# AN UNCERTAIN CAUCHY PROBLEM OF A NEW CLASS OF FUZZY DIFFERENTIAL EQUATIONS

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***Abstract**: The concept of fuzzy perception roaming process is entered. A new integral is built by the fuzzy perception roaming process. Properties of this integral are studied. The new class of differential equalizations is acquired. A theorem of existence and uniqueness of a solution for the new class of fuzzy perception differential equations are proved.*

***Keywords**: possibility-theoretical approach, dynamic processes, soft modeling, fuzzy perception processes, Cauchy problem.*

***ACM Classification Keywords**: G.1.7 – Ordinary Differential Equations.*

## Introduction

It is important to answer a question of how exact the result of design is at non-crisp supervision when we deal with mathematical modeling of complex systems. A stochastic approach is the most widespread to describe a non-crisp at the complex system design. However, such description in probability terms is unnatural for the unique phenomena. It appears that the most suitable for this are the ideas of possibility theory. First, the theory of possibility was most fully presented by Dubois D. Development of ideas [Dubois, 1988] is offered in this article.

There are different approaches to formalize non-crisp dynamics [Aubin, 1990]-[Buckley, 1992], [Friedmanan, 1994]-[Park, 1999], [Seikkala, 1987]. In this article new approaches to adequate description and analysis of non-crisp and uncertain information and dynamics of objects are proposed.

Methods of possibility theory allow to estimate an event truth with respect to other events and to take into account a subjective expert opinion. For example it is very important for prognostication of the social-economic phenomena, for medical diagnostic tasks, for mathematical modeling of human thinking process and other processes.

## Preliminaries

Let $(X, \mathbf{A})$ be a measurable space. According to [Pitiev, 2000], [Bychkov, 2005], [Bychkov, 2006] we consider some definitions. A new notations $+$, $\bullet$ are offered in [Pitiev, 2000]. We shall use them in the further.

**Definition 1.** Possibility scale is a semi-ring $L = \{[0,1], \leq, +, \bullet\}$, i.e. $[0,1]$ segment with usual order $\leq$ and two operations:

- $a + b = \max\{a, b\}$;

- $a \bullet b = \min\{a,b\}$.

Henceforth we consider only $A$ -measurable functions $f: X \to L$. Let's denote as $L(X)$ a class of functions that satisfies the following conditions:

- $f \in L(X), a \in L \Rightarrow a \bullet f = \min(a, f(x)) \in L(X)$;
- $f, g \in L(X) \Rightarrow f + g = \max(f(x), g(x)) \in L(X)$;

  $f \bullet g = \min(f(x), g(x)) \in L(X)$;
- $f \in L(X) \Rightarrow \neg f \equiv 1 - f(x) \in L(X)$;
- If a sequence of functions $f_1, \ldots, f_n, \ldots \in L(X)$, then $\underset{n=1}{\overset{\infty}{+}} f_n(x) \in L(X)$; $\underset{n=1}{\overset{\infty}{\bullet}} f_n(x) \in L(X)$.

**Definition 2.** Function $p: L(X) \to L$:

- $p((a \bullet f)(\cdot) + (b \bullet g)(\cdot)) = (a \bullet p(f(\cdot))) + (b \bullet p(g(\cdot)))$;
- $p\left(\left(\underset{n=1}{\overset{\infty}{+}} f_n\right)(\cdot)\right) = \underset{n=1}{\overset{\infty}{+}} p(f_n(\cdot))$;
- If $f(\cdot) \equiv 1$, $p(f(\cdot)) = 1$,

is called a possibility measure.

Let's note $\beta(X)$ - set of all subsets $X$, $\chi_A(\cdot)$ - characteristic function of $A$.

**Definition 3.** Let's define the function $P: \beta(X) \to L$ in the following manner: $P(A) \equiv p(\chi_A(\cdot))$. This function is called a possibility of crisp event $A$. Then let's call a triplet $(X, \mathbf{A}, P)$ a possibility space.

We will use a term **fuzzy perception** (from perceptio – lat.), in future, that will not contradict with term Zadeh L. – fuzzy.

**Definition 4**. Let $(X, \mathbf{A}, P)$ be a possibility space. Let's call an $A$ -measurable function $\tilde{A}: X \to \beta(Y)$ as fuzzy perception set.

Characteristic function of fuzzy perception set $\tilde{A}$ is calculated as $\mu_{\tilde{A}}(y) = P\{y \in \tilde{A}(x)\} = P(A^{-1}(y))$.

**Definition 5.** If $\tilde{A}$ is a fuzzy perception set, let's call a number $P(\tilde{A}) \equiv p(\mu_{\tilde{A}}(\cdot))$ as a possibility of fuzzy perception set $\tilde{A}$.

The definition implies the following properties:

- $P(A \cup B) = P(A) + P(B) = \max(P(A), P(B))$;
- $P(A \cap B) \le P(A) \bullet P(B) = \min(P(A), P(B))$.

**Definition 6.** If $P(A \cap B) = P(A) \bullet P(B)$ the events are called independent.

**Definition 7.** Conditional possibility of event $A$ in respect to event $B$ is the solution $P\{A \mid B\}$ of the equation

$$\min\{P(A\,|\,B), P(B)\} = P(A \cap B).$$

**Theorem 1.** [Bychkov, 2007] It is possible to extend possibility $P$ to all the subsets of set $X$ (let's call the extended possibility $\bar{P}: \beta(X) \to L$ ) and for all $A \in \mathbf{A}$ $\bar{P}(A) = P(A)$.

This theorem allows not to build the enclosed sequence of sigma algebras for the correct construction of fuzzy differential equation solution

**Definition 8**. With a given possibility space $(X, \mathbf{A}, P)$ and a measurable space $(Y, \mathbf{B})$ a fuzzy perception variable is an $(\mathbf{A}, \mathbf{B})$- measurable function $\xi: X \to Y$.

Theorem 1 implies that we may define a function $\mu_\xi(y) = \bar{P}\{\xi = y\}$ that is called distribution of a fuzzy perception variable. The possibility that a fuzzy perception variable $\xi$ will fall into set *B* can be expressed by distribution: $P\{\xi \in B\} = \sup\limits_{y \in B} \mu_\xi(y)$.

**Definition 9.** Fuzzy perception variables $\xi$ and $\eta$ are independent if distribution of perception vector $(\xi, \eta)$ equals to $\mu_{\xi\eta}(u, v) = \min\{\mu_\xi(u), \mu_\eta(v)\}$.

## Main results. *Modeling the uncertain dynamics*

Let's give some definitions.

**Definition 10.** [Bychkov, 2005] Fuzzy perception variable $\xi$ (scalar or vector) is normal if its distribution equals to

$$\mu_\xi(u) = \varphi\left\{\left\|\Xi^{-1/2}(u - u_0)\right\|^2\right\},$$

where $\varphi(x)$ – decreasing function, that's specified for $x \geq 0$ such as $\varphi(x) \xrightarrow[x \to \infty]{} 0$, $\varphi(0) = 1$.

**Definition 11.** [Bychkov, 2005] Fuzzy perception process is a function $\xi(x, t): X \times R \to Y$.

**Definition 12.** [Bychkov, 2005]Normal fuzzy perception process $\xi(t)$ is a process of fuzzy perception roaming if the following assumptions exist:

- Under independent increments, i.e. for all moments of time $t_1 < t_2 \leq t_3 < t_4$ fuzzy perception variables $\xi(t_2) - \xi(t_1)$ and $\xi(t_4) - \xi(t_3)$ are independent.

- Under fixed $t$ its transient possibility is $P\{\xi(t)=x \mid \xi(t_0)=x_0\}=\varphi\left(\dfrac{\left\|\Xi^{-1/2}(x-x_0)\right\|^2}{t-t_0}\right)$.
- $\xi(0)=0$.

**Definition 13.** A mode (or modal value) of a fuzzy perception variable is a point in which distribution function equals to 1, i.e.

$$\operatorname{mod}\xi=u_0;\quad \mu_\xi(u_0)=1.$$

**Definition 14.** $\alpha$-cut of fuzzy perception variable ξ is the following set:

$$[\xi]_\alpha=\{y: P\{\xi=y\}\geq\alpha\},\ \alpha\in(0,1]$$

An alternative way of specifying fuzzy distribution is to specify its α-cuts.

**Definition 15.** [Pitiev, 2000] Let's consider a semi-ring $\tilde{L}=([0,1];\ \tilde{\leq};\ \tilde{+};\ \tilde{\bullet})$, where $a\,\tilde{\leq}\,b$ if $b\geq a$; $a\,\tilde{+}\,b=\min(a,b)$; $a\,\tilde{\bullet}\,b=\max(a,b)$. We can define a necessity measure $n(f(\cdot)):L(X)\to\tilde{L}$ on $\tilde{L}$ in the same manner as on $L$. Then for any crisp set $A\in\beta(X)$ necessity of set $A$ is $N(A)=n(\chi_A(\cdot))$. For any fuzzy perception set necessity is $N(\tilde{A})=n(\mu_{\tilde{A}}(\cdot))$.

**Definition 16.** [Bychkov, 2006] Sequence of fuzzy perception variables $\xi_n$ converges to fuzzy perception variable $\xi$ with necessity 1 ($\underset{n\to\infty}{\mathrm{N1lim}}\,\xi_n=\xi$) if for each $x\in X$ for which $P(\{x\})\neq 0$, $\xi_n(x)\to\xi(x)$.
In this case $N\{\xi_n \xrightarrow[n\to\infty]{} \xi\}=1$.

**Definition 17.** Sequence of fuzzy perception variables $\xi_n$ converges to fuzzy perception variable $\xi$ by possibility ($\underset{n\to\infty}{\mathrm{Plim}}\,\xi_n=\xi$) if for any $c>0$ $P\{|\xi_n-\xi|>c\}\xrightarrow[n\to\infty]{}0$.
If any of these limits exists it is unique. $\underset{n\to\infty}{\mathrm{Plim}}\,\xi_n=\xi$ implies $\underset{n\to\infty}{\mathrm{N1lim}}\,\xi_n=\xi$ (unlike probability theory).

**Lemma 1.** If $\underset{n\to\infty}{\mathrm{N1lim}}\,\xi_n=\xi$ and for arbitrary $\alpha\in(0,1]$ sequence $\xi_n(x)$ is convergent to $\xi(x)$ on the set $[X]_\alpha=\{x\in X: P(\{x\})\geq\alpha\}$ uniformly by $x$, then $\underset{n\to\infty}{\mathrm{Plim}}\,\xi_n=\xi$. And conversely, if $\underset{n\to\infty}{\mathrm{Plim}}\,\xi_n=\xi$ then for any $\alpha\in(0,1]$ sequence $\xi_n(x)$ converges to $\xi(x)$ uniformly by $x$ on $[X]_\alpha$.

**Proof.** By definition of uniform convergence, for all $\alpha>0$ and $c>0$ an index $n(\alpha,c)\in N$ exists with following property: if $n'>n(\alpha,c)$ and $P(\{x\})\geq\alpha$ then $|\xi_{n'}(x)-\xi(x)|\leq c$. Possibility that difference between $\xi_n$ and $\xi$ will exceed $c$ may be estimated as

$$P\{|\xi_n - \xi| > c\} \le \sup\{\alpha : n \le n(\alpha, c)\} \equiv \alpha(n,c).$$

Let's prove that $\alpha(n,c) \xrightarrow[n\to\infty]{} 0$. Let's assume the contrary: $\alpha(n,c) \not\to 0$ when $n \to \infty$. Then such value $\alpha_0$ exists that for any sequence of indexes $n_k \to \infty$ holds the following: $n_k \le n(\alpha_0, c)$, that's impossible, because $n(\alpha_0, c)$ is finite. The direct lemma is proven.

Let's prove the converse lemma. Let's assume the contrary: such $c > 0$ and $\alpha > 0$ exist there that for any number $k$, there exist an element of possibility space $x$ and a number $n_k > k$, for which $P\{x\} \ge \alpha$ and $|\xi_{n_k}(x) - \xi(x)| > c$. We found such sequence $n_k \to \infty$ that $P\{|\xi_{n_k} - \xi| > c\} \ge P(\{x\}) \ge \alpha$, i.e. there's no convergence by possibility.

**Definition 18.** [Bychkov, 2005] Assume that a piecewise-constant function $f : f(t) = \{y_k : \quad t_k \le t < t_{k+1}\}$, $k = 0...N-1$, $t_0 = 0$, $t_N = T$ is given, $w(t)$ is a scalar process of fuzzy perception roaming. Denote

$$\int_0^T f(t)dw(t) = \sum_{k=0}^{N-1} y_k (w(t_{k+1}) - w(t_k)).$$

This fuzzy perception variable is called an integral by a process of fuzzy perception roaming of piecewise-constant function.

**Lemma 2.** For piecewise-constant function $f(t)$ the following equality exists:

$$\left[\int_0^T f(t)dw(t)\right]_\alpha = \left[-\int_0^T |f(t)|dt \cdot \sqrt{\phi^{-1}(\alpha)},\ \int_0^T |f(t)|dt \cdot \sqrt{\phi^{-1}(\alpha)}\right].$$

**Proof.** Let's use mathematical induction.

For $N = 1$ the equality is true by definition of process of fuzzy perception roaming.

Let's assume that the equality is true for $N$, and prove it for $N+1$.

Fuzzy perception variables $\int_0^{t_N} f(t)dw(t)$ and $\int_{t_N}^{t_{N+1}} f(t)dw(t)$ are independent. Therefore

$$\left[\int_0^{t_{N+1}} f(t)dw(t)\right]_\alpha = \left[\int_0^{t_N} f(t)dw(t)\right]_\alpha + \left[\int_{t_N}^{t_{N+1}} f(t)dw(t)\right]_\alpha =$$

$$= \left[-\int_0^{t_N} |f(t)|dt \cdot \sqrt{\varphi^{-1}(\alpha)},\ \int_0^{t_N} |f(t)|dt \cdot \sqrt{\varphi^{-1}(\alpha)}\right] +$$

$$+\left[-\int_{t_N}^{t_{N+1}} |f(t)|\,dt \cdot \sqrt{\varphi^{-1}(\alpha)},\ \int_{t_N}^{t_{N+1}} |f(t)|\,dt \cdot \sqrt{\varphi^{-1}(\alpha)}\right] =$$

$$= \left[-\int_{0}^{t_{N+1}} |f(t)|\,dt \cdot \sqrt{\varphi^{-1}(\alpha)},\ \int_{0}^{t_{N+1}} |f(t)|\,dt \cdot \sqrt{\varphi^{-1}(\alpha)}\right].$$

**Corollary 1.** Let's assume that a sequence of piecewise-constant functions $f_n(t) \xrightarrow[n\to\infty]{} 0$ in the mean. Then $\operatorname{Plim}_{n\to\infty} \int_0^T f_n(t)dw(t) = 0$.

This corollary implies the following theorem:

**Theorem 2.** [Bychkov, 2005] A measurable function $f(t)$ is given and two sequences of piecewise-constant functions $f_n(t)$ and $\overline{f}_n(t)$ converge to it in the mean. If the limit $Q = \operatorname{Plim}_{n\to\infty} \int_0^T f_n(t)dw(t)$ exists then the limit $\overline{Q} = \operatorname{Plim}_{n\to\infty} \int_0^T \overline{f}_n(t)dw(t)$ also exists, and $\overline{Q} = Q$.

Theorem 2 proves correctness of the following definition:

**Definition 19.** [Bychkov, 2005] Assume that a measurable function $f(t)$ is given and a sequence of piecewise-constant functions $f_n(t)$ converges to it in the mean. ( $\int_0^T |f_n(t) - f(t)|\,dt \xrightarrow[n\to\infty]{} 0$ ). If the sequence $\int_0^T f_n(t)dw(t)$ converges by possibility let's call its limit as an integral of a measurable function by the process of fuzzy perception roaming and

$$\int_0^T f(t)dw(t) = \operatorname{Plim}_{n\to\infty} \int_0^T f_n(t)dw(t).$$

**Definition 20.** Sequence of functions $a_n(x)$ is uniformly fundamental by $x$ if for arbitrary $\varepsilon > 0$ the independent from $x$ $n_0(\varepsilon)$ exists and such that when $m > n \geq n_0$: $|a_n(x) - a_m(x)| < \varepsilon$.

**Lemma 3.** If a sequence of functions $a_n(x)$ is uniformly fundamental it is uniformly convergent.

**Proof.** Let's denote $\lim_{n\to\infty} a_n(x) = a(x)$. From uniform fundamentality for any $\varepsilon > 0$ such $n_0$ exists that for $m > n \geq n_0$ $|a_n(x) - a_m(x)| < \frac{\varepsilon}{2}$ occurs. Hence, for all $m \geq n_0$ $a_m(x) \in [a_{n_0}(x) - \frac{\varepsilon}{2}, a_{n_0}(x) + \frac{\varepsilon}{2}]$ and $a(x) \in [a_{n_0}(x) - \frac{\varepsilon}{2}, a_{n_0}(x) + \frac{\varepsilon}{2}]$. Therefore, $|a(x) - a_m(x)| < \varepsilon$ that proves uniform convergence.

**Lemma 4.** For crisp integrable function $f(t)$ $\alpha$-cuts of the integral also equal to

$$\left[\int_0^T f(t)dw(t)\right]_\alpha = \left[-\int_0^T |f(t)|dt \cdot \sqrt{\varphi^{-1}(\alpha)}, \int_0^T |f(t)|dt \cdot \sqrt{\varphi^{-1}(\alpha)}\right].$$

**Proof.** As we know from functional analysis such sequence of piecewise-constant functions $f_n(t)$ exists that

$$|f_1(t)| \leq |f_2(t)| \leq \ldots \leq |f_n(t)| \leq \ldots \text{ and } \int_0^T f_n(t)dt \xrightarrow[n\to\infty]{} \int_0^T f(t)dt,$$

furthermore $|f_n(t)| \leq |f(t)|$ almost everywhere. In such a case $f_n(t) \to f(t)$ in the mean.
Let's prove the existence of the integral.

Let's fix an arbitrary $x_0 \in X$ and consider sequence $\int_0^T f_n(t)dw(t, x_0)$, then

$$\left|\int_0^T f_n(t)dw(t, x_0) - \int_0^T f_m(t)dw(t, x_0)\right| \leq \int_0^T |f_n(t) - f_m(t)|\, dw(t, x_0)$$

Since $f_n(t) \xrightarrow[n\to\infty]{} f(t)$ in the mean $|f_n(t) - f_m(t)| \xrightarrow[m,n\to\infty]{} 0$ in the mean and from Corollary 1

$$\operatorname*{Plim}_{m,n\to\infty} \int_0^T |f_n(t) - f_m(t)|\, dw(t) = \operatorname*{N1lim}_{m,n\to\infty} \int_0^T |f_n(t) - f_m(t)|\, dw(t) = 0.$$

Therefore $\lim_{m,n\to\infty} \int_0^T |f_n(t) - f_m(t)|\, dw(t, x_0) = 0$. So, the sequence $\int_0^T f_n(t)dw(t, x_0)$ is fundamental and hence it is convergent. We proved that $\operatorname*{N1lim}_{n\to\infty} \int_0^T f_n(t)dw(t, x) = Q(x)$ exists. Let's prove that $\operatorname*{Plim}_{n\to\infty}$ also exists.

By *Lemma 3* for any $x_0 \in X$, for which $P(\{x_0\}) \geq \alpha$ and $m > n$ from choice of $f_n(x)$

$$\left|\int_0^T |f_n(t)-f_m(t)|\,dw(t,x_0)\right| \le \int_0^T |f_n(t)-f_m(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)} \le$$

$$\le \int_0^T |f-f_n(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)}.$$

Holds, i.e. sequence $\int_0^T f_n(t)dw(t,x)$ is uniformly fundamental by $x$. According to *Lemma 3* this sequence uniformly converges by $x$. By *Lemma 1* $\mathrm{P}\lim_{n\to\infty}\int_0^T f_n(t)dw(t)$ exists.

We have proven that all Lebesgue integrable functions are also integrable by process of fuzzy roaming. However, the distribution of this integral is unknown. Let's calculate it.

By *Lemma 3*

$$\left[\int_0^T f_n(t)dw(t)\right]_\alpha = \left[-\int_0^T |f_n(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)},\ \int_0^T |f_n(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)}\right] = \overline{I}_n .$$

This segment increases with $n$, its limit equals to $I=\bigcup_n I_n = \left(-\int_0^T |f(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)},\ \int_0^T |f(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)}\right)$.

Let's denote closure of $I$ as $\overline{I}$. Obviously $\left[\int_0^T f(t)dw(t)\right]_\alpha \subseteq I$. We should prove that it is exactly $\overline{I}$.

For brevity we denote $M(f,\alpha)=\int_0^T |f(t)|\,dt\cdot\sqrt{\varphi^{-1}(\alpha)}$. For arbitrary $\varepsilon>0$ such $n_1$ exists that for $n>n_1$ $|M(f,\alpha)-M(f_n,\alpha)|<\varepsilon$. Because $\left[\int_0^T f_n(t)dw(t)\right]_\alpha = [-M(f_n,\alpha);M(f_n,\alpha)]$, that for each $n>n_1$ such $x_n\in X$ exists that $P(\{x\})\ge\alpha$ and $\left|\int_0^T f_n(t)dw(x,t)-M(f_n,\alpha)\right|<\varepsilon$. By *Lemma 1* (converse part) such $n_2\ge n_1$ exists that for each $n>n_2$ $\left|\int_0^T f_n(t)dw(x,t)-\int_0^T f(t)dw(x,t)\right|<\varepsilon$ independently of $x$. It means that for any $\varepsilon$ $\left(-M(f_n,\alpha)+3\varepsilon,\ M(f_n,\alpha)-3\varepsilon\right)\subseteq\left[\int_0^T f(t)dw(t)\right]_\alpha$. Therefore $I\subseteq\left[\int_0^T f(t)dw(t)\right]_\alpha$. Because $\pm M(f,\alpha)\in\left[\int_0^T f(t)dw(t)\right]_{\alpha-\delta}$ for any sufficiently small $\delta>0$, $P\left\{\int_0^T f(t)dw(x,t)=\pm M(f,\alpha)\right\}\ge\alpha-\delta$.

Hence, $P\left\{\int_0^T f(t)dw(x,t)=\pm M(f,\alpha)\right\}=\alpha$ and the ends of $\overline{I}$ segment also belong to cut.

**Theorem 3.** For a fuzzy perception process $\xi(t,\omega)$ which has integrable paths and is independent from $w(t,x)$, holds the following:

$$\left[\int_0^T \xi(t,x)dw(t,x)\right]_\alpha =$$

$$= \left[-\sup_{x:P\{x\}\geq\alpha}\int_0^T \left|\xi(t,x)\right|dt\cdot\sqrt{\varphi^{-1}(\alpha)},\ \sup_{x:P\{x\}\geq\alpha}\int_0^T \left|\xi(t,x)\right|dt\cdot\sqrt{\varphi^{-1}(\alpha)}\right].$$

**Proof.** From independence of processes $w(t,x),\xi(t,x)$ follows that:

$$P\{w(t,x)\in A\,|\,\xi(t,x)\in B\} = P\{w(t,x)\in A\}.$$

Therefore, process $\{w(t,x)\,|\,\xi(t,x)\in B\}$ is a process of fuzzy perception roaming identical to $w(t,x)$. Fixing $\xi(t,x)$ and using *Lemma 4* conclude proof.

Let's consider an initial-value problem

$$y(t,x) = y_0(x) + \int_{t_0}^{t} a(y(s),s)ds + \int_{t_0}^{t} b(y(s),s)dw(s,x), \tag{1}$$

$$y(t_0,x) = y_0. \tag{2}$$

The solution of the problem (1), (2) is a fuzzy perception process $y(t,x)$ that turns (1), (2) into equalities for each $x\in X$, for which $P(\{x\})\neq 0$.

## Main results. *A Cauchy problem for new class of differential equations*

**Theorem 4.** Let crisp functions $a(y,t)$ and $\mathrm{b}(y,t)$ be continuous by $t$ and satisfy Lipschitz condition by $y$ in the region $D = I\times[t_0,t_0+\Delta t]$, $I=[y_1,y_2]$, i.e.

$$\left|a(y,t)-a(z,t)\right|\leq L\left|y-z\right|, \qquad \left|b(y,t)-b(z,t)\right|\leq L\left|y-z\right|$$

for all $x,y\in I$, $t\in[t_0,t_0+\Delta t]$; initial value $y_0(x)$ is any fuzzy perception value.

If for fixed $x_0 \in X$, for which $P(\{x_0\}) \geq \alpha > 0$ the condition $[y_0(x_0) - \Delta y; y_0(x_0) + \Delta y] \subset I$ holds, then for this $x_0$ the problem (1), (2) has a unique solution in the segment $t \in [t_0; t_0 + h]$, where

$$h = \min\left[ k\frac{1}{2L} \cdot \min\left\{1, \frac{1}{\sqrt[4]{\varphi^{-1}(\alpha)}}\right\}, \frac{\Delta y}{\max\limits_t |a(x)| + \sqrt{\varphi^{-1}(\alpha)} \cdot \max\limits_t |b(x)|}, \Delta t \right], \ k \in (0,1).$$

**Proof.** Let's fix $x_0 \in X$, for which $P(\{x_0\}) \geq \alpha > 0$. The path of the fuzzy perception process $w(t)$ and the initial value $y_0$ depend on it.

Let's prove that for given $x_0$, $y_0(x_0)$ the unique solution exists in $[t_0, t_0 + h]$. Let's consider such space of all continuous functions $f(t)$, $t \in [t_0, t_0 + h]$ that $f(t_0) = y_0(x_0)$. Let's define a distance in this space:

$$\rho(f,g) = \max_{t \in [t_0; t_0 + \Delta t]} |f(t) - g(t)|.$$

We consider the following mapping:

$$\Phi(f(t)) = y_0(x_0) + \int_{t_0}^{t} a(f(s),s)ds + \int_{t_0}^{t} b(g(s),s)dw(s,x_0).$$

Let's prove that it's contracting. Lipschitz condition implies

$$\rho(\Phi(f),\Phi(g)) =$$

$$= \max_{t \in [t_0, t_0 + h]} \left| \int_{t_0}^{t} \big(a(f(s),s) - a(g(s),s)\big) ds + \int_{t_0}^{t} \big(b(f(s),s) - b(g(s),s)\big) dw(s,x_0) \right| \leq$$

$$= \max_{t \in [t_0, t_0 + h]} \left( \int_{t_0}^{t} \left| a\big(f(s),s\big) - a\big(g(s),s\big) \right| ds + \int_{t_0}^{t} \left| b\big(f(s),s\big) - b\big(g(s),s\big) \right| dw(s,x_0) \right) \leq$$

$$\leq Lh\left(1 + \int_{t_0}^{t_0+h} dw(s,x_0)\right)\rho(f,g) = Lh(1 + (w(t_0 + h, x_0) - w(t_0, x_0)))\rho(f,g).$$

Because of $P(\{x\}) \geq \alpha$, $w(t_0 + h, x_0) - w(t_0, x_0) \leq \sqrt{\varphi^{-1}(\alpha)}h$, where $\varphi(y) : [0; +\infty) \to [0,1]$ is the function that corresponds to distribution of possibilities of the process $w(t)$. We obtain:

$$\rho\big(\Phi(f), \Phi(g)\big) \leq$$

$$\leq Lh\left(1 + \sqrt{\varphi^{-1}(\alpha)} \cdot h\right) \cdot \rho(f,g) \leq 2L \cdot \max\left\{h, \sqrt{\varphi^{-1}(\alpha)} \cdot h^2\right\} \cdot \rho(f,g).$$

So, with $h \le k\frac{1}{2L}\cdot\min\left\{1,\frac{1}{\sqrt[4]{\varphi(\alpha)}}\right\}$, $k\in(0,1)$, the mapping is contracting and according to a contraction principle it has a unique fixed point. Obviously, any fixed point of mapping $\Phi$ is a solution of initial-value problem for given $x_0$.

At last, let's obtain conditions for $h$, which provide that the solution will remain within the domain $D$. First of all $h\le\Delta t$. Then with $h\le\dfrac{\Delta y}{\max\limits_t|a(x)|+\sqrt{\varphi^{-1}(\alpha)}\cdot\max\limits_t|b(x)|}$ we can estimate $y(t,x_0)$:

$$|y(t)-y(t_0)|=\left|\int_{t_0}^{t}a(y(s),s))ds+\int_{t_0}^{t}b(y(s),s)dw(s)\right|\le$$

$$\le h\cdot\left[\max_t|a(t)|+\max_t|b(t)|\cdot\sqrt{\varphi^{-1}(\alpha)}\right]\le\Delta t,$$

i.e. with these assumptions $y(t)\in I$.

If the functions $a(y,t)$ and $b(y,t)$ are defined in a finite domain $D$ it is possible that for any $h>0$ such $x_0\in X$ exists that $P(\{x_0\})\ne 0$ and in the time segment $[t_0,t_0+h]$ for $x=x_0$ the solution of (1), (2) will go beyond the bounds of $D$. In other words, for arbitrary small $h$ it is impossible to rely on a solution existence for all possible $x\in X$ in $[t_0,t_0+h]$ segment. Let's prove the theorem that is a sufficient condition of a solution existence for all $t\ge t_0$.

**Theorem 5.** If fuzzy perception functions $a(y,t,x)$ and $b(y,t,x)$, where $y,t\in R$, $x\in X$, for arbitrary $x\in X$ are continuous by $t$ and satisfy local Lipschitz condition by $y$ on $R$, i.e. for arbitrary segment $I$

$$|a(y,t,x)-a(z,t,x)|\le L(I)|y-z|,\qquad |b(y,t,x)-b(z,t,x)|\le L(I)|y-z|\ \text{ for } x,y\in I,$$

where $L(I)$ is a finite value that depends on chosen segment $I$, and growth condition

$$|a(y,t,x)|^2\le K\left(1+|y|^2\right),\qquad |b(y,t,x)|^2\le K\left(1+|y|^2\right),$$

then for any fuzzy $y_0$ the problem (1),(2) has a unique solution for $t\in[t_0;+\infty)$.

**Proof.** Crispness or fuzziness of functions $a$ and $b$ doesn't influence the proof.

Like in Theorem 4, we fix $x_0 \in X$, for which $P(\{x_0\}) \ge \alpha$. Let's take arbitrary great $T > 0$ and prove that on the segment $[t_0, t_0 + T]$ solution of the problem (1), (2) exists.

Although Theorem 4 proves that the solution exists for $t \in [t_0, t_0 + h]$, we can extend this solution beyond these limits. Indeed, let's consider the following problem

$$y(t,x_0) = y(t_0 + h, x_0) + \int_{t_0+h}^{t} a(y(s),s)ds + \int_{t_0+h}^{t} b(y(s),s)dw(s,x_0).$$

According to Theorem 4, this problem has a solution on $[t_0 + h, t_0 + h']$. Substitute $y(t_0 + h, x_0)$ with

$$y(t_0 + h_1, x_0) = y(t_0, x_0) + \int_{t_0}^{t_0+h_1} a(y(s),s)ds + \int_{t_0}^{t_0+h_1} b(y(s),s)dw(s,x_0)$$

and obtain that

$$y(t,x_0) = y(t_0,x_0) + \int_{t_0}^{t} a(y(s),s)ds + \int_{t_0}^{t} b(y(s),s)dw(s,x_0).$$

Hence, we got a solution of the system on $[t_0, t_0 + h + h']$ segment. This operation can be repeated as many times as needed. We must show that the solution will not run to infinity for finite $t$.

Assume that the solution of the system exists on the segment $[t_0, t_0 + T]$. Estimate $y(t, x_0)$:

$$\left| y(t,x_0) - y(t_0,x_0) \right| \le \int_{t_0}^{t} \left| a(y(s),s) \right| ds + \int_{t_0}^{t} \left| b(y(s),s) \right| dw(s,x_0).$$

Growth condition implies that

$$\left| y(t,x_0) - y(t_0,x_0) \right| \le \left| z(t,x_0) - y(t_0,x_0) \right|, \tag{3}$$

where $z(t, x_0)$ is a solution of crisp Cauchy problem

$$z(t,x_0) = y(t,x_0) + \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) \int_{t_0}^{t} \left(1 + K \left| z(s,x_0) \right|^2 \right)^{1/2} ds.$$

This problem has a solution for all $t \ge t_0$.

Thus, having fixed $x_0$, we fix the segment $I_T = [y(t_0,x_0) - 2\Delta; y(t_0,x_0) + 2\Delta]$, where $\Delta = \left| z(t_0 + T, x_0) - y(t_0, x_0) \right|$. According to Theorem 4, the solution of (1), (2) exists on $[t_0, t_0 + h_T]$, where $h_T \le k \frac{1}{2L(I_T)} \cdot \min\left\{1, \frac{1}{\sqrt[4]{\varphi^{-1}(\alpha)}}\right\}$. Extend this solution to $[t_0; t_0 + 2h_T]$, etc., until we cover the whole segment $\left[t_0, t_0 + T\right]$ or cross the bounds of $I_T$.

Let's assume that we cross the bounds for some $t_1 < t_0 + T$. For this $t_1$ (3) doesn't hold and that's impossible. Theorem 4 ensures uniqueness of the solution, that concludes the proof.

## Conclusions

In this article, we consider the problem of modeling non crisp dynamics. A new formalism for the design of uncertain dynamics – differential equation by fuzzy perception roaming processes is offered. Correctness of construction of this equation is shown and a Cauchy problem theorem is proved.

Local Lipschitz condition provides uniqueness of the solution; growth condition ensures that it won't run to infinity for finite $t$ .

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