INDEX MATRICES WITH FUNCTION-TYPE OF ELEMENTS

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Abstract: *Index Matrices (IMs) are extensions of the ordinary matrices. Some of the operations and relations defined over IMs are analogous of the standard ones, while there have been defined new ones, as well. Operators that do not have analogue in matrix theory have been defined, as well. In general, the elements of an IM can be real or complex numbers, as well as propositions or predicates. In the present paper, IMs with their elements being functions, are defined and some of their properties are discussed.*

Keywords: *Function, Index matrix, Operation, Operator, Relation.*

Introduction

The concept of Index Matrix (IM) was introduced in 1984 in [Atanassov, 1984], but during the next 25 years only some of its properties have been studied (see, e.g. [Atanassov, 1987]) and in general the concept has only been used as an auxiliary tool in generalized nets theory (see [Alexieva et al., 2007; Atanassov, 1991; Atanassov, 2007; Radeva et al., 2002]) and in intuitionistic fuzzy sets theory (see [Atanassov, 1999; Atanassov, 2012]). Some authors found an application of the IMs in the area of number theory, [Leyendekkers et al., 2007].

The first step towards developing the theory of IMs was done in [Atanassov, 2010a], where the concept was defined, as follows.

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}$ be the set of the real numbers. By IM with index sets K and L $(K, L \subset \mathcal{I})$, we denote the object:

$$[K, L, \{f_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \le i \le m$, and $1 \le j \le n : a_{k_i,l_j} \in \mathcal{R}$.

Six operations, six relations and ten operators are defined over IMs in [Atanassov, 2010a; Atanassov, 2013; Atanassov et al., 2013a]. In [Atanassov, 2010a] are discussed the cases when the elements a_{k_i,l_j} are: real numbers, elements of the set $\{0, 1\}$, and propositions or predicates. In [Atanassov, 2010b], the case is described, when the elements a_{k_i,l_j} are intuitionistic fuzzy pairs (see, e.g. [Atanassov, 2012; Atanassov et al., 2013b]). Here, we discuss a new case: when the elements a_{k_i,l_j} are functions.

Let the set of all used functions be $\mathcal{F}$.

The research over IMs with function-type of elements develops in the following two cases:

- each function of set $\mathcal{F}$ has one argument and it is exactly x (i.e., it is not possible that one of the functions has argument x and another function has argument y) – let us mark the set of these functions by $\mathcal{F}^1_x$;
- each function of set $\mathcal{F}$ has one argument, but that argument might be different for the different functions or the different functions of set $\mathcal{F}$ have different numbers of arguments.

Definition of the index matrix with function-type of elements

Let all used functions from $\mathcal{F}_x^1$ have one argument and let it be exactly x. Then, the IM with Function-type of Elements (IMFE) has the form

$$[K, L, \{f_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & f_{k_1,l_1} & f_{k_1,l_2} & \dots & f_{k_1,l_n} \\ k_2 & f_{k_2,l_1} & f_{k_2,l_2} & \dots & f_{k_2,l_n} \\ \vdots & & & & \\ k_m & f_{k_m,l_1} & f_{k_m,l_2} & \dots & f_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \le i \le m$, and $1 \le j \le n : f_{k_i,l_j} \in \mathcal{F}_x^1$.

Standard operations over IMFEs Introduction

For the IMFEs $A = [K, L, \{f_{k_i,l_j}\}], B = [P, Q, \{g_{p_r,q_s}\}]$, the operations that are analogous of the standard IM operations are the following.

(a) addition ($+$): $A \oplus_+ B = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ f_{k_i,l_j} + g_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \perp, & \text{otherwise} \end{cases},$$

where here and below, symbol $\perp$ denotes the lack of operation in the respective place.

(b) addition ($\times$): $A \oplus_\times B = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ f_{k_i,l_j} . g_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \perp, & \text{otherwise} \end{cases},$$

(c) termwise multiplication ($+$): $A \otimes_+ B = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{k_i,l_j} + g_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$;

(d) termwise multiplication ($\times$): $A \otimes_\times B = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{k_i,l_j}.g_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$;

(e) multiplication $A \odot B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q \\ \sum\limits_{l_j = p_r \in L \cap P} f_{k_i,l_j}.g_{p_r,q_s}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \perp, & \text{otherwise} \end{cases}$$

(f) structural subtraction: $A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}]$, where "–" is the set–theoretic difference operation and

$$h_{t_u,v_w} = f_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(g) multiplication with a constant: $\alpha.A = [K, L, \{\alpha.f_{k_i,l_j}\}]$, where α is a constant.

(h) termwise subtraction: $A - B = A \oplus_+ (-1).B$.

For example, if we have the IMs X and Y with elements

$$f_i(x) = x^i, \;\; g_i(x) = \frac{1}{i.x}$$

for $i = 1, 2, ..., 6$, then

$$X = \begin{array}{c|ccc} & c & d & e \\ \hline a & f_1 & f_2 & f_3 \\ b & f_4 & f_5 & f_6 \end{array}, \quad Y = \begin{array}{c|cc} & c & r \\ \hline a & g_1 & g_2 \\ p & g_3 & g_4 \\ q & g_5 & g_6 \end{array},$$

then

$$X \oplus Y = \begin{array}{c|cccc} & c & d & e & r \\ \hline a & f_1 + g_1 & f_2 & f_3 & g_2 \\ b & f_4 & f_5 & f_6 & \perp \\ p & g_3 & \perp & \perp & g_4 \\ q & g_5 & \perp & \perp & g_6 \end{array} = \begin{array}{c|cccc} & c & d & e & r \\ \hline a & x + \frac{1}{x} & x^2 & x^3 & \frac{1}{2x} \\ b & x^4 & x^5 & x^6 & \perp \\ p & \frac{1}{3x} & \perp & \perp & \frac{1}{4x} \\ q & \frac{1}{5x} & \perp & \perp & \frac{1}{6x} \end{array}.$$

The problem with the "zero"-IMFE is similar to the "zero-IM": if $f_{k_i,l_j} \in \mathcal{F}$, then

$$I_\emptyset = [\emptyset, \emptyset, \{f_{k_i,l_j}\}].$$

Relations over IMFSs

Let everywhere, variable x obtain values in set $\mathcal{X}$ (e.g., $\mathcal{X}$ being a set of real numbers) and let $a \in \mathcal{X}$ be an arbitrary value of x. It is suitable to define for each function f with n arguments: $\nu(f) = n$.

Let the two IMFEs $A = [K, L, \{f_{k,l}\}]$ and $B = [P, Q, \{g_{p,q}\}]$ be given. We introduce the following definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.

Definition 1.a: The strict relation "inclusion about dimension", when the IMFE-elements of both matrices are elements of $\mathcal{F}_x^1$, is

$$A \subset_d B \text{ iff } (((K \subset P) \,\&\, (L \subset Q)) \,\vee\, ((K \subseteq P) \,\&\, (L \subset Q))$$

$$\vee\ ((K \subset P) \ \&\ (L \subseteq Q))) \ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})(f_{k,l}(a) = g_{k,l}(a)).$$

Definition 1.b: The strict relation “inclusion about dimension", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subset_d B \text{ iff } (((K \subset P) \ \&\ (L \subset Q)) \ \vee\ ((K \subseteq P) \ \&\ (L \subset Q))$$

$$\vee\ ((K \subset P) \ \&\ (L \subseteq Q))) \ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) = g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

Definition 2.a: The non-strict relation “inclusion about dimension", when the IMFE-elements of both matrices are elements of $\mathcal{F}_x^1$, is

$$A \subseteq_d B \text{ iff } (K \subseteq P) \ \&\ (L \subseteq Q) \ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(f_{k,l}(a) = g_{k,l}(a)).$$

Definition 2.b: The non-strict relation “inclusion about dimension", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subseteq_d B \text{ iff } (K \subseteq P) \ \&\ (L \subseteq Q) \ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) = g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

Definition 3.a: The strict relation “inclusion about value", when the IMFE-elements of both matrices are elements of $\mathcal{F}_x^1$, is

$$A \subset_v B \text{ iff } (K = P) \ \&\ (L = Q) \ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(f_{k,l}(a) < g_{k,l}(a)).$$

Definition 3.b: The strict relation “inclusion about value", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subset_v B \text{ iff } (K = P) \ \&\ (L = Q) \ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) < g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

Definition 4.a: The non-strict relation “inclusion about value", when the IMFE-elements of both matrices are elements of $\mathcal{F}_x^1$, is

$$A \subseteq_v B \text{ iff } (K = P) \ \&\ (L = Q) \ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(f_{k,l}(a) \leq g_{k,l}(a)).$$

Definition 4.b: The non-strict relation “inclusion about value", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subseteq_v B \text{ iff } (K = P) \ \&\ (L = Q) \ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) \leq g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

Definition 5.a: The strict relation “inclusion", when the IMFE-elements of both matrices are elements of $\mathcal{F}_x^1$, is

$$A \subset B \text{ iff } (((K \subset P) \ \&\ (L \subset Q)) \ \vee\ ((K \subseteq P) \ \&\ (L \subset Q))$$

$$\vee\ ((K \subset P) \ \&\ (L \subseteq Q))) \ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})(f_{k,l}(a) < g_{k,l}(a)).$$

Definition 5.b: The strict relation “inclusion", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subset B \text{ iff } (((K \subset P) \ \&\ (L \subset Q)) \ \vee\ ((K \subseteq P) \ \&\ (L \subset Q))$$

$$\vee\ ((K \subset P)\ \&\ (L \subseteq Q)))\ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ \ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) < g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

Definition 6.a: The non-strict relation “inclusion", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subseteq B \text{ iff } (K \subseteq P)\ \&\ (L \subseteq Q)\ \&\ (\forall k \in K)(\forall l \in L)(\forall a \in \mathcal{X})$$

$$(f_{k,l}(a) \leq g_{k,l}(a)).$$

Definition 6.b: The non-strict relation “inclusion", when the IMFE-elements of both matrices are elements not only of $\mathcal{F}_x^1$, is

$$A \subseteq B \text{ iff } (K \subseteq P)\ \&\ (L \subseteq Q)\ \&\ (\forall k \in K)(\forall l \in L)(\nu(f_{k,l}) = \nu(g_{k,l}))$$

$$\&\ \ (\forall a_1, ..., a_{\nu(f_{k,l})} \in \mathcal{X})(f_{k,l}(a_1, ..., a_{\nu(f_{k,l})}) < g_{k,l}(a_1, ..., a_{\nu(f_{k,l})})).$$

It can be directly seen that for every two IMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

New operations over IMFEs

Three new operations are introduced, that are analogous of the operations over IMs. Now, the hierarhical operators over IM cannot be introduced over IMFE.

01 Operations “reduction" over an IMFE

First, we introduce operations $(k, \perp)$- and $(\perp, l)$-reduction of a given IM $A = [K, L, \{f_{k_i,l_j}\}]$:

$$A_{(k,\perp)} = [K - \{k\}, L, \{h_{t_u,v_w}\}]$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \text{ for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L$$

and

$$A_{(\perp,l)} = [K, L - \{l\}, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \text{ for } t_u = k_i \in K \text{ and } v_w = l_j \in L - \{l\}.$$

Second, we define

$$A_{(k,l)} = (A_{(k,\perp)})_{(\perp,l)} = (A_{(\perp,l)})_{(k,\perp)},$$

i.e.,

$$A_{(k,l)} = [K - \{k\}, L - \{l\}, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \text{ for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L - \{l\}.$$

Theorem 1. For every IMFE A and for every $k_1, k_2 \in K, l_1, l_2 \in L$,

$$(A_{(k_1,l_1)})_{(k_2,l_2)} = (A_{(k_2,l_2)})_{(k_1,l_1)}.$$

Third, let $P = \{k_1, k_2, ..., k_s\} \subseteq K$ and $Q = \{q_1, q_2, ..., q_t\} \subseteq L$. Finally, we define the following three operations:

$$A_{(P,l)} = (...((A_{(k_1,l)})_{(k_2,l)})...)_{(k_s,l)},$$

$$A_{(k,Q)} = (...((A_{(k,l_1)})_{(k,l_2)})...)_{(k,l_t)},$$

$$A_{(P,Q)} = (...((A_{(p_1,Q)})_{(p_2,Q)})...)_{(p_s,Q)} = (...((A_{(P,q_1)})_{(P,q_2)})...)_{(P,q_t)}.$$

Obviously,

$$A_{(K,L)} = I_{\emptyset},$$

$$A_{(\emptyset,\emptyset)} = A.$$

Theorem 2. For every two IMs $A = [K, L, \{f_{k_i,l_j}\}], B = [P, Q, \{g_{p_r,q_s}\}]$:

$$A \subseteq_d B \text{ iff } A = B_{(P-K,Q-L)}.$$

Proof: Let $A \subseteq_d B$. Therefore, $K \subseteq P$ and $L \subseteq Q$ and for every $k \in K, l \in L$, for every $a \in X$: $f_{k,l}(a) = g_{k,l}(a)$. From the definition,

$$B_{(P-K,Q-L)} = (...((B_{(p_1,q_1)})_{(p_1,q_2)})...)_{(p_r,q_s)},$$

where $p_1, p_2, ..., p_r \in P - K$, i.e., $p_1, p_2, ..., p_r \in P$, and $p_1, p_2, ..., p_r \notin K$, and $q_1, q_2, ..., q_s \in Q - L$, i.e., $q_1, q_2, ..., q_s \in Q$, and $q_1, q_2, ..., q_s \notin L$. Therefore,

$$B_{(P-K,Q-L)} = [P - (P - K), Q - (Q - L), \{g_{k,l}\}] = [K, L, \{g_{k,l}\}] = [K, L, \{f_{k,l}\}] = A,$$

because by definition the elements of the two IMs, which are indexed by equal symbols, coincide.

For the opposite direction we obtain, that if $A = B_{(P-K,Q-L)}$, then

$$A = B_{(P-K,Q-L)} \subseteq_d B_{\emptyset,\emptyset} = B.$$

02 Operation “projection" over an IM

Let $M \subseteq K$ and $N \subseteq L$. Then,

$$pr_{M,N}A = [M, N, \{g_{k_i,l_j}\}],$$

where

$$(\forall k_i \in M)(\forall l_j \in N)(g_{k_i,l_j} = f_{k_i,l_j}).$$

Theorem 3. For every IMFE A and sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$ the equality

$$pr_{M_1,N_1}pr_{M_2,N_2}A = pr_{M_1,N_1}A$$

holds.

03 "Inflating operation" over an IM

We can define "inflating operation" that is defined for index sets $K \subset P \subset \mathcal{I}$ and $L \subset Q \subset \mathcal{I}$ by

$$^{(P,Q)}A =^{(P,Q)} [K, L, \{a_{k_i,l_j}\}] = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$b_{p_r,q_s} = \begin{cases} a_{k_i,l_j}, & \text{if } p_r = k_i \in K \text{ and } q_s = l_j \in L \\ \perp, & \text{otherwise} \end{cases}$$

04 Operation "substitution" over an IM

Let IM $A = [K, L, \{f_{k,l}\}]$ be given.

First, local substitution over the IM is defined for the couples of indices (p, k) and/or (q, l), respectively, by

$$[\frac{p}{k}]A = [(K - \{k\}) \cup \{p\}, L, \{f_{k,l}\}],$$

$$[\frac{q}{l}]A = [K, (L - \{l\}) \cup \{q\}, \{f_{k,l}\}],$$

Second,

$$[\frac{p}{k}\frac{q}{l}]A = [\frac{p}{k}][\frac{q}{l}]A,$$

i.e.

$$[\frac{p}{k}\frac{q}{l}]A = [(K - \{k\}) \cup \{p\}, (L - \{l\}) \cup \{q\}, \{f_{k,l}\}].$$

Obviously, for the above indices k, l, p, q:

$$[\frac{k}{p}]([\frac{p}{k}]A) = [\frac{l}{q}]([\frac{q}{l}]A) = [\frac{k}{p}\frac{l}{q}]([\frac{p}{k}\frac{q}{l}]A) = A,$$

Let the sets of indices $P = \{p_1, p_2, ..., p_m\}$, $Q = \{q_1, q_2, ..., q_n\}$ be given.

Third, for them we define sequentially:

$$[\frac{P}{K}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_n}{k_n}]A,$$

$$[\frac{Q}{L}]A = ([\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A),$$

$$[\frac{K}{P}\frac{Q}{L}]A = [\frac{P}{K}][\frac{Q}{L}]A,$$

i.e.,

$$[\frac{P}{K}\frac{Q}{L}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_m}{k_m}\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A = [P, Q, \{f_{k,l}\}]$$

Obviously, for the sets K, L, P, Q:

$$[\frac{K}{P}]([\frac{P}{K}]A) = [\frac{L}{Q}]([\frac{Q}{L}]A) = [\frac{K}{P}\frac{L}{Q}]([\frac{P}{K}\frac{Q}{L}]A) = A.$$

Theorem 4. For every four sets of indices P_1, P_2, Q_1, Q_2

$$[\frac{P_2}{P_1}\frac{Q_2}{Q_1}][\frac{P_1}{K}\frac{Q_1}{L}]A = [\frac{P_2}{K}\frac{Q_2}{L}]A.$$

Operations over IMFEs and IMs

Let the IM $A = [K, L, \{a_{k_i,l_j}\}]$, where $a_{k_i,l_j} \in \mathcal{R}$ and IMFE $F = [P, Q, \{f_{p_r,q_s}\}]$ be given. Then

(a) $A \boxplus F = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} a_{k_i,l_j}.f_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \\ \perp, & \text{otherwise} \end{cases},$$

with elements of $\mathcal{F}^1$;

(b) $A \boxtimes F = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = a_{k_i,l_j}.f_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{F}^1$;

(c) $F \oplus A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{p_r,q_s}(a_{k_i,l_j}), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \perp, & \text{otherwise} \end{cases}$$

with elements of $\mathcal{R}$;

(d) $F \otimes A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{p_r,q_s}(a_{k_i,l_j}),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{R}$.

Let the IM $A = [K, L, \{\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle\}]$, for the natural number $n \geq 2$, where $a^1_{k_i,l_j}, ..., a^n_{k_i,l_j} \in \mathcal{R}$ and IMFE $F = [P, Q, \{f_{p_r,q_s}\}]$, where $f_{p_r,q_s} : \mathcal{F}^n \to \mathcal{F}$ be given. Then

(e) $F \diamondsuit_{\oplus} A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \perp, & \text{otherwise} \end{cases}$$

with elements of $\mathcal{R}$;

(f) $F \diamondsuit_{\otimes} A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{R}$.

Conclusion

In future, we will discuss the two other cases for the form of functions: when each function of set $\mathcal{F}$ has one argument, but it can be different for the different functions and when the functions of set $\mathcal{F}$ are different and they have different numbers of arguments.

We will use the new operations for description of some components of the artificial intelligence. For example, in [Atanassov & Sotirov, 2013] it was shown that the neural networks can be described in the form of IMs. On the basis of the present research, we will be able to introduce a new extension of the concept of neural networks, which will be described in the form of IMFE.

Acknowledgements

This work has been partially supported by the Bulgarian National Science Fund under the grant "Simulation of wild-land fire behavior" - I01/0006.

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