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OPTIMUM TRAJECTORY OF DYNAMIC INPUT-OUTPUT BALANCE MODEL FOR OPEN ECONOMY

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Abstract: When export and import is connected with output of basic production, and criterion functional represents a final state of economy, the generalization of classical qualitative results of the main-line theory on a case of dynamic input-output balance optimization model for open economy is given.

Keywords: dynamic input-output balance, optimization problem, main-line theory, open economy, trajectory of the balanced growth.

ACM Classification Keywords: J.1 Administrative Data Processing: Business, Financial, Government

Introduction

The basic difficulties at practical application of economic dynamics diversified models are connected with the large dimension of linear programming optimization problems. In this connection there was an interest of experts to qualitative methods of optimum trajectories research. On this way the interesting results concerning creation of the so-called main-line theory were received. One of the most beautiful results in the theory of extending economy is the theorem by Dorfman, Samuelson and Solow [Dorfman, 1958] and it assert that the effective trajectory of economic growth has the long-term tendency to come nearer to Neumann way of the steady balanced growth. After the publication of the book by Dorfman, Samuelson and Solow [Dorfman, 1958] the theorems on main-line were established by Hics, Morishima and Mac-Kensey - for model of Neumann-Leontiev; by Radner and Nikaido - for model of Neumann-Gail with strictly convex set of productions; by Mac-Kensey - for generalized Leontiev model including the capital boons.

Basic Concepts and Corollaries of the Main-line Theory

It is possible to show the basic concepts and corollaries of the main-line theory on an example of optimization problem for Neumann model [Ashmanov, 1984; Ponomarenko, 1995]:

$$\begin{aligned} & c_T x_T \to \max, \\ & Ax_t \le Bx_{t-1}, \quad x_t \ge 0, \quad t = 1,2,\ldots,T, \end{aligned} \tag{1}$$

where $A \ge 0, \quad B \ge 0$ - non-negative rectangular $n \times m$ matrixes of expenses and release accordingly, Bx_0 - given vector, $c_T > 0$ - given positive vector, x_t - vector of intensities of technological process in time interval t.

The stationary trajectory of intensities for Neumann model (*A*, *B*) is determined by rate of growth $\alpha = \overline{\lambda}^{-1}$ and Neumann beam $\overline{x}$ and looks like $x_t = \overline{\lambda}^{-t}\overline{x}$, where $\overline{\lambda} > 0$, $\overline{x} > 0$ - unique decision of inequalities system to within scalar multiplier

$$A\overline{x} \le \overline{\lambda} B\overline{x} \tag{2}$$

It is significant that the main-line $\overline{x}$ appears not sensitive to changing of coefficients of criterion functional $c_T > 0$, the problem (1) is reduced to the following Neumann problem consequently

$$\lambda \to \min, \quad Ax \le \lambda Bx, \quad x \ge 0 \tag{3}$$

The basic result concerning the minimal eigenvalue $\overline{\lambda} < 1$ of Neumann model (maximal rate of growth) is formulated as the following theorem.

<u>Theorem 1.</u> Let non-negative matrixes $A \ge 0, \quad B \ge 0$ are such that the matrix of production B has no zero lines, and the matrix of expenses A has no zero columns. Then the indecomposable productive Neumann model (3) has unique rate of growth $\overline{\lambda} < 1$ and main-line $\overline{x} > 0$.

Here indecomposability of Neumann model is understood as impossibility by simultaneous rearrangement of lines and columns in matrixes A and B to reduce them to the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

where the rectangular matrixes-blocks A11 and B11 have equal dimension, 0 - zero matrix of dimension more than 1. Indecomposability of Neumann model is equivalent to the condition: Frobenius number of model (3) is prime number, and Frobenius vector is strictly positive.

The *productivity* of Neumann model is understood as existence of the solution of inequalities system

$$Bx - Ax \geq c, \quad x \geq 0$$

at any $c \in R_+^n$. The productivity of Neumann model is equivalent to the condition: the Frobenius number of model (3) is less than 1.

General Scheme of π-model.

One of the most known scheme of dynamic input-output balance of closed economy is so-called general scheme of π-model (detailed scheme is developed by Yu.P.Ivanilov and A.A.Petrov [Ivanilov, 1971]):

$$\begin{gathered} Ax_t + D\eta_t + L_t c \leq x_t, \\ x_t \leq \xi_{t-1}, \quad \xi_t \leq \xi_{t-1} + \eta_t, \\ lx_t \leq L_t, \\ (x_t, \xi_t, \eta_t, L_t) \geq 0, \quad t = 1,2,...,T, \end{gathered} \tag{4}$$

where the index t is number of time interval (year), $x = (x_1, x_2, ..., x_n)^T$ - aggregate stock of goods, the technological expenses of every of n branches are described by Leontiev matrix *A*, $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ - greatest possible total output (capacities of branches), $\eta = (\eta_1, \eta_2, ..., \eta_n)^T$ - desirable increment of capacities, the material input on increment of the basic capacities of all n branches are described by matrix *D*, $l = (l_1, l_2, ..., l_n)^T$ - expense of manpower for unit of total output, L - total employment, $c = (c_1, c_2, ..., c_n)^T$ - vector of consumption on one worker (his natural wages), ξ_0 - basic capacities at the initial moment *t*=0.

For model (4) terminal criterion is used more often in the form

$$c_T x_T \to \max, \tag{5}$$

where $c_T > 0$, that directly contacts with maximal rate of economy growth.

The basic result concerning existence of main-line for the problem (4) - (5) is formulated as the following theorem [Ashmanov, 1984].

Theorem 2. *Let ξ_0 >0, matrix $R = (c_i l_j)_1^n$, matrix A+R is indecomposable and productive, matrix $Q(\lambda) = \lambda(A + R) + (1 - \lambda)D$ is primitive. Then the vector $(\bar{x}, \bar{\xi}, \bar{\eta}, \bar{L})$, where $\lambda = \bar{\lambda} < 1$ and $\bar{x} > 0$ are Frobenius number and right eigenvector of matrix $Q(\bar{\lambda})$ accordingly , is a main-line for model (4-5).*

Expansion of π-model.

The problem on expansion of the scheme (4) of model on a case of open economy is unsolved today, when export and import run up to such large volumes, that the refusal of them results in a situation of impossibility of economic development. The next problem is a problem on existence of a main-line open economy development, as it is shown for a case of closed economy. This paper is devoted to the solution of these two problems.

For open economy having export and import in large volumes, it is offered to allocate exporting branches (first group of branches) and importing branches (second group of branches), and on vectors of export e1 (t) and import i2 (t) to impose industrial restrictions on its volumes. The model is offered:

$$
\begin{gathered}
c_1^T x_1(T) + c_2^T x_2(T) \to \max, \\
A_{11}x_1(t) + A_{12}x_2(t) + D_{11}\eta_1(t) + D_{12}\eta_2(t) + c_1 L(t) + e_1(t) \le x_1(t), \\
A_{21}x_1(t) + A_{22}x_2(t) + D_{21}\eta_1(t) + D_{22}\eta_2(t) + c_2 L(t) - i_2(t) \le x_2(t), \\
x_1(t) \le \xi_1(t-1), \quad x_2(t) \le \xi_2(t-1), \\
\xi_1(t) \le \xi_1(t-1) + \eta_1(t), \quad \xi_2(t) \le \xi_2(t-1) + \eta_2(t), \\
l_1 x_1(t) + l_2 x_2(t) \le L(t), \\
e_1(t) \ge F_1 x_1(t), \quad i_2(t) \le H_2 x_1(t), \\
x_1(t) \ge 0, \quad x_2(t) \ge 0, \quad \xi_1(t) \ge 0, \quad \xi_2(t) \ge 0, \quad \eta_1(t) \ge 0, \quad \eta_2(t) \ge 0, \\
e_1(t) \ge 0, \quad i_2(t) \ge 0, \quad L(t) \ge 0, \quad t = 1,2,\dots,T.
\end{gathered}
\tag{6}
$$

In model (6) stocks of production of exporting group of branches x1 (t) in time interval of t provide direct industrial expenses A11x1 (t) and A12x2 (t), consumption L (t) c1, creation of capacities increments of both groups of branches $D_{11}\eta_1(t)$ and $D_{12}\eta_2(t)$, and export e1 (t) also. At the same time stock of production of importing group of branches x2 (t) in time interval t can ensure direct industrial expenses A21x1 (t) and A22x2 (t), consumption L (t) c2, creation of capacities increments $D_{21}\eta_1(t)$ and $D_{22}\eta_2(t)$, but with the help of import i2 (t) use. In model (6) F1 means a non-negative coefficients matrix of the minimal production export of the first group of branches, H2 - non-negative matrix of coefficients of the maximal import for maintenance of industrial needs of the first group of branches. In particular, the F1 can be diagonal matrix with diagonal elements which are smaller than one.

Model (6) is dynamic model, as a result of its functioning we receive at the initial data $\xi(0)$ a sequence of vectors $X(t) = (x_1(t), x_2(t), \xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t), e_1(t), i_2(t), L(t))$, t=1,2, ..., T, satisfying to all restrictions of the model. Such sequence represents a trajectory. At the end of the researched period (at the time moment T) the state of the model is characterized by a vector X (T) (so-called terminal state of model).

State of Equilibrium

Let's research a state of equilibrium in model (6). The appropriate stationary trajectory of intensities is determined by rate of growth $\alpha = \overline{\lambda}^{-1}$ > 1, by a Neumann beam $X = (x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, e_1, i_2, L)$ and looks like

$$
\begin{gathered}
x_1(t) = \lambda^{-t}x_1, \quad x_2(t) = \lambda^{-t}x_2, \quad \xi_1(t) = \lambda^{-t}\xi_1, \quad \xi_2(t) = \lambda^{-t}\xi_2, \\
\eta_1(t) = \lambda^{-t}\eta_1, \quad \eta_2(t) = \lambda^{-t}\eta_2, \quad e_1(t) = \lambda^{-t}e_1, \quad i_2(t) = \lambda^{-t}i_2, \quad L(t) = \lambda^{-t}L.
\end{gathered}
\tag{7}
$$

If correlations (7) to substitute to (6), for the state of equilibrium $(\lambda, x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, e_1, i_2, L)$ at large T we receive optimization problem:

$$\begin{gathered}
\lambda \to \min, \\
x_1 \geq A_{11}x_1 + A_{12}x_2 + D_{11}\eta_1 + D_{12}\eta_2 + Lc_1 + e_1, \\
x_2 \geq A_{21}x_1 + A_{22}x_2 + D_{21}\eta_1 + D_{22}\eta_2 + Lc_2 - i_2, \\
x_1 \leq \lambda\xi_1, \quad x_2 \leq \lambda\xi_2, \\
(1-\lambda)\xi_1 \leq \eta_1, \quad (1-\lambda)\xi_2 \leq \eta_2, \\
l_1x_1 + l_2x_2 \leq L, \\
e_1 \geq F_1x_1, \quad i_2 \leq H_2x_1, \\
x_1 \geq 0, \quad x_2 \geq 0, \quad \xi_1 \geq 0, \quad \xi_2 \geq 0, \quad \eta_1 \geq 0, \quad \eta_2 \geq 0, \\
e_1 \geq 0, \quad i_2 \geq 0, \quad L \geq 0.
\end{gathered} \tag{8}$$

Let's consider matrixes

$$R_{11} = \left(c_i^1 l_j^1\right), \quad R_{12} = \left(c_i^1 l_j^2\right), \quad R_{21} = \left(c_i^2 l_j^1\right), \quad R_{22} = \left(c_i^2 l_j^2\right).$$

Since at $0 < \lambda < 1$ we have

$$x_1 \leq \lambda\xi_1 \leq \frac{\lambda}{1-\lambda}\eta_1, \quad x_2 \leq \lambda\xi_2 \leq \frac{\lambda}{1-\lambda}\eta_2,$$

and consequently

$$\eta_1 \geq \frac{1-\lambda}{\lambda}x_1, \quad \eta_2 \geq \frac{1-\lambda}{\lambda}x_2,$$

then subject to $D_{11} \geq 0, \quad D_{12} \geq 0, \quad D_{21} \geq 0, \quad D_{22} \geq 0$ we receive

$$D_{11}\eta_1 \geq \frac{1-\lambda}{\lambda}D_{11}x_1, \quad D_{12}\eta_2 \geq \frac{1-\lambda}{\lambda}D_{12}x_2, \quad D_{21}\eta_1 \geq \frac{1-\lambda}{\lambda}D_{21}x_1, \quad D_{22}\eta_2 \geq \frac{1-\lambda}{\lambda}D_{22}x_2.$$

Further, since

$$(l_1x_1)c_1 = R_{11}x_1, \quad (l_2x_2)c_1 = R_{12}x_2, \quad (l_1x_1)c_2 = R_{21}x_1, \quad (l_2x_2)c_2 = R_{22}x_2,$$

then subject to non-negativity of matrixes F1 and H2, we come to inequalities

$$\begin{aligned}
& x_1 \geq A_{11}x_1 + A_{12}x_2 + D_{11}\eta_1 + D_{12}\eta_2 + Lc_1 + e_1 \geq \\
& \geq \left(A_{11} + R_{11} + \frac{1-\lambda}{\lambda}D_{11} + F_1\right)x_1 + \left(A_{12} + R_{12} + \frac{1-\lambda}{\lambda}D_{12}\right)x_2, \\
& x_2 \geq A_{21}x_1 + A_{22}x_2 + D_{21}\eta_1 + D_{22}\eta_2 + Lc_2 - i_2 \geq \\
& \geq \left(A_{21} + R_{21} + \frac{1-\lambda}{\lambda}D_{21}\right)x_1 + \left(A_{22} + R_{22} + \frac{1-\lambda}{\lambda}D_{22}\right)x_2 - H_2x_1.
\end{aligned} \tag{9}$$

After transferring the term H2x1 on the left and multiplication of both parts of these inequalities on $\lambda > 0$ we receive

$$\lambda(E+H)x \geq \left[\lambda(A+R+F) + (1-\lambda)D\right]x,$$

where $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ H_2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$
are non-negative square matrixes, E is diagonal identity matrix.

Thus, the problem on maximization of growth rate of open economy (6) is reduced by us to such generalized Neumann model

$$\lambda \to \min, \quad Q(\lambda)x \le \lambda Bx, \quad x \ge 0, \tag{10}$$

where

$$Q(\lambda) = \lambda(A + R + F) + (1 - \lambda)D, \qquad B = E + H. \tag{11}$$

Let's begin a finding of equilibrium state of model (6). The equilibrium trajectory of intensities is determined by rate of growth $\alpha = \lambda^{-1}$ and Neumann beam $X = (x_1, x_2, \xi_1, \xi_2, \eta_1, \eta_2, e_1, i_2, L)$. To ensure existence of the non-trivial solution of inequalities system (8), it is necessary to impose some restrictions on parameters of model.

Let's consider, that the matrix A is non-negative and indecomposable, $l > 0, \quad c \ge 0, \quad c \ne 0,$ and matrix A+R+F is productive, i.e. its Frobenius number is less than 1. This restriction means, that the existing technology (A, I, F) allows every worker "to support" itself, carrying out production and the foreign trade operations.

Besides, we consider, that $D \ge 0$ and if $\eta \ge 0, \quad D\eta = 0$, $\eta = 0$. The given assumption means, that any increment of the basic capacities requires material inputs. In other words, we consider, that in the matrix D there are'nt present zero columns.

Let's construct for (10) the dual problem

$$pQ(\lambda) \ge \lambda pB, \quad p \ge 0,$$

where $p = (p_1, p_2)$ is vector-line of dual estimations. As we are interested in a case $x > 0$, it is possible only then, when

$$pQ(\lambda) = \lambda pB. \tag{12}$$

System of linear algebraic equations (12) has non-trivial solution $p \ne 0$ only under condition of

$$\det\big(Q(\lambda) - \lambda B\big) = 0,$$

i.e.

$$\det\big[(1-\lambda)D - \lambda(E - A - R - F + H)\big] = 0. \tag{13}$$

Since the matrix $A + R + F$ is considered productive, then [Ponomarenko, 1995] there is a non-negative matrix $(E - A - R - F)^{-1} \ge 0$.

Let also matrix H is such, that $H \le A + R + F$ (it can be carried out, as $A \ge 0, \quad R > 0, \quad F \ge 0$). Then the matrix $\overline{A} = A + R + F - H \ge 0$ remains productive, i.e. there is a non-negative matrix $\big(E - \overline{A}\big)^{-1} = (E - A - R - F + H)^{-1} \ge 0$. The last statement appears the most important requirement.

Equation (13) is possible to rewrite as

$$\det\left[(E-A-R-F+H)^{-1}D-\mu E\right]=0,$$

where $\mu=\frac{\lambda}{1-\lambda}=\frac{1}{1-\lambda}-1>0$ at $0<\lambda<1$. To the least value λ corresponds the greatest value μ.

Matrix $(E-A-R-F+H)^{-1}\geq 0$. Then in accordance to the Perron-Frobenius theorem [Ponomarenko, 1995] there is a Frobenius number $\overline{\mu}>0$ and appropriate Frobenius vector $\overline{z}\geq 0$ such, that

$$(E-A-R-F+H)^{-1}D\overline{z}=\mu\overline{z}.$$

Let's note thus, that

$$(E-A-R-F+H)^{-1}D\leq(E-A-R-F)^{-1}D,$$

and therefore $\overline{\mu}\leq\mu^{*}$, where μ^{*} - Frobenius number of a matrix $(E-A-R-F)^{-1}D$.

Let's return now to problem (10), which rewrite as

$$\mu\to\min,\quad (E-A-R-F+H)^{-1}x\leq\mu x,\quad x\geq 0.$$

The solution of this problem is reached at $\mu=\overline{\mu},\quad x=\overline{z}$. Thus rate of growth $\overline{\lambda}^{-1}=1+\frac{1}{\overline{\mu}}\geq 1+\frac{1}{\mu^{*}}>1$, and also structure of output $\overline{x}\geq 0$.

The Basic Result

The basic result of this paper is formulated as such theorem.

__Theorem 3.__ If the matrix $A+R$ is productive, matrix $H\leq A+R+F$, and matrix D has no zero columns, then in model (6) there is a condition of equilibrium with rate of growth $\overline{\lambda}^{-1}=1+\frac{1}{\overline{\mu}}$, to which corresponds a unique Neumann beam $(x_1,x_2,\xi_1,\xi_2,\eta_1,\eta_2,y_2,L)$, and:

1) $\overline{\mu}$ - Frobenius number, $\overline{x}$ - right Frobenius vector of matrix $(E-A-R-F+H)^{-1}D$;

2) $\overline{\xi}=\overline{\lambda}^{-1}\overline{x},\quad \overline{\eta}=\frac{1-\overline{\lambda}}{\overline{\lambda}}\overline{x},\quad \overline{i}_1=F_1\overline{x}_1,\quad \overline{i}_2=H_2\overline{x}_1,\quad \overline{L}=l\overline{x}.$

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