WEB-BASED SIMULTANEOUS EQUATION SOLVER[12]

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Abstract: In this paper we present methods, theoretical basis of algorithms, and computer tools, which we have used for constructing our Web-based equation solver.

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Introduction

Many industrial and optimization tasks lead to the problem of finding all roots of (1) or arbitrary their part. One of branches for solving polynomial equations is parallel methods for simultaneous determination of all roots. With our solver automatically we can search simultaneously all or only one part of all roots of (1) (real, complex, lying in given area).

Iteration methods

Let us consider algebraic polynomial

$$\begin{aligned} A_n(x) &= x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = \\ &= \left(\ldots\left(\left(x + a_1\right)x + a_2\right)x + \ldots + a_{n-1}\right)x + a_n . \end{aligned} \tag{1}$$

The approximations of the k th iteration to zeroes $x_1, x_2, \ldots, x_w$ of (1) are denoted by $x_1^{[k]}, x_2^{[k]}, \ldots, x_w^{[k]}$ and their multiplicities by

$$\alpha_1, \alpha_2, \ldots, \alpha_w \ \left(1 \le \alpha_i \le n - w + 1,\, i = \overline{1, w},\, \sum_{i=1}^{w} \alpha_i = n\right).$$

Classical methods for individual searching of multiple roots of (1) can be written in this general way

$$\begin{aligned} & x_i^{[k+1]} = x_i^{[k]} - F\left(x_i^{[k]}, \alpha_i, a_1, a_2, \ldots, a_n\right), \\ & \quad i = \overline{1, w}\,, \ \ k = 0,1,2,\ldots . \end{aligned} \tag{2}$$

Other approach is given by methods for simultaneous extraction of all multiple roots and we can write them as

$$\begin{aligned} & x_i^{[k+1]} = x_i^{[k]} - F\left(x_1^{[k]}, x_2^{[k]}, \ldots, x_w^{[k]}, \alpha_1, \alpha_2, \ldots, \alpha_w, a_1, a_2, \ldots, a_n\right), \\ & \quad i = \overline{1, w}\,, \ \ k = 0,1,2,\ldots . \end{aligned} \tag{3}$$

Methods (3) are steadier and also they have a larger domain of convergence with comparison with methods (2). This is the main reason because these methods are object of detailed investigations in last twenty years.

For natural reasons we want to find simultaneously only one part of all roots of (1). Namely, we want to find simultaneously $p(\le w)$ different roots with multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_p$

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$\left(\alpha_1+\alpha_2+\ldots+\alpha_p=n-m, n-m-p+1\geq\alpha_j\geq 1, j=\overline{1,p}\right)$. In our Web-based equation solver we use [Iliev, Kyurkchiev, 2002a, 2002b, 2003], [Kyurkchiev, Iliev, 2002] type methods, which in common can be written as

$$x_i^{[k+1]}=x_i^{[k]}-F\left(x_1^{[k]},x_2^{[k]},\ldots,x_p^{[k]},\alpha_1,\alpha_2,\ldots,\alpha_p,a_1,a_2,\ldots,a_n\right),\quad i=\overline{1,p},\ k=0,1,2,\ldots. \tag{4}$$

Theoretical basis of our methods

Polynomial (1) can be presented in this way

$$A_n(x)=Q_{n-m}(x)T_m(x), \tag{5}$$

where $Q_{n-m}(x)$ is polynomial, whose zeroes we seek and $T_m(x)$ is polynomial, whose zeroes we ignore. Respectively

$$Q_{n-m}(x)=x^{n-m}+b_1x^{n-m-1}+b_2x^{n-m-2}+\ldots+b_sx^{n-m-s}+\ldots+b_{n-m}$$
$$T_m(x)=x^m+c_1x^{m-1}+c_2x^{m-2}+\ldots+c_px^{m-p}+\ldots+c_m.$$

Between the coefficients of polynomial (1) and the coefficients of polynomials (5) there exist the following relations

$$\begin{aligned}
a_1&=c_1+b_1\\
a_2&=c_2+b_2+c_1b_1\\
&\ldots\\
a_l&=c_l+b_l+c_1b_{l-1}+c_2b_{l-2}+\ldots+c_{l-1}b_1\\
&\ldots\\
a_m&=c_m+b_m+c_1b_{m-1}+c_2b_{m-2}+\ldots+c_{m-1}b_1\\
a_{m+1}&=c_2b_{m-1}+c_3b_{m-2}+\ldots+c_mb_1+b_{m+1}\\
&\ldots\\
a_n&=c_mb_{n-m}.
\end{aligned}$$

We want to find simultaneously $p(\leq w)$ different roots with multiplicities

$$\alpha_1,\alpha_2,\ldots,\alpha_p\ \left(\alpha_1+\alpha_2+\ldots+\alpha_p=n-m, n-m-p+1\geq\alpha_j\geq 1, j=\overline{1,p}\right)$$

and we set

$$\begin{aligned}
Q_{n-m}^{[k]}(x)&=\prod_{j=1}^{p}\left(x-x_j^{[k]}\right)^{\alpha_j}=\\
&=x^{n-m}+b_1^{[k]}x^{n-m-1}+b_2^{[k]}x^{n-m-2}+\ldots+b_s^{[k]}x^{n-m-s}+\ldots+b_{n-m}^{[k]},\\
T_m^{[k]}(x)&=x^m+c_1^{[k]}x^{m-1}+c_2^{[k]}x^{m-2}+\ldots+c_p^{[k]}x^{m-p}+\ldots+c_m^{[k]}.
\end{aligned} \tag{6}$$

From (6) it follows

$$\begin{aligned}
b_1^{[k]}&=-\sum_{j=1}^{p}\alpha_jx_j^{[k]}\\
b_2^{[k]}&=\sum_{j=1}^{p-1}\left[\alpha_jx_j^{[k]}\sum_{s=j+1}^{p}\alpha_sx_s^{[k]}\right]+\sum_{j=1}^{p}\frac{\alpha_j\left(\alpha_j-1\right)}{2}\left(x_j^{[k]}\right)^2\\
&\ldots\\
b_{n-m}^{[k]}&=(-1)^{n-m}\prod_{j=1}^{p}\left(x_j^{[k]}\right)^{\alpha_j}.
\end{aligned}$$

Combinative algorithms can be used for finding coefficients $b_1^{[k]}, b_2^{[k]}, \dots, b_{n-m}^{[k]}$.

We define $c_j^{[k]},\ j = \overline{1,m}$ using formulae

$$c_1^{[k]} = a_1 - b_1^{[k]}$$
$$c_2^{[k]} = a_2 - b_2^{[k]} - \left(a_1 - b_1^{[k]}\right)b_1^{[k]} = a_2 - b_2^{[k]} - c_1^{[k]}b_1^{[k]}$$
$$\dots$$
$$c_m^{[k]} = F(a_1, a_2, \dots, a_m, b_1^{[k]}, b_2^{[k]}, \dots, b_m^{[k]}) = a_m - b_m^{[k]} - \sum_{j=1}^{m-1} c_j^{[k]} b_{m-j}^{[k]}.$$

For simultaneous searching of roots of $Q_{n-m}(x)$ from (5) we give the following iteration algorithm

$$x_i^{[k+1]} = x_i^{[k]} - \frac{\alpha_i A_n\left(x_i^{[k]}\right)}{A_n'\left(x_i^{[k]}\right) - A_n\left(x_i^{[k]}\right)\left[\sum_{j=1, j\neq i}^{p} \frac{\alpha_j}{x_i^{[k]} - x_j^{[k]}} + \frac{T_m'^{[k]}\left(x_i^{[k]}\right)}{T_m^{[k]}\left(x_i^{[k]}\right)}\right]}, \qquad (7)$$

$i = \overline{1,p}$, $k = 0,1,2,\dots$.

When $m = n-1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ method (7) coincides with the classical Obreshkoff's method [Obreshkoff, 1963] for individual searching of one simple zero and if $p = w$ (7) is method for finding all roots of (1).

Theorem. Let $d \overset{\text{def}}{=\!=} \min_{i\neq j}\left|x_i - x_j\right|$, $c > 0$ and $1 > q > 0$ be real numbers such that

$$d > 2c,$$
$$2c^2\left[\left(n - m - \alpha_i\right)/(d-2c)^2 + \left(gP_2 + yP_1\right)G_1^{-2}\right] < \alpha_i,\ i = \overline{1,p},$$

where P_1, P_2, G_1, g and y are appropriate positive constants. If initial approximations $x_1^{[0]}, x_2^{[0]}, \dots, x_p^{[0]}$ to the real roots of (1) satisfy inequalities $\left|x_i^{[0]} - x_i\right| < cq,\ i = \overline{1,p}$ then for every $k \in N$ the inequalities

$$\left|x_i^{[k]} - x_i\right| < cq^{3^k}, i = \overline{1,p}$$

are satisfied.

From this theorem [Iliev, Kyurkchiev, 2003] it follows that iteration method (7) holds cubic convergence.

Localization technique for automatic determination of multiplicity of the roots and their initial approximations

For applying in practice in Obreshkoff's monograph [Obreshkoff, 1963] is given that Fujiwara prove that the circle with centre origin and radius $R = 2\max_{1\le p\le n}\left|a_{n-p}/a_n\right|^{1/p}$ in the complex plane contains all zeroes of polynomial (1).

We will use presentation

$$A_n(x) = (x - x_1)(x - x_2)\dots(x - x_n) = \rho_1\rho_2\dots\rho_n e^{i\varphi_1}e^{i\varphi_2}\dots e^{i\varphi_n},$$

where ρ_p are modulo of complex numbers $x - x_p$, $p = \overline{1,n}$ and φ_p, $p = \overline{1,n}$ are their arguments.

After one pass of contour with appropriate step in counter-clockwise direction every arguments of the roots in domain will be changed with 2π and every arguments out of contour will not be changed. Using Cauchy approach [Obreshkoff, 1963] if the argument change of $z = A_n(x)$, where x with appropriate step in counter-clockwise direction are different points from passed contour, is $2\pi s$, $s \in [1, n]$, it follows that in explored domain there are exactly s roots.

After first pass of localization of the roots with presented here algorithm in different domains we explore these domains, which have more than one root. For every such area arises the question whether or not in it there are localized one or several roots, which are sufficiently close. Confirmation or rejection of found multiplicity in "near" neighborhood could be made with Schröder's method [Schröder, 1870]. It will have quadratic convergence only when the multiplicity of the root is exact. Exactly we have in mind ε discernible roots (zeroes). If the roots are not multiple we will repeat Cauchy algorithm procedure. This is because we need fine localization of roots only in these areas, which contain more than one different roots.

Program description

The main modules are realized on Pascal program language [Krushkov, Iliev, 2002], using Delphi 5 environment. We realized specialized program units for complex numbers, multiple precision, input polynomial analyzer, specialized methods for finding a part of all roots [Iliev, Kyurkchiev, 2002a, 2002b, 2003], [Kyurkchiev, Iliev, 2002]. Also we have used dynamic structures for the economy of memory and for faster program code optimization. For Web-based input-output form interface for users is developed.

Conclusion

With this equation solver we try to give practical cover of theoretical improvements from classic, advanced techniques and iteration algorithms from last several years. It can be used simultaneously from many users with Internet connection without influence of distance.

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