Index matrices with function-type of elements. Part 2

Krassimir T. Atanassov

Abstract. *Index Matrices (IMs) are extensions of the ordinary matrices. Some of the operations and relations defined over IMs are analogous of the standard ones, while there have been defined new ones, as well. Operators that do not have analogues in matrix theory have been defined, as well. In general, the elements of an IM can be real or complex numbers, as well as propositions or predicates. In the present paper, new operation over IMs with elements being functions, is defined, and some of its properties are discussed.*

Keywords: *Function, Index matrix, Operation, Operator, Relation.*

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1 Introduction

The concept of Index Matrix (IM) was introduced in 1984, but the full description of the research over them was published in [1] exactly 30 years later. In [2], the concept of IM with elements being functions (IMFEs), is defined, and two specific operations over these IMs are introduced. Here, we introduce a new operation over IMEFs.

Firstly, we give the definition of a standard IM with elements being real (or complex) numbers.

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}$ be the set of the real numbers. By IM with index sets K and L $(K, L \subset \mathcal{I})$, we denote the object:

$$[K, L, \{f_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$.

Six operations, six relations and a lot of operators are defined over IMs in [1].

2 Definition of the index matrix with function-type of elements

Let the set of all used functions be $\mathcal{F}$.

The research over IMs with function-type of elements develops in the following two cases:

- each function of set $\mathcal{F}$ has one argument and it is exactly x (i.e., it is not possible that one of the functions has argument x and another function has argument y) – let us mark the set of these functions by $\mathcal{F}_x^1$;
- each function of set $\mathcal{F}$ has one argument, but that argument might be different for the different functions or the different functions of set $\mathcal{F}$ have different numbers of arguments.

The IM with Function-type of Elements (IMFE) has the form (see, [2])

$$[K, L, \{f_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & f_{k_1,l_1} & \ldots & f_{k_1,l_j} & \ldots & f_{k_1,l_n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & f_{k_i,l_1} & \ldots & f_{k_i,l_j} & \ldots & f_{k_i,l_n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & f_{k_m,l_1} & \ldots & f_{k_m,l_j} & \ldots & f_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \leq i \leq m$, and $1 \leq j \leq n : f_{k_i,l_j} \in \mathcal{F}_x^1$.

The IMFE has this form independently of the form of its elements. They can be functions from $\mathcal{F}_x^1$ having one, exactly determined argument (e.g., x), as well as functions with a lot of arguments. The set of n-argument functions will be marked by $\mathcal{F}^n$.

3 Standard operations over IMFEs

Here we give the definitions of the operations over IMFEs. The first three of them are in more general form than in [2].

Let the IMFEs $A = [K, L, \{f_{k_i,l_j}\}], B = [P, Q, \{g_{p_r,q_s}\}]$ be given. Then

Addition:

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ f_{k_i,l_j} \circ g_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \perp, & \text{otherwise} \end{cases},$$

where here and below, symbol "$\perp$" denotes the lack of operation in the respective place and $\circ \in \{+, \times, \max, \min, ...\}$.

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = f_{k_i,l_j} \circ g_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Multiplication

$$A \odot_{(\circ,*)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}],$$

where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q \\ \underset{l_j = p_r \in L \cap P}{\circ} (f_{k_i,l_j} * g_{p_r,q_s}), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \perp, & \text{otherwise} \end{cases},$$

where $(\circ, *) \in \{(+, \times), (\max, \min), (\min, \max), ...\}$.

Structural subtraction: $A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}]$, where "–" is the set–theoretic difference operation and

$$h_{t_u,v_w} = f_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

Multiplication with a constant: $\alpha.A = [K, L, \{\alpha.f_{k_i,l_j}\}]$, where α is a constant.

Termwise subtraction: $A - B = A \oplus_+ (-1).B$.

The problem with the “zero"-IMFE is similar to the “zero-IM": if $f_{k_i,l_j} \in \mathcal{F}$, then

$$I_\emptyset = [\emptyset, \emptyset, \{f_{k_i,l_j}\}].$$

4 Operations over IMFEs and IMs

Let the IM $A = [K, L, \{a_{k_i,l_j}\}]$, where $a_{k_i,l_j} \in \mathcal{R}$ and IMFE $F = [P, Q, \{f_{p_r,q_s}\}]$ be given. Then, we can define:

(a) $A \boxplus F = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} a_{k_i,l_j}.f_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ \perp, & \text{otherwise} \end{cases},$$

with elements of $\mathcal{F}^1$;

(b) $A \boxtimes F = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = a_{k_i,l_j}.f_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{F}^1$;

(c) $F \oplus A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{p_r,q_s}(a_{k_i,l_j}), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \perp, & \text{otherwise} \end{cases}$$

with elements of $\mathcal{R}$;

(d) $F \otimes A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{p_r,q_s}(a_{k_i,l_j}),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{R}$.

Let the IM $A = [K, L, \{\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle\}]$, for the natural number $n \geq 2$, where $a^1_{k_i,l_j}, ..., a^n_{k_i,l_j} \in \mathcal{R}$ and IMFE $F = [P, Q, \{f_{p_r,q_s}\}]$, where $f_{p_r,q_s} : \mathcal{F}^n \to \mathcal{F}$ be given. Then

(e) $F \diamondsuit_\oplus A = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \perp, & \text{otherwise} \end{cases}$$

with elements of $\mathcal{R}$;

(f) $F \diamondsuit_\otimes A = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{p_r,q_s}(\langle a^1_{k_i,l_j}, ..., a^n_{k_i,l_j}\rangle),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ with elements of $\mathcal{R}$.

Obviously, in some sense, operators $\boxplus$ and $\boxtimes$ are modifications/extensions of operation Multiplication with a constant.

An interesting **Open problem** is: Can there be defined new operations/operators that are modifications/extensions of operations Multiplication and Structural subtraction.

5 New operations over IMFEs

Let the IMFEs $F = [K, L, \{f_{k_i,l_j}\}]$ and $G = [P, Q, \{g_{p_r,q_s}\}]$ be given, where f_{k_i,l_j}, g_{p_r,q_s} are functions.

(g) $F \diamondsuit_\oplus G = [K \cup P, L \cup Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = \begin{cases} f_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ & \\ g_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ & \\ f_{k_i,l_j}(g_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q; \\ & \\ \perp, & \text{otherwise} \end{cases} .$$

The elements of this IM are functions, i.e., $F \diamondsuit_\oplus G$ is an IMFE.

(h) $F \diamondsuit_\otimes G = [K \cap P, L \cap Q, \{h_{t_u,v_w}\}]$, where

$$h_{t_u,v_w} = f_{p_r,q_s}(g_{p_r,q_s}),$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Therefore, the elements of this IM are functions, too.

Now, we can see that for every three IMFE $F = [K, L, \{f_{k_i,l_j}\}]$, $G = [P, Q, \{g_{p_r,q_s}\}]$ and $H = [T, U, \{h_{t_u,v_w}\}]$ operations $\boxplus, \boxplus, \oplus, \otimes, \diamondsuit_\oplus, \diamondsuit_\otimes$ are not commutative.

Theorem. For the above three IMFEs F, G, H and for two IMs A and B with elements real numbers:

(a) $(F \diamondsuit_\oplus G) \diamondsuit_\oplus H = F \diamondsuit_\oplus (G \diamondsuit_\oplus H)$,

(b) $(F \diamondsuit_\otimes G) \diamondsuit_\otimes H = F \diamondsuit_\otimes (G \diamondsuit_\otimes H)$,

(c) $(F \diamondsuit_\oplus G) \otimes A = F \diamondsuit_\oplus (G \otimes A)$,

(d) $(F \diamondsuit_\otimes G) \otimes A = F \diamondsuit_\oplus (G \otimes A)$,

(e) $(A \oplus_\times B) \boxplus F = A \boxplus (B \boxplus F)$,

(f) $(A \oplus_\times B) \boxtimes F = A \boxtimes (B \boxtimes F)$,

(g) $(F \oplus_\circ G) \diamondsuit_\otimes H = (F \diamondsuit_\otimes H) \oplus_\circ (G \diamondsuit_\otimes H)$.

Proof. Let IMFEs F, G, H be given. Then (g) is valid, because

$$(F \oplus_\circ G) \diamondsuit_\otimes H = [K \cap P, L \cap Q, \{f_{k_i,l_j} \circ g_{p_r,q_s}\}] \diamondsuit_\otimes H$$

$$= [K \cap P \cap T, L \cap Q \cap V, \{(f_{k_i,l_j} \circ g_{p_r,q_s})(h_{t_u,v_w})\}]$$

$$= [(K \cap T) \cap (P \cap T), (L \cap V) \cap (Q \cap V), \{f_{k_i,l_j}(h_{t_u,v_w}) \circ g_{p_r,q_s}(h_{t_u,v_w})\}]$$

$$= [(K \cap T), (L \cap V), \{f_{k_i,l_j}(h_{t_u,v_w})\}] \oplus_\circ [(P \cap T), (Q \cap V), \{g_{p_r,q_s}(h_{t_u,v_w})\}]$$

$$= (F \diamondsuit_\otimes H) \oplus_\circ (G \diamondsuit_\otimes H).$$

6 Conclusion

We will use the new operations for description of some components of the artificial intelligence. For example, in [3] it was shown that the neural networks can be described in the form of IMs. On the basis of this paper, a new extension of the concept of neural networks, was introduced in [4].

In future, the new type of IM will be used for description of other components of the artificial intelligence, e.g., genetic algorithms.

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8 Authors information

Krassimir T. Atanassov

Dept. of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, 105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria, e-mail: $krat@bas.bg$