) - \hat{f}(z)\right| \ge \left|f_u(z) - \hat{f}_0(z)\right|,$$

where $f_u(z)$ is the true signal and $\hat{f}(z)$ is the signal estimation.

Using the representation (6) for $\hat{u}(t)$, it is easy to show that the signal restoration by means of pseudoinversion is stable to small changes in the input information y. The value of deviation is:

$$\delta^2 = \int_Q |\hat{u}(t, y+\Delta y) - \hat{u}(t,y)|^2 dt = \lim_{\alpha\to 0} \frac{1}{|\Delta_\alpha|^2} \int_Q \begin{vmatrix} 0 & \bar{g}_1(t) & ... & \bar{g}_m(t) \\ ... & ... & ... & ... \\ \Delta y_m & k_{m1} & ... & k_{mm}+\alpha \end{vmatrix}^2 dt, \ \alpha > 0$$

and $\delta^2 \to 0$ at $\Delta y_i \to 0$, $i = 1,...,m$.

As a remark, we want to note that as a special case, the obtained results lead to Kotelnikov-Shannon theorem [Харкевич, 2007] (if there is no noises and $K = E$) and to various generalizations of this theorem for a finite number of counts. So, we can select as the counts not only the values of the function at the points z_k, but the values of its various derivatives either at these or other points that further should lead to a change of the vector $g(t)$.

Expansion by eigenfunctions of self-conjugate operator

When considering the pseudoinverse solution we don't use any information about the noise ν. Therefore, for the estimation (4)

$$\hat{u}(t) = G^{-}f + G^{-}\nu$$

the component $G^{-}\nu$ can uncontrollably effect upon the estimation. One can specify a way to reduce this effect, so, let's consider the estimation of the solution of the system (3) in the form of expansion on some basis. By multiplying (3) by G^{*} we transform the equation (3) to the following form:

$$G^{*}y = G^{*}Gu + G^{*}\nu \tag{7}$$

Then let's find the eigenvalues and eigenfunctions of the operator $G^{*}G$ from the following ratio:

$$G^{*}Ge_k(t) = \lambda_k e_k(t) \tag{8}$$

After denoting $Ge_k(t) = c_k$, we transform (8) to the form $GG^{*}c_k = \lambda_k Ge_k(t)$ or $(K - \lambda_k E)c_k = 0$. Therefore, the problem of finding the eigenvalues and eigenfunctions of the operator $G^{*}G$ is reduced to the problem of finding the eigenvalues and eigenvectors of the matrix K, which is Hermitian and exactly has m eigenvalues λ_k (taking multiplicity into account). Let's renumber them in nonincreasing order. The eigenvectors of the matrix K (the orthonormal system, $(c_j, c_k) = \delta_{jk}$) define the eigenfunctions of the operator $G^{*}G$ by the following equality:

$$e_k(t) = \lambda_k^{-1} g^{*}(t) c_k,\ k = 1,\dots,p \le m,\ \lambda_k \ne 0.$$

Clearly, that

$$(e_k(t), e_j(t)) = e_j^{*}(t)e_k(t) = (\lambda_k\lambda_j)^{-1} c_j^{*}GG^{*}c_k = (\lambda_k\lambda_j)^{-1} c_j^{*}Kc_k = \lambda_j^{-1}c_j^{*}c_k = \begin{cases} 0, & j \ne k \\ \lambda_k^{-1}1, & j = k \end{cases},$$

i.e. the system of eigenfunctions $e_k(t)$ is orthogonal. It is easy to make it orthonormal by multiplying it by a scalar $\lambda_k^{1/2}$:

$$\tilde{e}_k(t) = \lambda_k^{-1/2} g^{*}(t) c_k = \lambda_k^{-1/2} e_k(t).$$

Let's find the solution of the system (3) in the following form:

$$u(t) = \sum_{k=1}^{p} \gamma_k \tilde{e}_k(t) \tag{9}$$

Then, to find the coefficients γ_k we substitute (9) at (7):

$$G^{*}(y - \nu) = \sum_{k=1}^{p} \gamma_k G^{*}G\tilde{e}_k(t) = \sum_{k=1}^{p} \gamma_k \lambda_k \tilde{e}_k(t)$$

whence, it follows by multiplying on the scalarwise left:

$$\gamma_j = \lambda^{-1}(G^{*}(y - \nu), \tilde{e}_j(t)),\ j = 1,\dots,p.$$

Then

$$u(t) = \sum_{k=1}^{p} \lambda_k^{-1}(y - \nu, G\tilde{e}_k(t))\tilde{e}_k(t),$$

and the estimation of the solution of the equation (3) takes the following form:

$$\hat{u}(t)=\sum_{k=1}^{p}\lambda_k^{-1}(f,G\tilde{e}_k(t))\tilde{e}_k(t)+\sum_{k=1}^{p}\lambda_k^{-1}(\nu,G\tilde{e}_k(t))\tilde{e}_k(t) \quad (10)$$

The first summand in (10) determines the estimation of the signal spectrum in the absence of noise (the useful component), the second one represents the contribution to the solution of the noise component. Since λ_k are arranged in nonincreasing order $\lambda_1 \ge \ldots \ge \lambda_p$, the noise component after achieving a sufficiently large index k begins to dominate over the useful component of the estimation of the solution.

The problem of choosing the value $p_1 \le p$ can be solved as follows. Let's consider the normalized parameter:

$$\mu(k)=\left(\frac{\lambda_1^2+\ldots+\lambda_k^2}{\lambda_1^2+\ldots+\lambda_p^2}\right)^{1/2},\ 1\le k\le p.$$

When approaching k to p, we have $\mu(k)\to 1$.

For some situations $\mu(k)$ is close to unity even for k significantly lower than p. The desired value p_1 can be defined by the minimum value k under which one can consider $\mu(k)$ sufficiently close to unity. The question of what is meant by sufficient proximity to unity is subjective and the answer to it should be determined by means of the computational experiment taking into account the specifics of the problem being solved.

Let's return to the estimation (10):

$$\hat{u}(t)=\sum_{k=1}^{p}\lambda_k^{-1}(y,G\tilde{e}_k(t))\tilde{e}_k(t)=\sum_{k=1}^{p}(y,Ge_k(t))e_k(t)=\sum_{k=1}^{p}e_k(t)c_k^*y=\sum_{k=1}^{p}\lambda_k^{-1}G^*c_kc_k^*y.$$

Using the property of pseudoinverse solution of minimizing the square of the residual norm $\|Gu-y\|_{C^m}^2$ it is easy to show that:

$$G^-=\sum_{k=1}^{p}\lambda_k^{-1}G^*c_kc_k^* \quad (11)$$

Thus, defining G^- in the form (11) by selecting the appropriate value p makes it possible to reduce the impact of component $G^-\nu$ at the estimation $\hat{u}(t)$.

Pareto-optimal estimations

It is possible to go into further details of the influence of the noise component on the restoring quality using the idea [Пытьев, 1989] about reduction of measurement up to the output from the specified device. Unlike to the above-mentioned approaches, we shouldn't look for estimations of solutions $\hat{u}(t)$ and should estimate $f(z)$ at once. Following the idea of Pitiev, we consider the system (2) as a linear scheme of measurement by devices with characteristic functions $g_k(t)$, $k=1,\ldots,m$, and we consider the data y_k, $k=1,\ldots,m$ as the output from these devices, which is distorted by noise. Then the problem of estimating the value of the functional $f(z)$ at the arbitrary point $z\in D$ can be considered as the problem of estimating the output from the device Π_z:

$$\Pi_z u=\int_Q \pi(t)u(t)dt$$

with the characteristic function $\pi(t)=g_z(t)=g(z,t)$ by the data (2). Let's look for the desired estimation at the class of estimations that are linear relatively the mentioned estimations, i.e. we should construct such vector

$B \subset C^m$ that $\hat{\Pi}_z u = By$. For that, we transform (3) to the following form:

$$By = \Pi_z u + (BG - \Pi_z)u + B\nu ,$$

which shows that the By is different from $\Pi_z u$ by two summands: the displacement $(BG - \Pi_z)u$ and the noise background $B\nu$. Since

$$\|By - \Pi_z u\|^2 \le \|BG - \Pi_z\|^2 \|u\|^2 + M\|B\nu\|^2 ,$$

and $u(t)$ is unknown, then for finding the vector B we should solve the following two-criteria problem of minimization as follows:

$$\begin{cases} \varphi(B) = \|BG - \Pi_z\|^2 \to \min\limits_B \\ h(B) = M\|B\nu\|^2 \to \min\limits_B \end{cases} \quad (12)$$

The first criterion (the operator residual), presented here characterizes the proximity of the characteristic function of the synthesized device BG , whose input should be adopted as the desired estimation $\hat{\Pi}_z u$, to the characteristic function $\pi(t)$, the second criterion characterizes the level of noise background (dispersion) of the desired estimation. Let's consider two-criteria decision of the problem in the sense of Pareto optimization [Ногин, 2005], i.e. we should look for such vector B that both criteria (12) are nondecreased at the same time. In view of convexity of the optimization criteria the problem (12) can be reduced [Глинкин, 1981] to a one-criterion problem of the following form:

$$\lambda\varphi(B) + (1-\lambda)h(B) \to \min_B ,\ \lambda \in (o,1)$$

$$\varphi(B) = tr\int_Q Bg(t) - \pi(t))(Bg(t) - \pi(t))^* dt ,$$

$$h(B) = trBRB^* .$$

By solving this problem we should obtain the continuum family of the desired estimations as follows:

$$\hat{\Pi}_z u = By = \Pi G^*(K + \alpha R)^{-1} y ,\ 0 < \alpha < \infty \quad (13)$$

or

$$\hat{f}(z) = \int_Q g(z,t)g^*(t)dt(K + \alpha R)^{-1} y ,\ 0 < \alpha < \infty \quad (14)$$

which are naturally named the Pareto-optimal estimations.

In a view of the solution of the problem (12) all the estimations (13) are equal.

Every estimation from (13) has both the nondecreased value of the operator residual:

$$\varphi(\alpha) = tr\Pi_z G^*((K + \alpha R)^{-1} K(K + \alpha R)^{-1} - 2(K + \alpha R)^{-1})(\Pi_z G^*)^*$$

and the level (dispersion)of noise background:

$$h(\alpha) = tr\Pi_z G^*(K + \alpha R)^{-1} R(K + \alpha R)^{-1}(\Pi G^*)^* .$$

Analyzing these functions shows that $h(\alpha)$ monotonically decreases and $\varphi(\alpha)$ monotonously increases with increasing α . Besides, $h'(\alpha)$ and $\varphi'(\alpha)$ are connected by the “conservation law”

$$\alpha h'(\alpha) + \varphi'(\alpha) = 0 .$$

To select the specific estimation $\hat{\Pi}_z u$ from the family (13) one can use, for example, the principle of the guaranteed result, according to which $\alpha = \arg\min_{\alpha} \max_{(h,\varphi)}(h(\alpha), \varphi(\alpha))$. As a result of the monotonicity of the optimization criteria relative to α and because of opposite trends to change these criteria such value of the parameter is unique. To realize a certain compromise between the optimization criteria one can use other general principles of multi-criteria optimization, namely, the principle of uniform optimality $\alpha = \arg\min_{\alpha}(\varphi(\alpha) + h(\alpha))$, the principle of reasonable compromise $\alpha = \arg\min_{\alpha}(\varphi(\alpha)h(\alpha))$, "Eldorado" principle $\alpha = \arg\min_{\alpha}(\varphi^2(\alpha) + h^2(\alpha))$.

In fact, choosing the specific estimation of the family (13) needs clearly understanding of what is preferable: to minimize the noise dispersion, which leads to increasing the operator residual generated by the data incompleteness, or contrariwise. Therefore, to select the appropriate value of the parameter α let's construct the Pareto set (parametric dependence $\varphi(h)$, $\alpha \in (0, \infty)$), where the decision-maker, being acquainted with the ratio between the noise level and the operator residual value should select a suitable point which is corresponding to one or another value of the parameter α.

As a remark, we want to note that the set of the Pareto-optimal estimations of the functions-signals (14) contains the estimation, obtained by the pseudoinversion tools (5), if the system is previously decorrelated by multiplying by $R^{-1/2}$, then it will coincide with the estimation (14) at $\alpha = +0$.

Maximally plausible estimations

Let's briefly dwell on another approach to restoring the function-signal $f(z)$, namely, the maximally plausible restoration. In that case, we should find the estimation $\hat{u}(t)$ as a function that maximizes the functional of plausibility.

Let's suppose the random noise vector is distributed by the normal law with the following probability density distribution:

$$p(\nu) = ((2\pi)^m \det R)^{-1/2} \exp(-\tfrac{1}{2}\nu^* R^{-1}\nu),$$

and then we obtain the functional of plausibility as follows:

$$\Phi(u) = ((2\pi)^m \det R)^{-1/2} \exp\left\{-\tfrac{1}{2}(y - \int_Q g(t)u(t)dt)^* R^{-1}(y - \int_Q g(t)u(t)dt)\right\}, \tag{15}$$

The problem of maximizing the functional (15) is equivalent to the problem of minimizing the following quadratic functional:

$$J(u) = (y - \int_Q g(t)u(t)dt)^* R^{-1}(y - \int_Q g(t)u(t)dt), \tag{16}$$

By calculating the variation of the functional (16) and equating it to zero, we obtain the Euler equation:

$$\int_Q g^*(t)R^{-1}g(\tau)u(\tau)d\tau = g^* R^{-1} y.$$

This is a Fredholm equation of the 1st kind. The problem of finding of its solutions belongs to a class of incorrect ones and requires regularization. For the Tikhonov regularizer:

$$T_\alpha(u) = J(u) + \alpha \int_Q |u(t)|^2 dt, \; \alpha > 0$$

the Euler equation that determines the desired estimation $\hat{u}(t)$, is as follows:

$$\int_Q g^*(t)R^{-1}g(\tau)u(\tau)d\tau = g^*(t)R^{-1}y - \alpha u(t),$$

where $\alpha > 0$ is a regularization parameter , which implies that:

$$\hat{u}(t) = g^*(t)(k + \alpha R)^{-1}y \text{ , } \alpha > 0$$

and in that case, $\hat{f}(z)$ coincides with Pareto-optimal estimations (14).

Restoration under fuzzy data errors

Let's consider the case when there is no statistical information about the vector of errors. In this case, we can assume that the data y are distorted by fuzzy noise ν. At the same time, a problem formulation should be added by information on distribution of the fuzzy element $\nu - \varphi^{\nu}(\cdot): C^m \to L$ (it is assumed that an expert should specify this distribution). One believes that this distribution is such that a bigger noise corresponds to a smaller possibility, and this is a natural assumption for many measurement processes. In accordance with specifying the distribution of fuzzy value [Пытьев, 2000], the scale L is defined at the segment $[0,1]$ in natural order, specified by the inequality $\le$ and by two rules of composition: addition and multiplication (max and min, respectively).

In that case, to construct the optimal estimation let's use the results from the paper [Zavorotnyy, 2008]: we should find the estimation $\Pi_z u$ in the form of By , where B is a solution of the problem of simultaneous minimizing a norm of the operator residual and maximizing the need of correctness of fuzzy value estimating, or the integral of necessity by $I(\Pi_z u, d(x))$, written in terms of the theory of possibilities.

$$\begin{cases} \|BG - \Pi_z\| \to \min\limits_B \\ \theta \sup\limits_x \sup\limits_u \min(\varphi^y(x,u), \theta I(\Pi_z u, d(x))) \to \max\limits_B \end{cases} \quad (17)$$

In (9) $I(\Pi_z u, d(x))$ is a possibility of lack of the error that occurs under selecting the estimating strategy $d(x)$ as a value $\Pi_z u$ for each value u, θ is an involution [Пытьев, 2000], $\varphi^y(z,u)$ is the joint distribution of the values y and u. Solving the problem leads to the following result: the solution of the problem (2) under fuzzy errors in the data y coincides with (4), and the corresponding estimation of the function-signal $f(z)$ is in the form of (5).

Conclusion

In this paper we constructed and thoroughly investigated (relatively the data), linear estimations of functions-signals on a finite set of data containing errors. The authors showed that all constructed estimations (namely, pseudoinverse, maximally plausible at normal distribution of the vector of noise, in the form of an expansion by eigenfunctions of self-conjugate operator) are contained in the set of Pareto-optimal estimations (14). One can control the effect of errors by selecting the parameter value $\alpha \in (0,\infty)$, the smaller α the smaller a noise background of the estimation. However, the decrease of α leads to increasing the operator residual, generated by data incompleteness, and the expert has to seek a suitable compromise between these two components which distort the desired estimation. The authors also showed the following fact: if there is no statistical information about noise and one can naturally consider the errors as fuzzy values, the desired estimation coincides with the Pareto-optimal estimation (14) on the predecorrelated data, which corresponds to $\alpha = +0$.

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