ON THE ALGORITHMIC ASPECT OF THE MODIFIED WEIGHTED HAUSDORFF DISTANCE

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Abstract: *In this paper we introduce three formulas for calculation of the weights of the MWHD through the notion of a degree of friendship/relationship Γ. Further investigations and applications of this decision making procedures are also proposed, which are based on the few versions of weights determination procedures in the the MWHD formula stated in the current paper. As an application example we employ this formulas for Intuitionistic Fuzzy Distances.*

Keywords: *Modified weighted Hausdorff distance, Degree of friendship, Intuitionistic fuzzy sets.*

Preliminaries

In this section we give some preliminary information about modified distances and in particular, the modified Hausdorff distance. It has been used as a basis for the introduction of the modified weighted Hausdorff distance (MWHD), firstly introduced in [Marinov et al., 2012]. At the of the section we provide a brief introduction to intuitionistic fuzzy sets (IFS) and distances between them, as an example of application of the derived formulas.

01 Modified Hausdorff distance.

Let us introduce now the definition and provide some information about *modified distance* and in particular, the *modified Hausdorff distance*.

Definition 1. A mapping $\rho : Y \times Y \to \mathbb{R}_{\geq 0}$ is told *modified metric* or *modified distance* in Y if there exists a function $\phi : Y \to \mathbb{R}_{\geq 0}$ (see [Kuratowski, 1966], p. 209) with the following conditions satisfied for all $x, y, z \in Y$:

1. $\rho(x, y) \geq 0$, and equality holds iff $x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y) + \phi(z)$

In the last definition, the third axiom is known as the ϕ-modified triangular inequality. One easily remarks that if $\phi \equiv 0$, then the mapping ρ turns into a usual metric in Y. Dubuisson and Jain, investigating different properties of the (directed) Hausdorff metric, introduced 24 different distance measures classified according to their behaviour in presence of noise in image matching and pattern recognition. They introduced in [Dubuisson M. & Jain A.1994] a new definition of the directed Hausdorff distance h, for which the corresponding

$$H(A, B) = \max(h(A, B), h(B, A))$$

does not produce a usual metric but modified metric, called *modified Hausdorff distance*.

Definition 2. Let (X, d) be a metric space and $\mathcal{F}(X) \subset \mathcal{P}(X)$ the collection of finite subsets of X. The *directed modified Hausdorff distance* from $A = \{a_i\}_{i=1}^{|A|}$ to $B = \{b_i\}_{i=1}^{|B|}$ $(A, B \in \mathcal{F}(X))$ is given by:

$$h_X^{'}(A, B) = \frac{1}{|A|} \sum_{a \in A} \min_{b \in B} d(a, b) \tag{1}$$

and the formula for the *modified Hausdorff distance* (MHD) between A and B by:

$$H_X^{'}(A, B) = \max\{h_X^{'}(A, B), h_X^{'}(B, A)\}. \tag{2}$$

A complete proof that the so defined $H_X^{'}(A, B)$ provides a modified distance, where $\phi(C) = \sup\{d(x, y) \mid x, y \in C\}$ from Definition 1 is the diameter of the considered subset $C \in \mathcal{F}(X)$, can be found in a more general case in [Marinov et al., 2012]. Taking into account that $d(a, B) = \min\{d(a, b) \mid b \in B\}$, the directed modified distance takes the form $h_X^{'}(A, B) = \frac{1}{|A|} \sum\limits_{a \in A} d(a, B)$. From the last expression we get that the directed modified distance between two finite subsets A and B is exactly the *arithmetic mean* of the distances from all points $a \in A$ of the first subset to the second subset B.

Remark 1. In the above notations suppose now that A has a point a' that is pretty far from his nearest point of B, i.e. $d(a', B)$ is considerably larger in comparison with the distance of the other elements of A to B. In such a case the usual directed distance $h_X(A, B)$ is exactly $d(a', B)$, even if all points of A except a' are much closer to B. On the other hand, the modified directed distance $h_X^{'}(A, B)$ won't be affected in such a degree if there is one (or only a few) isolated points too far from B, which provides a more realistic similarity measure for the distance between A and B.

In practice and applications it is important to be able to compare portions of subsets (objects, images, shapes) instead of looking for exact matches. That is why the above introduced modification of the Hausdorff distance compared to the usual Hausdorff distance provides an improved measure, which is less sensitive to noise. More detailed discussions about its applications and results are given in [Dubuisson M. & Jain A.1994; Takács1998].

02 Degree of friendship

Let us remind the formulas for realistic and adequately calculation priorities/weights assigned in the points of elements of $\mathcal{F}(X)$ (the collection of all finite subsets of X), on which the modified weighted Hausdorff distance (MWHD) has been introduced. The reader may refer to [Marinov et al., 2012] for more detailed information about the MWHD, its properties and application in intuitionistic fuzzy decision making procedures.
Let us consider again (X, d) and $A, B \in \mathcal{F}(X)$ as in Definition 2, in order to recall the definition of MWHD.

Definition 3. Let (X, d) be a metric space and $\mathcal{P}_0 \subset \mathcal{F}(X)$ such that $\forall A \in \mathcal{P}_0$ we choose $\{\rho_a^A\}_{a \in A} \subset (0, 1]$ such that $\sum_{a \in A} \rho_a^A = 1$. Thereby, we are given the pair $(\mathcal{P}_0, \rho)$ and for any $A, B \in \mathcal{P}_0$, let us define

$$h_\rho(A, B) := \sum_{a \in A} \rho_a^A d(a, B) \tag{3}$$

the *weighted directed distance* between A and B with weights ρ_a^A in a and ρ_b^B in b. We also introduce the *modified weighted Hausdorff distance* (MWHD) H_ρ in $\mathcal{P}_0$ between A and B as:

$$H_\rho(A, B) := \max\{h_\rho(A, B), h_\rho(B, A)\}, \tag{4}$$

which gives us obviously a map $H_\rho : \mathcal{P}_0 \times \mathcal{P}_0 \longrightarrow \mathbb{R}_{\geq 0}$.

One can easily remark that $H^{'}(A, B)$ is actually a special case of $H_\rho(A, B)$ where $\mathcal{P}_0$ coincides with $\mathcal{F}(X) = \{U \subset X | card(U) < \infty\}$ and for every $A \in \mathcal{P}_0$ all points of A have the same weights, namely $\forall a \in A : \rho_a^A =$

$\frac{1}{|A|}$. Thus, all the points $a \in A$ of any $A \in \mathcal{P}$ are given the same weights/priorities in a regarding the membership of a to A. Taking now $B \in \mathcal{P}_0$ such that $a \in A \cap B$, it is clear that the weight in a regarding its membership to $A(\rho_a^A = \frac{1}{|A|})$ and its membership to $B(\rho_a^B = \frac{1}{|B|})$ respectively are not equal if $|A| \neq |B|$. In the general case, where the weights in the points of A are not homogeneous, ρ_a^A may be interpreted as the degree of importance or the priority of membership of the element a to the set A.

Remark 2. In decision making procedures, for instance, we may have a group of experts $A \in \mathcal{P}_0$ ($\mathcal{P}_0$ - the collection of all groups) and $a \in A$ is any individual expert belonging to the group A. Then, the opinions of the experts may differ between each other according to the priority of every expert in the given group, i.e. they have different weights. As in the above example, we may have $a \in A \cap B$, which means that an expert belongs to two different groups simultaneously and regarding his membership to any of the group his opinion has a priority ρ_a^A in the group A but priority ρ_a^B in the group B. The priorities of all members from any expert group of $\mathcal{P}_0$ are normalized, i.e. have sum equal to 1.

Considering a member a of the group A we can ask what would be the priority of his opinion/decision/judgment about any given problem? We may suppose that there is some kind of selforganization within the group. The decisions of some members are more influential, whereas other members are of little importance. Intuitively, we say that the experts, whose decisions are most important stay in more **central positions** than others. And literally that is the geometric meaning regarding the defined metric d in the underlying set X.

1. If for an expert a there are many other members of $A \subset X$, which are very close to a with respect to d, we mean that a has relatively influential decision.

2. On the other hand, if the expert a is more isolated, i.e. $d(a, A \setminus \{a\})$ is relatively large or there are only a few members that are close to a, we conclude that he is not very important as a member of the group.

In the train of thought from the above interpretations let us introduce a very natural and intuitive method for generation of the priorities/weights.

Suppose we are given a function:

$$\gamma\colon (0, \infty) \longrightarrow [0, \infty] \tag{5}$$

that is non-increasing or decreasing.

Definition 4. On the base of (5) and the underlying metric space (X, d), let us introduce the *degree of relationship* or *degree of friendship* through the following map:

$$\begin{array}{rcl} \Gamma_{d,\gamma}\colon X \times X \setminus D_X & \longrightarrow & [0, \infty] \\ (x_1, x_2) & \mapsto & \gamma(d(x_1, x_2)). \end{array} \tag{6}$$

Whereas $D_X = \{(x, x) \mid x \in X\}$ is the diagonal of the Cartesian product of X.

The above expression means that $\Gamma_{\gamma,d} := (\gamma \circ d)$. Thereby, if x_1 and x_2 are close then the value of $\Gamma_{\gamma,d}(x_1, x_2)$ is large. Otherwise, if they are far away from each other the value is small. And moreover, the value of $\Gamma_{\gamma,d}(x_1, x_2)$ is supposed to be close to 0 or equal to 0 only if x_1 and x_2 are extremely far from each other. One may note that for all $x \in X$ we have that $(x, x) \notin Dom(\Gamma_{\gamma,d})$ because $d(x, x) = 0 \notin Dom(\gamma)$. Thereby, the function $\Gamma_{\gamma,d}$ is correctly interpreted as *degree of relationship* or *degree of friendship*. When it is not misleading, we will write just Γ and omit the index γ and/or d as well .

As an example for γ can be taken $\gamma(z) = \frac{1}{z}$ and therefore for the above defined degree of friendship $\Gamma_{\gamma,d}$ we would have

$$\Gamma_{\gamma,d}(a_1, a_2) = \frac{1}{d(a_1, a_2)}, \tag{7}$$

which satisfies (5) and (6). Actually, (5) can be seen as an generalization of the last defined (7).

Definition 5. The set $A \in \mathcal{P}_0$, with $\mathcal{P}_0$ as described at the beginning of the paragraph, will be called $\Gamma_{\gamma,d}$-*degenerated* or just *degenerated* if

$$(\forall a_1, a_2 \in A)(\Gamma_{\gamma,d}(a_1, a_2) = 0) \tag{8}$$

i.e. $\Gamma_{\gamma,d} \equiv 0$ on $A \times A$, which means that all members of A are too isolated from each other. On the other hand, if all sets from $\mathcal{P}_0$ are non-degenerated we say that $\mathcal{P}_0$ is non-degenerated, that is:

$$(\forall A \in \mathcal{P}_0 \exists a_1, a_2 \in A)(\Gamma_{\gamma,d}(a_1, a_2) \neq 0)$$

Remark 3. In our interpretation example with the groups of experts a degenerated group A can be described as a very strange group, where all experts of A would have pretty different judgments and do not respect each other. They may in this case be experts with competencies in different areas of knowledge.

We suppose that only the weights $\{\rho_a^A\}_{a \in A}$ are yet unknown and namely that is what we want to define. We will then state practical algorithms assigning appropriate weights on the basis of the given distance function d and γ.

We allow some (but not all) of the normalized weights ρ_a^A, such as non-normalized weights w_a^A of A (to be defined below) to be zero. The members of A with zero weights correspond to very isolated points that can be regarded as unimportant and therefore can be neglected in the calculation of the directed modified weighted Hausdorff distances. Such points $a \in A\colon \rho_a^A = 0$ can be omitted in the calculation of the directed distance from A to any other set B. But they can not be removed completely from the underlying set as they may belong to any other set B where their weight in regard of their membership to B may be positive. That is, they may appear to be important in the set B.

03 Intuitionistic fuzzy sets and distances

As opposed to a fuzzy set in X [Zadeh, 1965], given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \tag{9}$$

where $\mu_{A'}(x) \in [0, 1]$ is the membership function of the fuzzy set A', an intuitionistic fuzzy set (IFS) [Atanassov, 1999] A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{10}$$

where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{11}$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively. (Two approaches to the assigning memberships and non-memberships for IFSs are proposed in [Szmidt & Baldwin, 2006]).

An additional concept for each IFS in X, that is not only an obvious result of (10) and (11) but also relevant for applications, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{12}$$

a *degree of uncertainty* of $x \in A$. It expresses a lack of knowledge of whether x belongs to A or not (see [Atanassov, 1999]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

Distances between IFSs are calculated in the literature in two ways, using two parameters only (e.g. [Atanassov, 1999]) or all three parameters (see [Szmidt & Kacprzyk, 2000; Atanassov et al., 2005; Szmidt & Baldwin, 2003; Szmidt & Baldwin, 2004; Deng-Feng, 2005] and [Narukawa, 2006]) describing elements belonging to the sets. Both ways are proper from the point of view of pure mathematical conditions concerning distances (all properties are fulfilled in both cases). One cannot say that both ways are equal when assessing the results obtained by the two approaches. In [Szmidt & Kacprzyk, 2000; Szmidt & Baldwin, 2003; Szmidt & Baldwin, 2004] it is shown why in the calculation of distances between IFSs one should prefer all three parameters describing IFSs. Examples of the distances between any two IFSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$ while using three parameter representation (see [Szmidt & Kacprzyk, 2000; Szmidt & Baldwin, 2003; Szmidt & Baldwin, 2004]).

A *normalized distance* or *normalized metric* d in X is a metric such that $d \colon X \times X \to [0,1] \subset \mathbb{R}_{\geq 0}$. Sometimes it is more convenient and easier to work with normalized metrics. Every metric can be normalized (see [Adams & Franzosa, 2008]). For more detailed general properties and proofs about distances for IFSs the reader may refer to [Marinov et al., 2012].

Definition 6. Let $A, B \in IFS(X)$ be two intuitionistic fuzzy sets on the finite universe X. We can state the following standard distances between IFSs

1. With two parameters:

   - Hamming distance:

$$l_{2,IFS}(A,B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)$$

   - Normalized Hamming distance:

$$L_{2,IFS}(A,B) = \frac{1}{2n} l_{2,IFS}(A,B)$$

   - Euclidean distance:

$$e_{2,IFS}(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2}$$

   - Normalized Euclidean distance:

$$E_{2,IFS}(A,B) = \sqrt{\frac{1}{2n}} e_{2,IFS}(A,B)$$

2. With three parameters:

   - Hamming distance:

$$l_{3,IFS}(A,B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$

   - Normalized Hamming distance:

$$L_{3,IFS}(A,B) = \frac{1}{2n} l_{3,IFS}(A,B)$$

- Euclidean distance:

$$e_{3,IFS}(A,B) = \sqrt{\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i)^2)}$$

- Normalized Euclidean distance:

$$E_{2,IFS}(A,B) = \sqrt{\frac{1}{2n}}e_{3,IFS}(A,B)$$

Remark 4. It is almost evident that for the above defined distances the following inequalities hold:

$$\begin{array}{llll} 0\leq & l_{2,IFS}(A,B) & \leq l_{3,IFS}(A,B) & \leq 2n \\ 0\leq & L_{2,IFS}(A,B) & \leq L_{3,IFS}(A,B) & \leq 1 \\ 0\leq & e_{2,IFS}(A,B) & \leq e_{3,IFS}(A,B) & \leq \sqrt{2n} \\ 0\leq & E_{2,IFS}(A,B) & \leq E_{3,IFS}(A,B) & \leq 1 \end{array}$$

Formulas for the weights determination in MWHD

Let us now introduce a few formulas for calculation of the weights in the MWHD degree of friendship algorithm through the above stated function γ and friendship degree Γ.

04 First version of weights determination formula

The following

$$w^A_{\gamma,d,a_0} := \begin{cases} 1 & \text{if } |A|=1 \\ \sum\limits_{a\in A\setminus\{a_0\}}\Gamma_{\gamma,d}(a_0,a) & \text{if } |A|>1 \end{cases} \tag{13}$$

we call *non-normalized weight* in a_0 regarding d,γ and its membership to A and

$$W^A_{\gamma,d} := \sum_{a_0\in A} w^A_{\gamma,d,a_0} = \sum_{a'\neq a''\in A}\Gamma_{\gamma,d}(a',a'') \tag{14}$$

will be called *general sum* of A with respect to d and γ, where d and γ will be omitted if it is not misleading. But as $\forall x_1,x_2\in\cup\mathcal{P}_0\colon \Gamma_{\gamma,d}(x_1,x_2)=\Gamma_{\gamma,d}(x_2,x_1)$, i.e. $\Gamma_{\gamma,d}$ is symmetric, and rewriting (14) we get that $W^A_{\gamma,d}$ equals $\sum\limits_{a_0\in A}(\sum\limits_{a\in A\setminus\{a_0\}}\Gamma_{\gamma,d}(a_0,a))$. If $|A|=n$ and $A=\{a_1,\ldots,a_n\}$ to emphasize the algorithmic aspect the expression of $W^A_{\gamma,d}$ may be stated in the following form:

$$W^A_{\gamma,d} = 2(\sum_{1\leq i<j\leq|A|}\Gamma_{\gamma,d}(a_i,a_j)) \tag{15}$$

because $\forall a',a''\in A\colon a'\neq a''$ the value of $\Gamma_{\gamma,d}(a',a'')$ is taken twice in the calculation of $W^A_{\gamma,d}$, first in regard of a' and second in regard of a'' in the sum (14). For the *normalized weights* let us introduce now the formula:

$$\rho^A_{\gamma,d,a} := \frac{w^A_{\gamma,d,a}}{W^A_{\gamma,d}} = \frac{w^A_{\gamma,d,a}}{2(\sum\limits_{1\leq i<j\leq|A|}\Gamma_{\gamma,d}(a_i,a_j))} \tag{16}$$

Remark 5. Suppose now that γ (5) is constant, therefore Γ (6) is constant as well, i.e. $\Gamma_{\gamma,d} \equiv \alpha > 0$. Applying this in (16) we get that for any $A \in \mathcal{P}_0$:
$W^A_{\gamma,d} = 2(\sum\limits_{1\le i<j\le n} \alpha) = 2\alpha.C^2_n = 2\alpha.n(n-1)/2 = \alpha.n(n-1)$, where C^2_n is the number of 2-combinations without repetitions from a collection of n elements. As for any $a \in A$: $w^A_{\gamma,d,a} = \sum\limits_{a'\in A\setminus\{a\}} \alpha = \alpha.(n-1)$, for the normalized weight in a we get $\rho^A_{\gamma,d,a} = \alpha.(n-1)/\alpha.n(n-1) = 1/n = 1/|A|$, which is expected for the particular case with equal weights imposed on the elements of A.

Supposing now that $\mathcal{P}_0$ is degenerated with respect to $\Gamma_{\gamma,d}$, i.e. there are degenerated subsets $A \in \mathcal{P}_0$, for which $\forall a \in A$: $w^A_{\gamma,d,a} = 0$ and therefore $W^A_{\gamma,d} = \sum_{a\in A} w^A_{\gamma,d,a} = 0$. The last expression implies that the formula (16) for $\rho^A_{\gamma,d,a}$ could not be applied for degenerated subsets of X.

05 Second version of weights determination formula

There may be some cases, when we would like some groups of experts to be designated as "degenerated" or to impose some penalty value $P_0 \ge 0$ on the too isolated members belonging to a group. Then we will certainly have that

$$P_0 = \Gamma_{\gamma,d}(x_1, x_2)$$

for all $x_1, x_2 \in \cup\mathcal{P}_0$ such that $d(x_1, x_2) \ge d_0$. Where d_0 is an appropriately chosen large enough positive constant. Therefore, we can assume for $\Gamma_{\gamma,d}$ that the following inequality is satisfied

$$0 \le P_0 \le \min\{\Gamma_{\gamma,d}(x_1, x_2) \mid x_1, x_2 \in \cup\mathcal{P}_0\} \tag{17}$$

The meaning of the penalty value P_0 is that if for $a \in A$, $d(a, A\setminus\{a\})$ is too large, say greater or equal to $d_0 > 0$, then automatically

$$\forall a' \in A \setminus \{a\}\colon \gamma(d(a, a')) = P_0,$$

i.e. $\gamma([d_0, \infty)) = \{P_0\}$. Note that the introduction of penalties may significantly improve the performance and the quality of results in algorithms for practical applications (see [Takács1998]). That way we have slightly modified the function γ imposing the additional condition $\gamma \ge P_0$ on its domain of definition. Therefore, for the so modified γ and $\Gamma_{\gamma,d}$ in particular we can use (13), (15) and (16) to define the non-normalized weights, general sums and the normalized weights, respectively. Note that as in the first version of the weights generation algorithm every expert group has to contain at least one element.

06 Third version of weights determination formula

Let us give now another version of the above formulas for the weights. Suppose that we want to define the function γ from (5) in the point 0 as well, i.e.

$$\gamma\colon [0, \infty) \longrightarrow [0, \infty] \tag{18}$$

In (18) γ is supposed to be non-increasing in $(0, \infty)$ and $\gamma(0)$ has to be defined in an appropriate way if applied in real problem solutions. The expression (6) and the above introduced γ provide that $Dom(\Gamma_{d,\gamma}) = X \times X$, i.e. $D_X \subset Dom(\Gamma_{d,\gamma})$. Therefore, we have that

- $\forall x \in X\colon \gamma(d(x, x)) = \gamma(0) = \sup Range(\gamma)$
- $\forall x \in X\colon (x, x) \in Dom(\Gamma_{\gamma,d})$ and $\Gamma_{\gamma,d}(x, x) = \gamma(0)$

By analogy of (13) –(16), let us introduce

$$\widetilde{w}^A_{\gamma,d,a_0} := \sum_{a\in A} \Gamma_{\gamma,d}(a_0, a), \tag{19}$$

which we also call *non-normalized weight* in a with respect to d, γ and its membership to A and

$$\widetilde{W}^A_{\gamma,d} := \sum_{a_0\in A} \widetilde{w}^A_{\gamma,d,a_0} = \sum_{a',a''\in A} \Gamma_{\gamma,d}(a', a'') \tag{20}$$

is the *general sum* of A corresponding to d and γ. Again because of the symmetry property of $\Gamma_{\gamma,d}$, we have that $\widetilde{W}^A_{\gamma,d} = 2(\sum\limits_{1\le i<j\le |A|} \Gamma_{\gamma,d}(a_i, a_j)) + \sum\limits_{i=1}^{|A|} \Gamma_{\gamma,d}(a_i, a_i)$ and therefore the normalized weights could be defined in the following way

$$\widetilde{\rho}^A_{\gamma,d,a} := \frac{\widetilde{w}^A_{\gamma,d,a}}{\widetilde{W}^A_{\gamma,d}} = \frac{\widetilde{w}^A_{\gamma,d,a}}{2(\sum\limits_{1\le i<j\le |A|} \Gamma_{\gamma,d}(a_i, a_j)) + \sum\limits_{i=1}^{|A|} \Gamma_{\gamma,d}(a_i, a_i)}. \tag{21}$$

It is obvious that the closest point of $x_0 \in X$ is x_0 itself. Thereby, if we like γ to be non-increasing on the whole domain $[0, \infty)$, $\gamma(0)$ should take the maximum value of its range. And so in the sum (19) the member $\Gamma_{\gamma,d}(a_0, a_0)$ has the greatest impact in the determination of the weight in a_0.

In the so introduced weights, if needed, we can apply the penalty value from the second version of the algorithm. Let us now check what happens when one applies the above expressions for a degenerated set $A \subset X$ whereas $P_0 = 0$. First of all, let us recall that for the chosen maximal diameter $M_0 > 0, P_0 = 0$ and $S_0 := \gamma(0) = \sup Range(\gamma) = \sup Range(\Gamma_{\gamma,d}) = \Gamma_{\gamma,d}(x, x)$ for any $x \in X$. The degeneration of A provides that $\forall a', a'' \in A\colon a' \neq a'' \Rightarrow \Gamma_{\gamma,d}(a', a'') = 0$. Thereby,

$$\forall a \in A\colon \widetilde{w}^A_{\gamma,d,a} = S_0 \text{ and } \widetilde{W}^A_{\gamma,d} = |A|S_0,$$

and hence, $\widetilde{\rho}^A_{\gamma,d,a} = 1/|A|$ which is exactly what we expect.

Remark 6. Let us take any A, B such that A is degenerated and B is non-degenerated. It follows almost straightforward, that

- $\forall a \in A\colon \widetilde{w}^A_{\gamma,d,a} = S_0 \le \widetilde{w}^B_{\gamma,d,a}$
- $\widetilde{W}^A_{\gamma,d} = |A|S_0 < \widetilde{W}^B_{\gamma,d}$

As in *Remark 5* the reader can easily check that the above introduced formula for the normalized weights is again a relevant generalization for the standard modified Hausdorff distance (with equal weights) when A is non-degenerated and γ is a constant function.

Conlusion

In this paper we have introduced few versions of algorithms for calculation th the weights through the above stated function γ and friendship degree Γ. As a start point we employ the modified weighted Hausdorff distance (MWHD) notion, which was firstly defined in [Marinov et al., 2012]. Let us remind how we can apply the MWHD notion and especially the introduced here algorithms in IFS models.

Supposing that there is a problem to be estimated with respect to a few criteria $C = \{C_0, \ldots, C_n\}$. The criteria will be considered as a universe for IFS. That is, every intuitionistic fuzzy set $E \in X = IFS(C)$ can be considered to an expert estimation about the problem with respect to the above criteria. Taking a finite subset of the power set of X : $\mathcal{P}_0 \subset \mathcal{P}(X)$ to be a group of estimations of the chosen experts, one could estimate how far from each other are the decisions of the the different groups of experts.

As another exmple, we can take the universe $P = \{P_0, \ldots, P_n\}$ to be a collection of problems to be estimated. In this situation every intuitionistic fuzzy set $E \in IFS(P)$ can be considered as an expert estimation of the collection of the chosen problems P and groups of expert estimations with respect of P can be also modeled through the introduced MWHD formulas in this paper.

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