DESCRIPTION REDUCTION FOR RESTRICTED SETS OF (0,1) MATRICES [1]

Hasmik Sahakyan

Abstract: Any set system can be represented as an n-cube vertices set. Restricted sets of n-cube weighted subsets are considered. The problem considered is in simple description of all set of partitioning characteristic vectors. A smaller generating sets are known as "boundary" and "steepest" sets and finally we prove that the intersection of these two sets is also generating for the partitioning characteristic vectors.

ACM Classification Keywords: G.2.1 Discrete mathematics: Combinatorics

1. Introduction

In recent years, the processing of data flows has become a topic of active research in several fields of computer science. Continuous arrival of data items in rapid, potentially unbounded flows raises new challenges and research problems. The study of known combinatorial algorithms and their computational complexity for data flow conditions become an important issue.

Consider a $(0,1)$-matrix A of size $m \times n$. Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ denote the row and column sums of A respectively, and let $U(R,S)$ be the set of all $(0,1)$-matrices with row sums R and column sums S.

It was found by Gale and Ryser [R,1966] a necessary and sufficient condition for the existence of a $(0,1)$ matrix of the class $U(R,S)$. This result has found a recent revival in the field of discrete tomography [H, 1997]. In discrete tomography the problem is to reconstruct a discrete valued function f from knowledge of weighted sums of function values over subsets of the domain. A much studied special case is $m \times n$ $(0,1)$-matrices with known row and column sums, precisely matrices in the class $U(R,S)$.

As the number of matrices in this class may be high, it is of interest to study the reconstruction problem where with additional constraints on the $(0,1)$-matrices, which could either lead to a unique realization, or reduce the number of alternative solutions. The restrictions may be of different nature: requirements on rows of reconstructed matrices – to be different, some geometrical requirements such as convexity and connectivity, etc. It is proven ([B,1996], [W,2001], [D,1999] that the existence problems of horizontal and vertical convex matrices and the existence problem for connected) matrices (polyominoes are NP-complete; and the reconstruction problem for horizontal and vertical convex polyominoes can be solved in polynomial time. At the same time the complexity of the existence problem for matrices with different rows is still an open problem [BL,1988].

We assume now that we consider the last mentioned problem for data flow conditions and the coordinates of column sum vector S might varied slowly by the data flow. Then - which are the allowable values for coordinates of S to correspond to column sum vectors?

Complete description of the set of all integer-value vectors, which serve as column sum vectors for $(0,1)$-matrices with different rows, is given through its boundary elements [S,1997]. An alternative description of this set is known through its special elements - "steepest" vectors. The main result of this research states: the description might be given by the common (intersecting) elements of these sets - of upper boundary and steepest vectors, which minimizes the descriptor set size.

[1] The research is supported partly by INTAS: 04-77-7173 project, http://www.intas.be

2. Problem Description

Let consider the problem of existence of a $(0,1)$-matrix by the given column sums vector S and with different rows. Let assume that the coordinates of vector S is varying slightly by data flow, and then the problem is in description of all integer vectors, which serve as column sums vectors for $(0,1)$-matrices of fixed size and with different rows. .

This problem has an equivalent formulation in terms of unit cube E^n.

Let $M \subseteq E^n$ be a vertex subset of fixed size $|M| = m$, $0 \le m \le 2^n$. An integer, nonnegative vector $S = (s_1, s_2, \cdots, s_n)$ is called the **characteristic vector of partitions** of set M, if its coordinates equal to the partition-subsets sizes of M by coordinates $x_1, x_2, \cdots, x_n$ - the Boolean variables composing E^n. s_i equals the size of one of the partition-subsets of M by the i-th direction and then $m - s_i$ is the size of the complementary part of partition. To make this notation precise we will later assume that s_i is the size of the partition subset with $x_i = 1$. Then the problem is in description of all integer-coordinate vectors, which serve as characteristic vectors of partitions for vertex subsets of size m.

3. Description through the Boundary Elements

Let Ξ_{m+1}^n denotes the set of all vertices of n dimensional, $m+1$ valued discrete cube, i.e. the set of all integer-vectors $S = (s_1, s_2, \cdots, s_n)$ with $0 \le s_i \le m$, $i = 1, \cdots, n$. The vertices are distributed schematically on the $m \cdot n + 1$ layers of Ξ_{m+1}^n according to their weights – sums of all coordinates. The L-th layer contains all vectors $S = (s_1, s_2, \cdots, s_n)$ with $L = \sum_{i=1}^{n} s_i$.

Let ψ_m denotes the set of all characteristic vectors of partitions of m-subsets of E^n. It is evident, that - $\psi_m \subseteq \Xi_{m+1}^n$. Let $\hat{\psi}_m$ and $\breve{\psi}_m$ are subsets of ψ_m, consisting of all its upper and lower boundary vectors, correspondingly: $\hat{\psi}_m$ ($\breve{\psi}_m$) is the set of all “upper” (“lower”) vectors $S \in \psi_m$, for which for all $R \in \Xi_{m+1}^n$ greater than S (less than S), $R \notin \psi_m$.

These sets of all “upper” and “lower” boundary vectors have symmetric structures - for each upper vector there exists a corresponding (opposite) lower vector, and vice versa; so that also the numbers of these vectors are equal:

$$\hat{\psi}_m = \{ \hat{S}_1, \cdots, \hat{S}_r \} \text{ and } \breve{\psi}_m = \{ \breve{S}_1, \cdots, \breve{S}_r \}.$$

Let $\hat{S}_j$ and $\breve{S}_j$ be an arbitrary pair of opposite vectors from $\hat{\psi}_m$ and $\breve{\psi}_m$ $\breve{\psi}_m$ correspondingly. $I(\hat{S}_j)$ (equivalently $I(\breve{S}_j)$) will denote the minimal sub-cube of Ξ_{m+1}^n, passing through this pair of vectors. Then,

$$I(\hat{S}_j) = \{ Q \in \Xi_{m+1}^n \mid \hat{S}_j \le Q \le \breve{S}_j \} \text{ (the coordinate-wise comparison is used).}$$

The following theorem states that the minimal sub-cubes passing the pairs of corresponding opposite vectors of the boundary subsets are continuously and exactly filling the vector area ψ_m.

Theorem 1 [S,1997]: $\psi_m = \bigcup_{j=1}^{r} I(\hat{S}_j)$.

It follows that the description of ψ_m is provided through the set of upper boundary vectors $\hat{\psi}_m = \{ \hat{S}_1, \cdots, \hat{S}_r \}$ (correspondingly, the set of lower boundary vectors $\breve{\psi}_m = \{ \breve{S}_1, \cdots, \breve{S}_r \}$). Let assume that the upper boundary vectors are distributed between the layers L_{min} and L_{max} of Ξ_{m+1}^n. Then for each layer L, $L_{min} \le L \le L_{max}$, it is sufficient to have all upper boundary vectors situated on that layer..

4. Description through the ”Steepest” elements

Let introduce a concept of “steepest” vectors, defined for each layer.

Definition 1 [B,1988]

Let $S=(s_1,s_2,\cdots,s_n)$ and $S'=(s_1',s_2',\cdots,s_n')$ be two vectors of length n with integer, nonnegative components, and let $s_1 \ge s_2 \ge \cdots \ge s_n$ and $s_1' \ge s_2' \ge \cdots \ge s_n'$. S' is an **elementary flattening** of S if and only if S' can be obtained from S by:

(1) finding i,j such that $s_i \ge s_j+2$; and then

(2) transferring 1 from the larger to the smaller; that is, $s_i' = s_i - 1$ and $s_j' = s_j + 1$; and then

(3) re-ordering the resulting sequence so that it is decreasing.

We mention that operation of elementary flattening doesn’t move vector from one layer of Ξ_{m+1}^n to another.

Definition 2 [B,1988]

Let $S=(s_1,s_2,\cdots,s_n)$ and $S'=(s_1',s_2',\cdots,s_n')$ be two vectors of length n with integer, nonnegative components, and let $s_1 \ge s_2 \ge \cdots \ge s_n$ and $s_1' \ge s_2' \ge \cdots \ge s_n'$. S' is **flatter** than S, and S is **steeper** than S', if and only if S' can be obtained from S by a non-empty sequence of elementary flattening.

S is a **steepest** vector if and only if there is no vector in ψ_m, which is steeper than S.

The following theorem is an extension of similar result [B, 1988], which is in terms of hypergraphs and degree sequences:

Theorem 2. If S belongs to ψ_m then all vectors flatter than S also belong to ψ_m.

Proof:

Let $S=(s_1,s_2,\cdots,s_n)\in\psi_m$ and $S'=(s_1',s_2',\cdots,s_n')$ is flatter than S. It follows that there exists a sequence of elementary flattening, which transfers S to S'. We prove that after each elementary flattening, the obtained vector belongs to ψ_m. Let $s_i \ge s_j+2$ and after an elementary flattening we receive the vector $(s_1,\cdots,s_i-1,\cdots,s_j+1,\cdots,s_n)$.

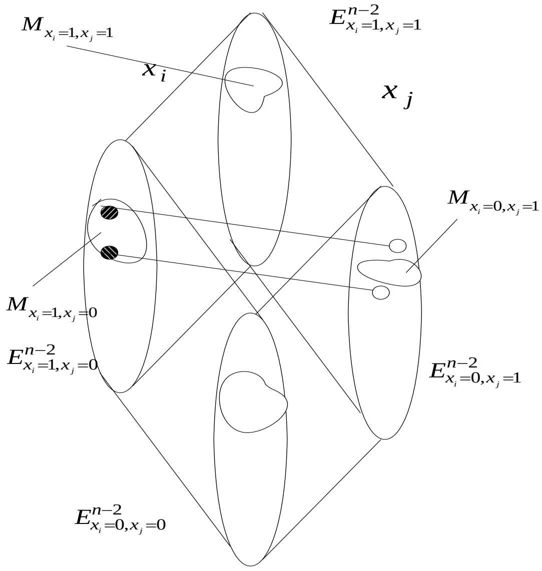
Consider the partitioning of E^n by i th and j th directions. Let $E^{n-2}_{x_i=1,x_j=1}$, $E^{n-2}_{x_i=1,x_j=0}$, $E^{n-2}_{x_i=0,x_j=1}$, $E^{n-2}_{x_i=0,x_j=0}$ be the corresponding sub-cubes, and $M_{x_i=1,x_j=1}$, $M_{x_i=1,x_j=0}$, $M_{x_i=0,x_j=1}$, $M_{x_i=0,x_j=0}$ - the corresponding subsets of M, belonging to these sub-cubes. Then we have: $\left|M_{x_i=1,x_j=1}\right|+\left|M_{x_i=1,x_j=0}\right|=s_i$

$$\left|M_{x_i=1,x_j=1}\right|+\left|M_{x_i=0,x_j=1}\right|=s_j$$

Hence $\left|M_{x_i=1,x_j=0}\right|-\left|M_{x_i=0,x_j=1}\right|\ge 2$

Therefore there exist two vertices in $M_{x_i=1,x_j=0}$ such that the corresponding vertices in $E^{n-2}_{x_i=0,x_j=1}$ don't belong to $M_{x_i=0,x_j=1}$. Moving one of them from $M_{x_i=1,x_j=0}$ to $M_{x_i=0,x_j=1}$, will provide the necessary s_i-1 and s_j+1 values.

The geometrical visualisation is through the following picture:



It follows from the above theorem that the steepest vectors of each layer L, $L_{min} \le L \le L_{max}$ of Ξ^n_{m+1} provide the description of all vectors from ψ_m belonging to that layer.

5. Description through the Boundary Steepest Elements

On one hand, ψ_m can be described through the set of all upper boundary vectors, and on the other hand - through the set of all "steepest" vectors. Below we prove that ψ_m can be described having only the intersection of these two sets – which is the set of all "boundary steepest" vectors.

The theorem below states that if some layer of Ξ^n_{m+1} contains more than one upper boundary vector, then only the steepest ones of them are necessary for the description of ψ_m, or the same – if among the steepest vectors are both boundary and non-boundary, then only the boundary ones are necessary to describe the whole set of partitioning characteristic vectors.

Theorem 2. If a layer of ψ_m contains a boundary vector, then it can be obtained by operations of flattening from only an other boundary vector.

The theorem has been proved by contradiction, considering all possible cases.

Conclusion

Any set system can be represented as a subset of n-cube vertices set. For a given subset it is important to know the partition sizes, - the coordinates of partitioning characteristic vectors. A smaller generating sets are known as "boundary" and "steepest" sets and finally we prove that the intersection of these two sets is also generating for the partitioning characteristic vectors.

Bibliography

[S, 1997] H. Sahakyan. On a class of (0,1)-matrices connected to the subsets partitioning of E^n, Doklady NAS Armenia, v. 97, 2, 1997, pp. 12-16.

[B, 1988] Billington D., Conditions for degree sequences to be realisable by 3-uniform hypergraphs". The Journal of Combinatorial Mathematics and Combinatorial Computing". 3, 1988, pp. 71-91.

[D, 1999] Durr Ch., Chrobak M., Reconstructing hv-convex polyominoes from orthogonal orojections. Information Processing Letters 69 (1999) pp. 283-291.

[R, 1966] H. J. Ryser. Combinatorial Mathematics, 1966.

[H, 1997] G.T. Herman and A. Kuba, editors. Discrete Tomography: Foundations, Algorithms and Applications. Birkhauser, 1999.

[B, 1996] E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani. Reconstructing convex polyominoes from horizontal and vertical projections. Theoret. Comput. Sci., 155:321{347, 1996.

[W, 2001] G.J. Woeginger. The reconstruction of polyominoes from their orthogonal projections. Inform. Process. Lett., 77:225{229, 2001.

Author's Information

Hasmik Sahakyan – Institute for Informatics and Automation Problems, NAS Armenia, P.Sevak St. 1, Yerevan-14, Armenia; e-mail: hasmik@ipia.sci.am

ALGORITHMIC MINIMIZATION OF NON-ZERO ENTRIES IN 0,1-MATRICES

Adriana Toni, Juan Castellanos, Jose Erviti

Abstract: In this paper we present algorithms which work on pairs of 0,1- matrices which multiply again a matrix of zero and one entries. When applied over a pair, the algorithms change the number of non-zero entries present in the matrices, meanwhile their product remains unchanged. We establish the conditions under which the number of 1s decreases. We recursively define as well pairs of matrices which product is a specific matrix and such that by applying on them these algorithms, we minimize the total number of non-zero entries present in both matrices. These matrices may be interpreted as solutions for a well known information retrieval problem, and in this case the number of 1 entries represent the complexity of the retrieve and information update operations.

Keywords: zero-one matrices, analysis of algorithms and problem complexity, data structures, models of computation

Introduction

We introduce some notation and concepts that will be useful from now on.

Let $I^m_{i,j}$ denote the matrix resulting from permuting the i^{th} and j^{th} rows in the identity matrix of dimensions m × m, denoted I^m. For any matrix *M* of dimensions *m × n*, $I^m_{i,j} \times M$ returns the matrix *M* in which rows *i, j* have switched position.

Generally, if $I^m_\sigma = I^m_{i_1,j_1} \times I^m_{i_2 j_2} \times ... \times I^m_{i_k,j_k}$, the effect of the multiplication $I^m_\sigma \times M$ is to switch the position of rows i_k and j_k of *M*, *then* do the same thing with rows i_{k-1} and j_{k-1}, *then* with rows i_{k-2} and j_{k-2}... until finally rows i_1, j_1 have been switched.

Let H be the matrix of dimensions $\frac{n(n+1)}{2} \times n$ *defined by:*

$$H_{ij} = \begin{cases} 1 & l \le j \le l + (i - w_{l-1} - 1) \\ 0 & otherwise \end{cases}$$

where