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OPTIMAL CONTROL OF A SECOND ORDER PARABOLIC HEAT EQUATION

Mahmoud Farag, Mainouna Al-Manthari

***Abstract**: In this paper, we are concerned with the optimal control boundary control of a second order parabolic heat equation. Using the results in [Evtushenko, 1997] and spatial central finite difference with diagonally implicit Runge-Kutta method (DIRK) is applied to solve the parabolic heat equation. The conjugate gradient method (CGM) is applied to solve the distributed control problem. Numerical results are reported.*

***Keywords**: Distributed control problems, Second order parabolic heat equation, Runge-Kutta method, CGM.*

***ACM Classification Keywords**: F.2.1 Numerical Algorithms and Problems; G.4 Mathematical Software*

Introduction

In the recent years, optimal control of systems governed by partial differential equations have been extensively studied. We refer for instance to [Lions, 1971], [Farag, 2004] for parabolic problems and to [Wu,2003], [Borzi, 2002] for numerical studies. In this paper, we are concerned with the optimal control boundary control of a second order parabolic heat equation. Using the results in [Evtushenko, 1997] and spatial central finite difference with diagonally implicit Runge-Kutta method of order 2 in 2 stages is applied to solve the parabolic heat equation. The conjugate gradient method (CGM) is applied to solve the distributed control problem. Numerical results are reported.

Consider the second order heat equation

(1) $\frac{\partial y(x,t)}{\partial t} = a^2 \frac{\partial^2 y(x,t)}{\partial x^2} + u(x,t), (x,t) \in \Omega = (0,l) \, x \, (0,T)$ where $y(x,t)$ is the temperature at time t and at a point x and $u(x,t)$ is a distributed control.

The initial and boundary conditions are given by

(2) $y(x,0) = \varphi(x), \; x \in [0,l],$

(3) $\frac{\partial y(0,t)}{\partial x} = 0, \; \frac{\partial y(l,t)}{\partial x} = \nu [g(t) - y(l,t)], \; t \in (0,T)$ where $g(t)$ is a boundary control.

The problem is to find control functions $u(x,t)$ and $g(t)$ that minimize the cost functional

(4) $J = \int_0^l \Phi(y(s,T)) \, ds$

where Φ is continuously differentiable with respect to its argument.

DIRK Method

In this section we present some basic results about the Runge-Kutta methods, the diagonally implicit Runge-Kutta method of order 2 in 2 stages (DIRK). The reader is referred to [Alexander, 1977], [Shamardan, 1998].

In [Alexander, 1977], he has given the A-stable DIRK methods of maximum order in two stages and derived new methods with stronger stability properties, from this work one can extract the following theorem.

<u>Theorem 1</u>: There are exactly two strongly s-stable DIRK formulae of order two in two stages and exactly are strongly s-stable DIRK formulae of order three in three stages. They are

$$(5)\qquad \begin{array}{cc|c} \alpha & 0 & \alpha \\ 1-\alpha & \alpha & 1 \\ - & - & - \\ 1-\alpha & \alpha & \end{array}\ , \qquad \begin{array}{ccc|c} \alpha & 0 & 0 & \alpha \\ c_2-\alpha & \alpha & 0 & c_2 \\ b_1 & b_2 & \alpha & 1 \\ - & - & - & - \\ b_1 & b_2 & \alpha & \end{array}$$

with $\alpha = 1 \pm 0.5\sqrt{2}$ in the first and α is the root of $x^3 - 3x^2 + \frac{2}{3}x - \frac{1}{6} = 0$ lying in $(\frac{1}{6}, \frac{1}{2})$ and,

$b_1 = -0.25\,(6\alpha^2 - 16\alpha + 1)$, $b_2 = 0.25(6\alpha^2 - 20\alpha + 5)$ in the second.

In [Shamardan, 1998], we presented the numerical solution of the linear equation

$$(6)\quad \begin{cases} \dfrac{\partial y(x,t)}{\partial t} = a\,\dfrac{\partial^2 y(x,t)}{\partial x^2} - c\,\dfrac{\partial y(x,t)}{\partial x} \\ y(x,0) = \sin \pi x\,(x,t),\ \ x \in (0,1),\ y(0,t) = y(1,t) = 0\ ,\ t \in (0,1) \end{cases}$$

using DIRKB (Diagonally Implicit Runge-Kutta Backword Method), DIRKC (Diagonally Implicit Runge-Kutta Central Method), DIRKF (Diagonally Implicit Runge-Kutta Forward Method) and given the following figures.

In the figures (1.a),(1.b),(1.c), (2.a),(2.b),(2.c) the wave rapidly decreases as there is no shock wave but in the figures (3.a),(3.b),(3.c), (4.a),(4.b),(4.c) there is a medium boundary region near x=1.

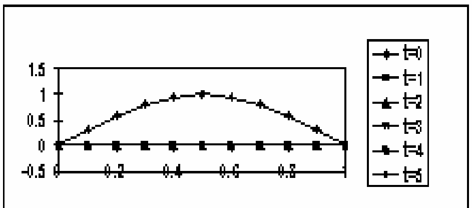


Fig. (1.a)

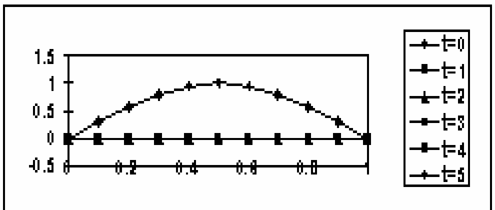


Fig. (2.a)

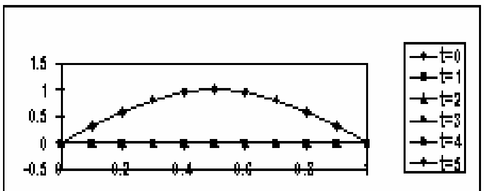


Fig. (1.b)

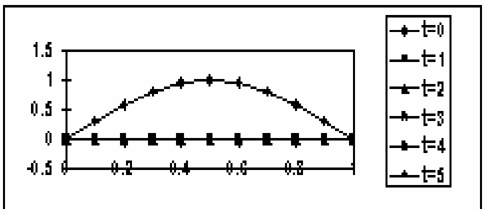


Fig. (2.b)

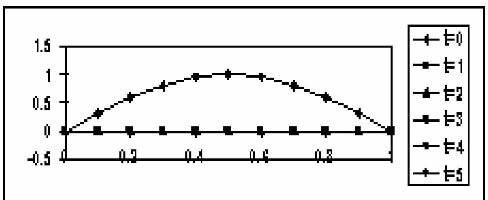


Fig. (1.c)

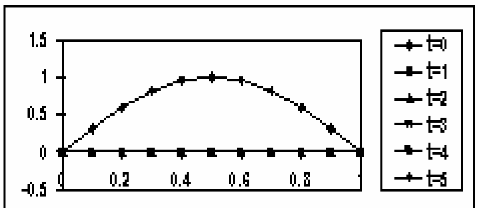


Fig. (2.c)

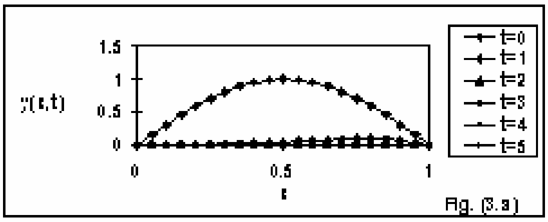


Fig. (3.a)

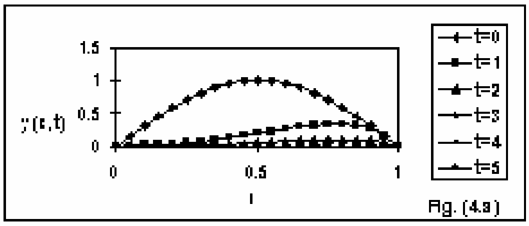


Fig. (4.a)

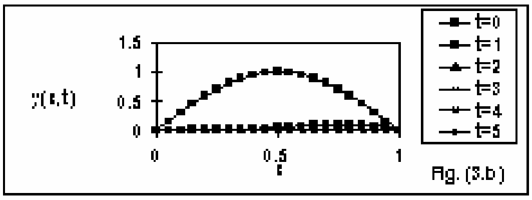


Fig. (3.b)

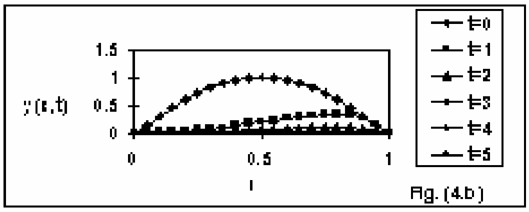


Fig. (4.b)

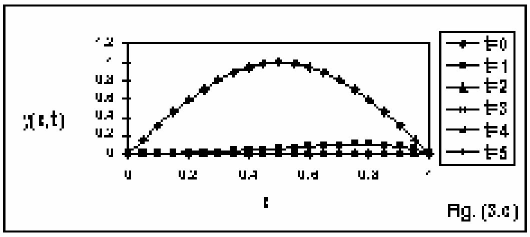


Fig. (3.c)

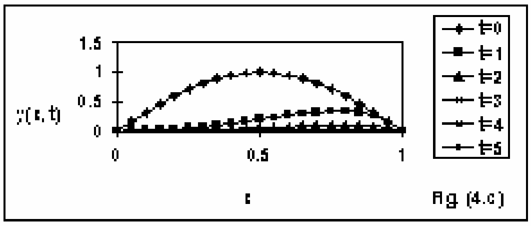


Fig. (4.c)

Differentiation of the Cost Functional

For given control functions $u(x,t)$ and $g(t)$, we solve equation (1) with conditions (2), (3), and then substitute this solution into (4) to evaluate J. This value is a composite function of $u(x,t)$ and $g(t)$. Denote it by $\Theta(u,g)$. Since the optimal control cannot be obtained as an analytic solution of the necessary and sufficient optimality conditions, we attempt to find it numerically by minimizing $\Theta(u,g)$ via a gradient algorithm. We are thus faced with computing the gradient of the cost functional [Evtushenko, 1997]. The problem is discredited by a finite difference approximation scheme. We use a uniform grid denote

$$(7)\quad \begin{cases} x_i = i\Delta x \ , t_j = j\Delta t,\ i = 0,..,k,\ j = 0,...,m, \\ \Delta x = \dfrac{l}{k},\ \Delta t = \dfrac{t}{m},\ y_i^j = y(i\Delta x, j\Delta t),\ u_i^j = u(i\Delta x, j\Delta t), \\ \varphi_i = \varphi(i\Delta x),\ g^j = g(j\Delta t),,\ i = 0,..,k,\ j = 0,...,m \end{cases}$$

Using an explicit forward Euler scheme in time, then the cost functional and the system (1)-(3) are replaced by

$$(8)\qquad \overline{J} = \Delta x \sum_{i=0}^{k} \alpha_i\ \Phi(y_i^m),$$

$$(9)\quad y_i^j = \begin{cases} (1-2\lambda)\,y_i^{j-1} + \lambda(y_{i-1}^{j-1} + y_{i+1}^{j-1}) + \Delta t\, u_i^{j-1} & 1\le i\le k-1 \quad 1\le j\le m \\ y_1^j & i=0 \quad 1\le j\le m \\ \mu y_{k-1}^j + \mu\,\nu\,\Delta x\, g^j & i=k \quad 1\le j\le m \\ \varphi_i & 0\le i\le k \quad j=0 \end{cases}$$

where α_i are the quadrature coefficients, $\lambda = \dfrac{a^2\,\Delta t}{(\Delta x)^2}$, $\mu = \dfrac{1}{1+\nu\,\Delta x}$.

We introduce the adjoint variables Ψ_i^j and the auxiliary function

$$(10)\qquad E=\overline{J}+\sum_{j=1}^{m}[\mu(y_{k-1}^{j}+\nu\,\Delta x\,g^{j})\Psi_{k}^{j}+y_{1}^{j}\Psi_{0}^{j}]+\sum_{i=0}^{k}\varphi_{i}\Psi_{i}^{0}$$
$$+\sum_{i=1}^{k-1}\sum_{j=1}^{m}[(1-2\lambda)y_{i}^{j-1}+\lambda(y_{i-1}^{j-1}+y_{i+1}^{j-1})+\Delta t\,u_{i}^{j-1}]\Psi_{i}^{j}$$

Appling {formula 11, in [Evtushenko, 1997]}, we obtain

$$(11)\quad \Psi_{i}^{j}=\begin{cases}(1-2\lambda)\Psi_{i}^{j+1}+\lambda(\Psi_{i-1}^{j-1}+\Psi_{i+1}^{j+1}) & 2\le i\le k-2 \quad 0\le j\le m-1\\ (1-2\lambda)\Psi_{1}^{j+1}+\lambda\,\Psi_{2}^{j+1}+\Psi_{0}^{j} & i=1 \quad 1\le j\le m-1\\ (1-2\lambda)\Psi_{k-1}^{j+1}+\mu\,\Psi_{k}^{j}+\lambda\,\Psi_{k-2}^{j+1} & i=k-1 \quad 1\le j\le m-1\\ \mu\,\Psi_{k}^{m}\,\delta_{i}^{k-1}+\Psi_{0}^{m}\,\delta_{i}^{1}+\Delta x\,\alpha_{i}\,\Phi(y_{i}^{m}) & 0\le i\le k \quad j=m\\ (1-2\lambda)\Psi_{i}^{1}+\lambda\,\Psi_{i+1}^{1} & i=1,i=k-1 \quad j=0\\ \lambda\Psi_{k}^{j+1} & i=0 \quad 0\le j\le m-1\\ \lambda\Psi_{k-1}^{j+1} & i=k \quad 0\le j\le m-1\end{cases}$$

Then using {formula 12, in [Evtushenko, 1997]}, we obtain

$$(12)\quad \begin{aligned}&\frac{d\,\Theta}{d\,u_{i}^{j}}=\Delta t\,\Psi_{i}^{j+1} && 1\le i\le k-1 && 0\le j\le m-1\\ &\frac{d\,\Theta}{d\,u_{0}^{j}}=\frac{d\,\Theta}{d\,u_{k}^{j}}=0 && && 0\le j\le m-1\\ &\frac{d\,\Theta}{d\,u_{i}^{m}}=0 && 0\le i\le k &&\\ &\frac{d\,\Theta}{d\,g^{j}}=\mu\,\nu\,\Delta x\,\Psi_{k}^{j} && && 1\le j\le m-1\end{aligned}$$

If we let $k\to\infty$, $\Delta x\to 0$, $\Delta t\to 0$, then in both cases we find that the function satisfies the following conditions:

$$(13)\quad \begin{aligned}&\frac{\partial\Psi}{\partial t}+a^{2}\frac{\partial^{2}\Psi}{\partial x^{2}}=0\,,(x,t)\in(0,1)\,x\,(0,T)\\ &\Psi(x,t)=\Phi_{y}(y(x,T)),\quad x\in[0,l]\\ &\frac{\partial\Psi(0,t)}{\partial x}=0,\ \frac{\partial\Psi(l,t)}{\partial x}=-\nu\,\Psi(l,t)\,,t\in(0,T)\end{aligned}$$

The gradients of the cost functional for the continuous problem are given by

$$(14)\qquad \frac{d\,\Theta}{d\,u(x,t)}=\Psi(x,t)\quad,\quad\frac{d\,\Theta}{d\,g(t)}=\nu\,a^{2}\,\Psi(l,t)\,.$$

Solution Algorithm

With the gradient obtained, the following gradient type algorithm [Farag, 2003] can then be developed for the optimal values of u^{*}, g^{*} based on the conjugate gradient method (CGM). The direct and adjoint systems are converted to ordinary differential equations and solving by DIRK method. The outlined of the algorithm for solving control problem are as follows:

Step 1: Choose an initial guess $u^{(n)}(x,t),\ g^{(n)}(t)\in U$.

Step 2: Solve the direct problem to obtain $y(x,t,u^{(n)},g^{(n)})$.

Step 3: Solve the adjoint problem to find the gradient of the cost functional

$$(15)\quad \begin{cases} [\dfrac{dJ}{du(x,t)}]^{(n)} = [\dfrac{d\Theta}{du(x,t)}]^{(n)} = \Psi(x,t,u^{(n)},g^{(n)}) \\ [\dfrac{dJ}{dg(t)}]^{(n)} = [\dfrac{d\Theta}{dg(t)}]^{(n)} = \nu\, a^2\, \Psi(l,t,u^{(n)}(l,t),g^{(n)}(t)). \end{cases}$$

Step 4: Compute the conjugate coefficient by:

$$(16)\quad \chi^{(n)} = \frac{\iint_{\Omega} \{ [\dfrac{dJ}{du(x,t)}]^{(n)} \}^2 dx\,dt + \int_0^T \{ [\dfrac{dJ}{dg(t)}]^{(n)} \}^2 dt}{\iint_{\Omega} \{ [\dfrac{dJ}{du(x,t)}]^{(n-1)} \}^2 dx\,dt + \int_0^T \{ [\dfrac{dJ}{dg(t)}]^{(n-1)} \}^2 dt}$$

Step 5 : Calculate the direction of descent :

$$(17)\quad u^{(n)} = [\frac{dJ}{du(x,t)}]^{(n)} + \chi^{(n)} u^{(n-1)},\ g^{(n)} = [\frac{dJ}{dg(t)}]^{(n)} + \chi^{(n)} g^{(n-1)}$$

Step 6: Test the optimality of $u^{(n+1)}$, $g^{(n+1)}$.

If $u^{(n+1)}$, $g^{(n+1)}$ are optimum, stop the process. Otherwise, go to Step 7.

Step 7: Set $u^{(n+1)} = u^{(n)}$, $g^{(n+1)} = g^{(n)}$, $n = n+1$ and go to Step 2.

Numerical Example

Let us present a numerical example. The programs were written in FORTRAN. We choose

$$a=1\ ,\ \nu=1\ ,\ \varphi(t)=x^2\ ,\ g(t)=50\,e^t+t+3\ ,\ x\in(0,1)\ ,\ t\in(0,1)$$

In Figure 3.1 the bold curve is the exact optimal boundary control and the other curves are the values of optimal control $g(t)$ via iterations. In figure 3.2 $u(x,t)$ is plotted for the approximation optimal control of the control problem. Figure 3.3 shows the values of cost functional via iterations.

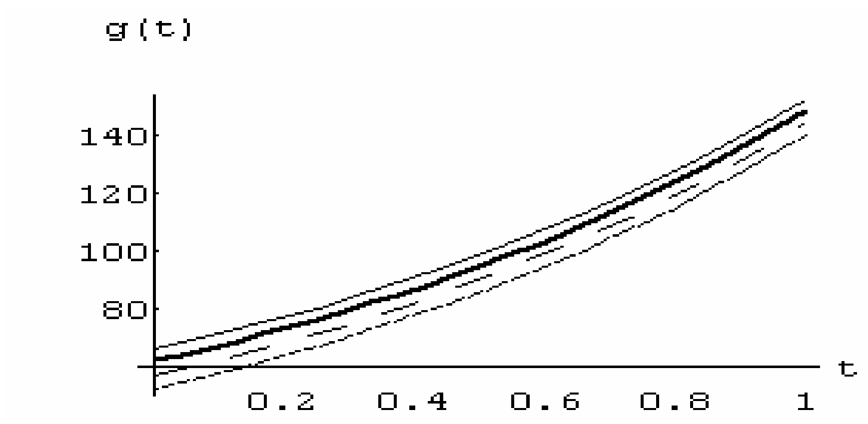


Figure 3.1 The optimal boundary Control via iterations

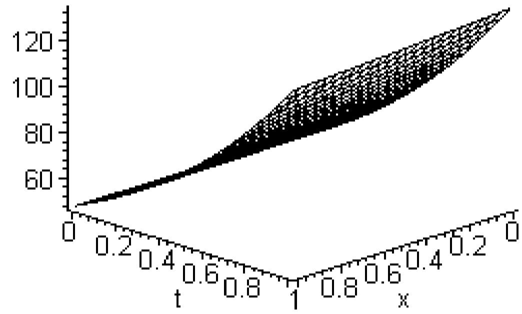


Figure 3.2 The exact optimal control $u(x,t)$

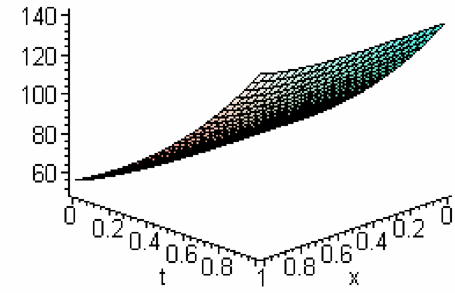
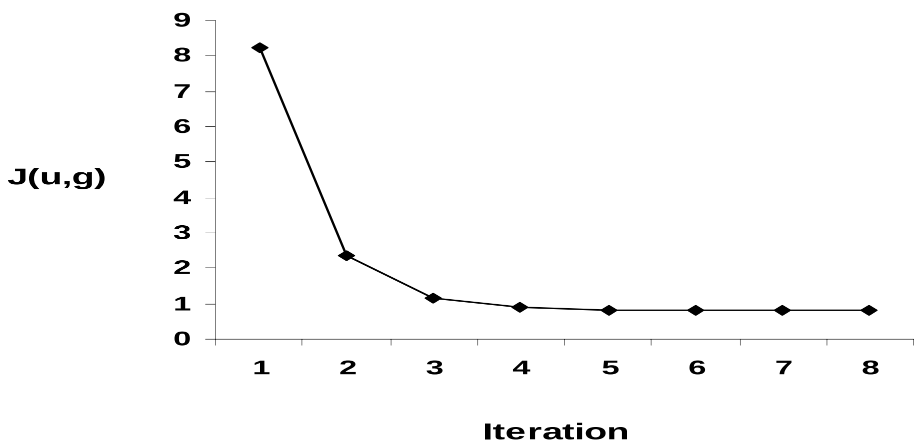


Figure 3.3 The approximation optimal control $u(x,t)$

Figure 3.3 The Value of Cost functional for different iterations



Conclusion

Optimal control problems for partial differential equations are currently of much interest. A large amount of the theoretical concept which governed by quasilinear parabolic equations has been investigated in the field of optimal control problems. These problems have dealt with the processes of hydro- and gas dynamics, heat physics, filtration, the physics of plasma and others. In this paper; we are concerned with the optimal control boundary control of a second order parabolic heat equation. Using the results in [Evtushenko, 1997] and spatial central finite difference with diagonally implicit Runge-Kutta method (DIRK) is applied to solve the parabolic heat equation. The conjugate gradient method (CGM) is applied to solve the distributed control problem. Numerical results are reported.

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THE MATRIX METHOD OF DETERMINING THE FAULT TOLERANCE DEGREE OF A COMPUTER NETWORK TOPOLOGY

Sergey Krivoi, Miroslaw Hajder, Pawel Dymora, Miroslaw Mazurek

Abstract: *This work presents a theoretical-graph method of determining the fault tolerance degree of the computer network interconnections and nodes. Experimental results received from simulations of this method over a distributed computing network environment are also presented.*

Keywords: *computer network, fault tolerance, coherent graph, regular graph, network topology, adjacency matrix.*

ACM Classification Keywords*: C.2.1 Network Architecture and Design - network topology, F.2.1 Numerical Algorithms and Problems - matrix methods, B.8.1 Reliability, Testing, and Fault-Tolerance - fault tolerance degree*

Introduction

Computer networks plays an extremely important role in today's information technologies, because by its means it's possible to accelerate processes like i.e. transmission, processing and storage of information in computer systems. In such a process the most crucial issues are related with protecting a correct work of a computer network and its interconnection and node fault tolerance. The solution of these problems is related with examining the network topological characteristics and its topological structures. In this work the theoretical-graph method of determining the computer network topology critical points which refers to computer network interconnections and nodes failures is proposed.