HYPER-RANDOM PHENOMENA: DEFINITION AND DESCRIPTION

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Abstract: The paper is dedicated to the theory which describes physical phenomena in non-constant statistical conditions. The theory is a new direction in probability theory and mathematical statistics that gives new possibilities for presentation of physical world by hyper-random models. These models take into consideration the changing of object's properties, as well as uncertainty of statistical conditions.

Keywords: uncertainty, random, hyper-random, phenomenon, probability, statistics.

Introduction

The most of physical phenomena (electrical, electromagnetic, mechanical, acoustics, nuclear, and others) are an indeterminate type. Usually, different stochastic methods are used to describe them. However, possibilities of such methods are limited. There are serious problems, when the observation conditions are changed in space or time and it is impossible to determine the statistical regularity, even by a large experimental sample size.

The changed conditions are met everywhere. It is impossible to image any real event, value, process or field in absolutely fixed conditions. All mass measures are led in the variable conditions, controlled only partly.

The fixed statistical condition and the probability measure are linked together. When it is said the fixed (constant) condition about it is meant that there is the probability measure for every samples of the researched set.

When a physical phenomenon is observed in more or less invariable condition, there is possibility to achieve the statistical regular results. However, if the condition is changed in wide bounds, the statistical estimates are not stable and it is impossible to obtain probable estimates.

To image a depth of the problem, let us apply to well known classic task with tossing a coin. The stability of head or tail (A or B) essentially depends from the style of tossing [1]. In a fixed statistical condition there are stable event frequencies $p_N(A)$, $p_N(B)$, which tend to any probabilities $P(A)$, $P(B)$, when the number of experiments N is tend to infinity. In case of variable condition, the frequencies $p_N(A)$ and $p_N(B)$ are continuously changed. They oscillate in any intervals and not tend to any fixed probabilities.

The condition stability plays the important role in the probability theory that marked by a number of scientists, beginning from Jakob Bernoulli [2]. R. von Mises proposed even to define [3] the probability conception on the base of the event frequency in fixed condition.

It is not simple to determine correctly a probability measure for real physical phenomenon. This fact was marked in many works, for instance, in the article [4].

Difficulties and often impossibility to use the probability theory stimulate the developing of new theories, such as fuzzy logic [5], neural network [6], chaotic dynamical systems [7], and others. The new theory of hyper-random phenomena, the bases of which are presented below, may be included to this list.

The aim of the paper is to review the original author's researches published in articles [8 – 15] and generalized in the monograph [16].

The theory is oriented to description of different type uncertainty, as a contingency, when the probability measure exists, as another one, when the probability measure does not exist.

In modern mathematics, the random phenomena are defined by the probability field that assigned by the triad ($\Omega, \Im, P$), where Ω represents the set of the simple events $\omega \in \Omega$, $\Im$ – the Borel field, P – the probability measure of the subsets.

The hyper-random phenomena may be defined by the tetrad $(\Omega, \Im, G, P_g)$ [9], where Ω and $\Im$ are the set of simple events and the Borel field (as in the case of the probability field), G is the set of the conditions $g \in G$, and P_g – the probability distributions for the condition g.

Any hyper-random phenomena (events, variables, functions) may be regarded as a set (family) of random subsets. In this construction, every subset is associated with any fixed observation condition. The probability measures are determined for the elements of each subset; however, the measures are not determined for the subsets of the set.

Hyper-random Events and Variables

The hyper-random event A from the Borel field $\Im$ cannot be described by any probability. However, the event A/g under the condition $g \in G$ may be presented by the probability $P_g(A) = P(A/g)$. This probability oscillates when condition is changed. The range of the oscillation may be described by the supremum $P_S(A)$ and the infimum $P_I(A)$ of the event probability defined as

$$P_S(A) = \sup_{g \in G} P(A/g), \quad P_I(A) = \inf_{g \in G} P(A/g).$$

In the constant condition ($g = \text{const}$) these bounds are congruent and the hyper-random event degenerates to the random one with the probability $P(A) = P_S(A) = P_I(A)$.

The bounds $P_S(A)$, $P_I(A)$ are half-measures. There has been obtained the expressions that are similar to the formulas, describing the product and the addition rules, the Bayes' and other theorems of the probability theory.

To describe the scalar hyper-random variable X a number of the characteristics have been proposed. They are similar to the probability characteristics of a random variable. The main of them are the supremum $F_S(x)$ and the infimum $F_I(x)$ bounds of the distribution function and also the probability density functions $f_S(x)$ and $f_I(x)$ of these bounds. They are determined by the following expressions:

$$F_S(x) = \sup_{g \in G} P\left\{X \le x/g\right\}, \quad F_I(x) = \inf_{g \in G} P\left\{X \le x/g\right\},$$

$$f_S(x) = \frac{dF_S(x)}{dx}, \qquad f_I(x) = \frac{dF_I(x)}{dx},$$

where $P\left\{X \le x/g\right\}$ is the probability of the inequality $X \le x$ for the condition g.

It has been found that the bounds of the distribution function and the probability density functions of the bounds for hyper-random variable have the same particularities as according characteristics for a random variable and in addition $F_S(x) \ge F_I(x)$.

Among the bounds of the distribution function there is a zone of the ambiguity (fig. 1). For a random variable X its width $\Delta F(x) = F_S(x) - F_I(x)$ equals to zero for all x. If the supremum $F_S(x)$ of the distribution function is tend for all x to unit and the infimum $F_I(x)$ – to zero, zone of the ambiguity is tend to maximum. In this case, the hyper-random variable approaches to a chaos one.

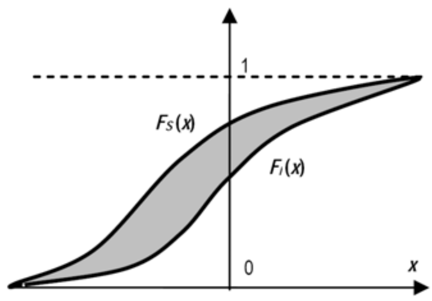

Fig. 1. The bounds of the distribution function and the zone of the ambiguity (the black-out part)

To describe the hyper-random variable, the characteristics similar to random variable ones, may be used. They are the bound's crude and the central moments determined for the hyper-random variable X on the base of the bound's expectation $\mathrm{M}_S\left[\varphi(X)\right]$, $\mathrm{M}_I\left[\varphi(X)\right]$ of the function $\varphi(X)$:

$$\mathrm{M}_S\left[\varphi(X)\right] = \int_{-\infty}^{\infty} \varphi(x) f_S(x) \mathrm{d}x, \quad \mathrm{M}_I\left[\varphi(X)\right] = \int_{-\infty}^{\infty} \varphi(x) f_I(x) \mathrm{d}x.$$

In particular, the bound's means m_{Sx}, m_{Ix} are $m_{Sx} = \mathrm{M}_S\left[X\right]$, $m_{Ix} = \mathrm{M}_I\left[X\right]$. For the real hyper-random variable X the bound's variance D_{Sx}, D_{Ix} are $D_{Sx} = \mathrm{M}_S\left[(X - m_{Sx})^2\right]$, $D_{Ix} = \mathrm{M}_I\left[(X - m_{Ix})^2\right]$. The bound's crude moments are determined by the expressions $m_{Sx\nu} = \mathrm{M}_S\left[X^\nu\right]$, $m_{Ix\nu} = \mathrm{M}_I\left[X^\nu\right]$ and the bound's central moments – by the ones $\mu_{Sx\nu} = \mathrm{M}_S\left[(X - m_{Sx})^\nu\right]$, $\mu_{Ix\nu} = \mathrm{M}_I\left[(X - m_{Ix})^\nu\right]$, where ν is the order of the moment.

To describe the hyper-random variables other type characteristics may be used too. They are supremum and infimum of the crude moments and the same bounds of the central moments. These characteristics are determined on the base of the expectation of the function $\varphi(X/g)$:

$$\mathrm{M}_s[\varphi(X/g)] = \sup_{g\in G}\int_{-\infty}^{\infty}\varphi(x/g)f(x/g)\mathrm{d}x, \quad \mathrm{M}_i[\varphi(X/g)] = \inf_{g\in G}\int_{-\infty}^{\infty}\varphi(x/g)f(x/g)\mathrm{d}x,$$

where $f(x/g)$ is the probability density function in condition g.

In particular, for the hyper-random variable X the supremum m_{sx} and the infimum m_{ix} of the mean are $m_{sx} = \mathrm{M}_s[X]$, $m_{ix} = \mathrm{M}_i[X]$, and the supremum D_{sx} and the infimum D_{ix} of the variance are $D_{sx} = \mathrm{M}_s[(X - m_{x/g})^2]$, $D_{ix} = \mathrm{M}_i[(X - m_{x/g})^2]$, where $m_{x/g}$ represents the mean of the random variable X/g. The crude moment's bounds $m_{sx\nu}$ and $m_{ix\nu}$ are described by the expressions $m_{sx\nu} = \mathrm{M}_s\left[X^\nu\right]$, $m_{ix\nu} = \mathrm{M}_i\left[X^\nu\right]$, and the central moment's bounds $\mu_{sx\nu}$ and $\mu_{ix\nu}$ – by the following ones $\mu_{sx\nu} = \mathrm{M}_s[(X - m_{x/g})^\nu]$, $\mu_{ix\nu} = \mathrm{M}_i[(X - m_{x/g})^\nu]$.

In general, the operators $\mathrm{M}_S\left[\cdot\right]$, $\mathrm{M}_I\left[\cdot\right]$ differ from the operators $\mathrm{M}_s\left[\cdot\right]$, $\mathrm{M}_i\left[\cdot\right]$ and the bound's moments differ from the moment's bounds, although in some particular cases they may be expressed by each other, for instance, when the distribution functions $F(x/g)$ for different conditions g have not interception points. Then, if the variance $D_{x/g}$ is raised with raising the mean $m_{x/g}$ ("a" type distribution) there are the following equalities: $m_{Sx} = m_{ix}$, $m_{Ix} = m_{sx}$, $D_{Sx} = D_{ix}$, $D_{Ix} = D_{sx}$; if the variance $D_{x/g}$ is reduced with raising the mean $m_{x/g}$ ("b" type distribution) there are the equalities: $m_{Sx} = m_{ix}$, $m_{Ix} = m_{sx}$, $D_{Sx} = D_{sx}$, $D_{Ix} = D_{ix}$.

The results were generalized to complex $\dot{X}$ and vector $\vec{X}$ hyper-random variables, to real $X(t)$, complex $\dot{X}(t)$, and vector $\vec{X}(t)$ functions.

Hyper-random Functions

The scalar hyper-random process $X(t)$ has been presented as a family of the random processes $X(t)/g$ determined for a set conditions $g \in G$. The process described by the supremum $F_S(\vec{x};\vec{t})$ and the infimum $F_I(\vec{x};\vec{t})$ of the distribution function, probability density functions $f_S(\vec{x};\vec{t})$, $f_I(\vec{x};\vec{t})$ of these bounds, the bound's moments $m_{Sx\vec{\nu}}(\vec{t})$, $m_{Ix\vec{\nu}}(\vec{t})$, $\mu_{Sx\vec{\nu}}(\vec{t})$, $\mu_{Ix\vec{\nu}}(\vec{t})$ and the moment's bounds $m_{sx\vec{\nu}}(\vec{t})$, $m_{ix\vec{\nu}}(\vec{t})$, $\mu_{sx\vec{\nu}}(\vec{t})$, $\mu_{ix\vec{\nu}}(\vec{t})$, where $\vec{\nu} = (\nu_1, \ldots, \nu_L)$ is the order vector of the moment, L is the measure of the distribution. These characteristics are described by expressions that are similar to ones for hyper-random variable:

$$F_S(\vec{x};\vec{t}) = \sup_{g\in G} P\{X(t_1) \le x_1, ..., X(t_L) \le x_L / g\}, \quad F_I(\vec{x};\vec{t}) = \inf_{g\in G} P\{X(t_1) \le x_1, ..., X(t_L) \le x_L / g\},$$

$$f_S(\vec{x};\vec{t}) = \frac{\partial^L F_S(\vec{x};\vec{t})}{\partial x_1 ... \partial x_L}, \quad f_I(\vec{x};\vec{t}) = \frac{\partial^L F_I(\vec{x};\vec{t})}{\partial x_1 ... \partial x_L},$$

$$m_{Sx}(t) = \mathrm{M}_S[X(t)] = \int_{-\infty}^{\infty} x f_S(x;t)dx, \quad m_{Ix}(t) = \mathrm{M}_I[X(t)] = \int_{-\infty}^{\infty} x f_I(x;t)dx,$$

$$m_{Sx\nu_1\ldots\nu_L}(t_1,\ldots,t_L) = \mathrm{M}_S[X^{\nu_1}(t_1)\ldots X^{\nu_L}(t_L)] = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} x_1^{\nu_1}\ldots x_L^{\nu_L} f_S(x_1,\ldots,x_L;t_1,\ldots,t_L)dx_1\ldots dx_L,$$

$$m_{Ix\nu_1\ldots\nu_L}(t_1,\ldots,t_L) = \mathrm{M}_I[X^{\nu_1}(t_1)\ldots X^{\nu_L}(t_L)] = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} x_1^{\nu_1}\ldots x_L^{\nu_L} f_I(x_1,\ldots,x_L;t_1,\ldots,t_L)dx_1\ldots dx_L,$$

$$\mu_{Sx\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_S[(X(t_1) - m_{Sx}(t_1))^{\nu_1}...(X(t_L) - m_{Sx}(t_L))^{\nu_L}],$$

$$\mu_{Ix\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_I[(X(t_1) - m_{Ix}(t_1))^{\nu_1}...(X(t_L) - m_{Ix}(t_L))^{\nu_L}],$$

$$m_{sx}(t) = \mathrm{M}_s[X(t)],\ m_{ix}(t) = \mathrm{M}_i[X(t)],$$

$$m_{sx\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_s[X^{\nu_1}(t_1)...X^{\nu_L}(t_L)], \quad m_{ix\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_i[X^{\nu_1}(t_1)...X^{\nu_L}(t_L)],$$

$$\mu_{sx\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_s[(X(t_1) - m_{x/g}(t_1))^{\nu_1}...(X(t_L) - m_{x/g}(t_L))^{\nu_L}],$$

$$\mu_{ix\nu_1...\nu_L}(t_1,...,t_L) = \mathrm{M}_i[(X(t_1) - m_{x/g}(t_1))^{\nu_1}...(X(t_L) - m_{x/g}(t_L))^{\nu_L}].$$

The bound's correlation functions and the bound's covariance functions are

$$K_{Sx}(t_1,t_2) = \mathrm{M}_S[X(t_1)X(t_2)], \quad K_{Ix}(t_1,t_2) = \mathrm{M}_I[X(t_1)X(t_2)],$$

$$R_{Sx}(t_1,t_2) = \mathrm{M}_S[(X(t_1) - m_{Sx}(t_1))(X(t_2) - m_{Sx}(t_2))],$$

$$R_{Ix}(t_1,t_2) = \mathrm{M}_I[(X(t_1) - m_{Ix}(t_1))(X(t_2) - m_{Ix}(t_2))]$$

and the correlation function's bounds and covariance function's bounds are

$$K_{sx}(t_1,t_2) = \mathrm{M}_s[X(t_1)X(t_2)], \quad K_{ix}(t_1,t_2) = \mathrm{M}_i[X(t_1)X(t_2)],$$

$$R_{sx}(t_1,t_2) = \mathrm{M}_s[(X(t_1) - m_{x/g}(t_1))(X(t_2) - m_{x/g}(t_2))],$$

$$R_{ix}(t_1,t_2) = \mathrm{M}_i[(X(t_1) - m_{x/g}(t_1))(X(t_2) - m_{x/g}(t_2))].$$

Stationary and Ergodic Hyper-random Functions

It has been found that some hyper-random functions have special stationary and ergodic properties. A function $X(t)$ has been called a stationary hyper-random one if the bound's mean do not depend from time and bound's correlation functions depend only from time interval $\tau = t_2 - t_1$: $K_{Sx}(t_1,t_2) = K_{Sx}(\tau)$, $K_{Ix}(t_1,t_2) = K_{Ix}(\tau)$.

A function $X(t)$ has been called stationary hyper-random one for all conditions if the mean $m_{x/g}(t) = \int_{-\infty}^{\infty} x f(x;t/g)\mathrm{d}x$ does not depend from time t ($m_{x/g}(t) = m_{x/g}$) and the correlation function

$$K_{x/g}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1,x_2;t_1,t_2/g)\mathrm{d}x_1\mathrm{d}x_2$$

depends only from the interval τ and the condition g: $K_{x/g}(t_1,t_2) = K_{x/g}(\tau)$.

The bound's correlation functions $K_{Sx}(\tau)$, $K_{Ix}(\tau)$ are determined by bound's spectral density $S_{Sxx}(f)$, $S_{Ixx}(f)$ that linked each other by the following expressions:

$$S_{Sxx}(f) = \int_{-\infty}^{\infty} K_{Sx}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau,\ S_{Ixx}(f) = \int_{-\infty}^{\infty} K_{Ix}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau,$$

$$K_{Sx}(\tau)=\int_{-\infty}^{\infty} S_{Sxx}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f,\ K_{Ix}(\tau)=\int_{-\infty}^{\infty} S_{Ixx}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f,$$

where f is a frequency.

The spectral density's bounds are determined by expressions $S_{sxx}(f)=\sup_{g\in G} S_{xx/g}(f)$, $S_{ixx}(f)=\inf_{g\in G} S_{xx/g}(f)$, where $S_{xx/g}(f)$ is the spectral density for condition g:

$$S_{xx/g}(f)=\int_{-\infty}^{\infty} K_{x/g}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau,\ K_{x/g}(\tau)=\int_{-\infty}^{\infty} S_{xx/g}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f.$$

For two hyper-random functions $X(t)$, $Y(t)$ stationary linked each other the bound's correlation functions are determined by the following expressions:

$$K_{Sxy}(\tau)=\int_{-\infty}^{\infty} \dot{S}_{Sxy}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f,\ K_{Ixy}(\tau)=\int_{-\infty}^{\infty} \dot{S}_{Ixy}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f,$$

where $\dot{S}_{Sxy}(f)$, $\dot{S}_{Ixy}(f)$ are the bound's spectral density: $\dot{S}_{Sxy}(f)=\int_{-\infty}^{\infty} K_{Sxy}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau$, $\dot{S}_{Ixy}(f)=\int_{-\infty}^{\infty} K_{Ixy}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau$.

The spectral density's bounds are $\dot{S}_{sxy}(f)=\sup_{g\in G}\dot{S}_{xy/g}(f)$, $\dot{S}_{ixy}(f)=\inf_{g\in G}\dot{S}_{xy/g}(f)$, where $\dot{S}_{xy/g}(f)$ is the spectral density for condition g: $\dot{S}_{xy/g}(f)=\int_{-\infty}^{\infty} K_{xy/g}(\tau)\exp(-\mathrm{j}2\pi f\tau)\mathrm{d}\tau$, $K_{xy/g}(\tau)=\int_{-\infty}^{\infty} \dot{S}_{xy/g}(f)\exp(\mathrm{j}2\pi f\tau)\mathrm{d}f$.

It has been determined the particularities of these characteristics and introduced a number of new conceptions, in particular hyper-random white noise.

Some hyper-random function $X(t)$ may be presented as a set of the random functions determined on the disjoint intervals $T_g=[Tg,T(g+1))$ with longitude T on that the conditions are not changed ($g=0,\pm1,\pm2,\dots$). Let $X_g(t)$ is the part of the function $X(t)$ according to interval T_g and reduced to interval $[-T/2,T/2)$:

$$X_g(t-T(g+0,5))=\begin{cases} X(t), & \text{if} \quad t\in T_g,\\ 0, & \text{if} \quad t\notin T_g.\end{cases}$$

The function $X_g(t)$ in a fixed condition $g=0,\pm1,\pm2,\dots$ is the random function determined on the interval $t\in[-T/2,T/2)$. The set of these functions in uncertainty conditions is a hyper-random function $Y(t)=\{X_g(t),\quad g=0,\pm1,\dots\}$. A hyper-random function is any function $\varphi(Y(t_1),...,Y(t_L))$ too, where $t_1,\dots,t_L\in[-T/2,T/2)$.

A hyper-random function $X(t)$, that is stationary for all conditions and $\lim_{T\to\infty}\bar{m}_\varphi(T)=m_\varphi$, has been called an ergodic one. Here $\bar{m}_\varphi(T)$ is the sample mean:

$$\bar{m}_\varphi(T)=\bar{\mathrm{M}}_T[\varphi(Y(t_1),...,Y(t_L))]=\frac{1}{T}\int_{-T/2}^{T/2}\varphi(Y(t_1+t),...,Y(t_L+t))\mathrm{d}t$$

and $m_\varphi = \mathrm{M}[\varphi(Y(t_1),...,Y(t_L))]$ is the mean of the function $\varphi(Y(t_1),...,Y(t_L))$.

A hyper-random ergodic function $X(t)$ may be presented by the following serious: $X(t) = \lim_{T\to\infty} \sum_g X_g(t - T(g+0,5))$.

When $T \to \infty$ the mean's bounds and the correlation and the covariance function's bounds are described by the following expressions:

$$\bar{m}_{sx_T} = \sup_{g\in G} \frac{1}{T} \int_{T_g} x_g(t)\,\mathrm{d}t,\ \bar{m}_{ix_T} = \inf_{g\in G} \frac{1}{T} \int_{T_g} x_g(t)\,\mathrm{d}t,$$

$$\bar{K}_{sx_T}(\tau) = \sup_{g\in G} \frac{1}{T} \int_{T_g} x_g(t+\tau)\, x_g(t)\mathrm{d}t,\ \bar{K}_{ix_T}(\tau) = \inf_{g\in G} \frac{1}{T} \int_{T_g} x_g(t+\tau)\, x_g(t)\mathrm{d}t,$$

$$\bar{R}_{sx_T}(\tau) = \sup_{g} \frac{1}{T} \int_{T_g} [x_g(t+\tau) - \bar{m}_{x_T/g}][x_g(t) - \bar{m}_{x_T/g}]\mathrm{d}t,$$

$$\bar{R}_{ix_T}(\tau) = \inf_{g} \frac{1}{T} \int_{T_g} [x_g(t+\tau) - \bar{m}_{x_T/g}][x_g(t) - \bar{m}_{x_T/g}]\mathrm{d}t,$$

where $\bar{m}_{x_T/g} = \frac{1}{T} \int_{T_g} x_g(t)\,\mathrm{d}t$.

Hyper-random Models

Developed approaches give possibilities to model different types of real physical objects and their estimates under uncertainty changing of object's properties and statistical observation conditions. It has been proposed different measure models: determine – hyper-random, random – hyper-random, and hyper-random – hyper-random ones, in that the objects are presented by determine, random, and hyper-random models and their estimates – by hyper-random models.

In case of determine – hyper-random measure model, in the fixed condition g the accuracy of vector estimation $\vec{\Theta}^*$ of parameter $\vec{\theta}$ may be described by the expectation of error's square $\Delta_g^2 = \mathrm{M}[\left|\vec{\Theta}^* - \vec{\theta}\right|^2 / \vec{\theta}, g]$, where M is expectation operator. For the indefinite condition the accuracy is characterized by the interval where the value Δ_g^2 may be situated. The bounds of this interval are $\Delta_{\min}^2 = \min[\Delta_S^2, \Delta_I^2]$, $\Delta_{\max}^2 = \max[\Delta_S^2, \Delta_I^2]$ where $\Delta_S^2 = \mathrm{M}_S[\left|\vec{\Theta}^* - \vec{\theta}\right|^2 / \vec{\theta}]$, $\Delta_I^2 = \mathrm{M}_I[\left|\vec{\Theta}^* - \vec{\theta}\right|^2 / \vec{\theta}]$ are the bound's quadratic estimate.

The accuracy of point estimation may be characterized by bounds of quadratic estimate:

$$\Delta_s^2 = \sup_{g\in G} \mathrm{M}[\left|\vec{\Theta}^* - \vec{\theta}\right|^2 / \vec{\theta}, g],\quad \Delta_i^2 = \inf_{g\in G} \mathrm{M}[\left|\vec{\Theta}^* - \vec{\theta}\right|^2 / \vec{\theta}, g].$$

In scalar case the volumes Δ_S^2, Δ_I^2 and Δ_s^2, Δ_i^2 may be presented as $\Delta_S^2 = \sigma_S^2 + \varepsilon_{S0}^2$, $\Delta_I^2 = \sigma_I^2 + \varepsilon_{I0}^2$ and $\Delta_s^2 = \sup_{g\in G}[\sigma_g^2 + \varepsilon_{0/g}^2]$, $\Delta_i^2 = \inf_{g\in G}[\sigma_g^2 + \varepsilon_{0/g}^2]$ where $\sigma_S^2 = \mathrm{M}_S\left[\left(\Theta^* - m_S\right)^2 / \theta\right]$, $\sigma_I^2 = \mathrm{M}_I\left[\left(\Theta^* - m_I\right)^2 / \theta\right]$ are the variances of error bounds, $\sigma_g^2 = \mathrm{M}\left[\left(\Theta^* - m_{\theta^*/g}\right)^2 / \theta, g\right]$ is the error variances for condition g, $\varepsilon_{S0} = (m_S - \theta)$, $\varepsilon_{I0} = (m_I - \theta)$ are the systematic errors for estimation distribution bounds, and $\varepsilon_{0/g} = (m_{\theta^*/g} - \theta)$ is the systematic error for condition g (fig. 2).

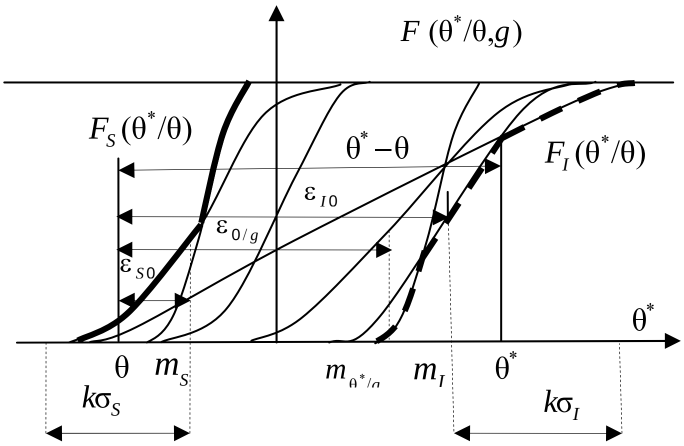

Fig 2. The fan of distribution functions $F\ (\theta^*/\theta,g)$ (thin curves) for different conditions g and supremum $F_S(\theta^*/\theta)$ and the infimum $F_I(\theta^*/\theta)$ bounds of the distribution function (bold curves)

To characterize the error $\Delta = \Theta^* - \theta$ of scalar parameter θ and it's estimate Θ^* the intervals $[\varepsilon_{S0} - k\sigma_S,\ \varepsilon_{I0} + k\sigma_I]$ and $[m_S - k\sigma_S,\ m_I + k\sigma_I]$ may be used correspondently, where σ_S, σ_I are the bound's error standard deviation, and k is the constant (fig. 2). If conditional distributions of random values $\Theta^*/\theta, g$ are not penetrates and the variance $D_{x/g}$ is raised or reduced with rising the mean $m_{x/g}$, the last interval is determined by error mean's bounds m_s, m_i and error standard deviation's bounds σ_s, σ_i. For "a" type distribution it may be presented as $[m_i - k\sigma_i,\ \ m_s + k\sigma_s]$ and for "b" type distribution – as $[m_i - k\sigma_s,\ \ m_s + k\sigma_i]$.

A hyper-random estimate $\vec{\Theta}^*$ of fixed parameter $\vec{\theta}$ was called consistent one if it converged in probability to this parameter under all conditions $g \in G$: $\lim_{N\to\infty} P\{|\vec{\Theta}^* - \vec{\theta}| > \varepsilon / \vec{\theta}, g\} = 0 \quad \forall g \in G$, where N is a sample size for every condition g and $\varepsilon > 0$.

The necessary condition, that the hyper-random estimate is a consistent type, is that it degenerates to random estimate when $N \to \infty$. So, estimates are not consistent if they stay hyper-random type when $N \to \infty$.

It was made a hypothesis (hyper-random hypothesis) that all real physical phenomena are existed in continuously changed statistic conditions and therefore all physical phenomena, usually considered as a random type, really are the hyper-random type. This particularity exists not only in case of finite but infinite interval observation. It is followed from this that all real estimates are not consistent and it is impossible to achieve infinitely large accuracy in any conditions.

The bounds of error's square expectation Δ_s^2, Δ_i^2 formed on the base of sample $\vec{X}$ size N and bounds $\mathrm{D}_s\lfloor \Theta^*/\theta \rfloor$, $\mathrm{D}_i\lfloor \Theta^*/\theta \rfloor$ of estimate's variance $\mathrm{D}\lfloor \Theta^*/\theta,g \rfloor$ are described by the inequalities

$$\Delta_s^2 \ge \mathrm{D}_s\lfloor \Theta^*/\theta \rfloor \ge \sup_{g\in G}\left[\left(1+\frac{\partial \varepsilon_{0/g}}{\partial \theta}\right)^2 J_{N/g}^{-1}\right], \qquad \Delta_i^2 \ge \mathrm{D}_i\lfloor \Theta^*/\theta \rfloor \ge \inf_{g\in G}\left[\left(1+\frac{\partial \varepsilon_{0/g}}{\partial \theta}\right)^2 J_{N/g}^{-1}\right],$$

where $J_{N/g}$ – Fisher intrinsic accuracy for random value $\Theta^*/\theta,g$:

$$J_{N/g} = \mathrm{M}\left[\left(\frac{\partial \ln f_N(\vec{X}/\theta,g)}{\partial \theta}\right)^2\right] = -\mathrm{M}\left[\frac{\partial^2 \ln f_N(\vec{X}/\theta,g)}{\partial \theta^2}\right],$$

$f_N(\vec{x}/\theta, g)$ – probability density function of sample $\vec{X}/\theta, g$.

The bound's quadratic estimate Δ_S^2, Δ_I^2 and the bound's variance $D_S\lfloor\Theta^*/\theta\rfloor$, $D_I\lfloor\Theta^*/\theta\rfloor$ are defined by inequalities

$$\Delta_S^2 \geq D_S\lfloor\Theta^*/\theta\rfloor \geq \frac{\left(1+\frac{\partial\varepsilon_{S0}}{\partial\theta}\right)^2}{M_S\left[\left(\frac{\partial\ln f_{SN}(\vec{X}/\theta)}{\partial\theta}\right)^2\right]}, \qquad \Delta_I^2 \geq D_I\lfloor\Theta^*/\theta\rfloor \geq \frac{\left(1+\frac{\partial\varepsilon_{I0}}{\partial\theta}\right)^2}{M_I\left[\left(\frac{\partial\ln f_{IN}(\vec{X}/\theta)}{\partial\theta}\right)^2\right]},$$

were $f_{SN}(\vec{x}/\theta)$, $f_{IN}(\vec{x}/\theta)$ are bound's probability density function of sample $\vec{X}/\theta$.

Analogues results were obtained for hyper-random – hyper-random measure model too.

On the base of hyper-random hypothesis was shown that in any case accuracy of any real physical measurements is limited, all real estimates are not consistent ones, and therefore all real physical phenomena are hyper-random type.

Processing of Hyper-random Signals

Developed body of mathematics may be effectively used for signal processing. The example illustrating such possibilities presents below.

Let us look the measure process of the level noise in the production area when there is a lot of production equipment which time to time switch on and switch off and therefore the noise condition is changed in widely boundaries. The measurement is done on the basis of the data obtained for a long time.

This task may be concretized by different manner. If the noise in fixed condition and the rule of changing condition may be regarded as random processes, the task becomes a classic one that consists of estimation of a random variable or some random variables. To solve this task it is requested to know the distribution functions type or at least have information that such distributions exist.

If it is impossible to propose, the changing conditions may be described by any distribution, the task is a hyper-random type. In this case, the recorded data is a sample from a general population of the hyper-random function $X(t)$.

By the processing of this data it is possible to obtain estimates of different characteristics. The image of the recorded data and estimates of some characteristics give the fig. 3 – 4. It has been proposed that the process is an ergodic type.

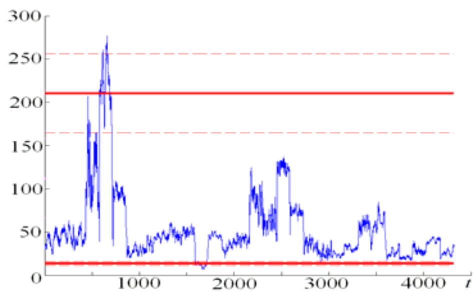

Fig. 3. Current noise level in the production area (solid line), estimates of the bounds of the expected values m_{sx}^*, m_{ix}^* (straight solid bold lines), and the bounds of the standard deviations $m_{sx}^* \pm \sigma_{sx}^*$, $m_{ix}^* \pm \sigma_{ix}^*$ (dashed lines).

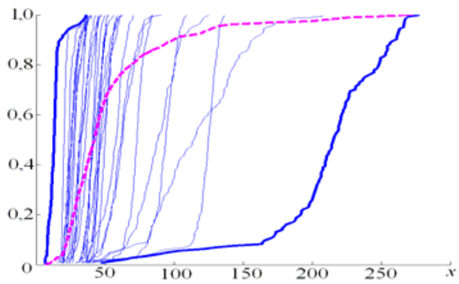

Fig. 4. The estimates of the distribution functions $F^*(x/g)$ (solid lines), the estimates of the bounds of the distribution function $F_S^*(x)$, $F_I^*(x)$ (solid bold lines), and the estimate of the distribution function $F^*(x)$ calculated in the hypothesis that the data are random type (bold dashed line).

It is followed from the figures that presented parameters and functions give a lot of useful information that is essentially more informative than characteristics usually used for describing of random processes.

Conclusion

1. Any hyper-random phenomena (events, variables, functions) may be regarded as a set (family) of random subsets. In this construction, every subset is associated with any fixed observation condition. The probability measures are determined for the elements of each subset; however, the measures are not determined for the subsets of the set.

2. Hyper-random variables and functions may be described by the supremum and the infimum bounds of the distribution function. Among the bounds of the distribution function there is a zone of the ambiguity. Random and chaotic phenomena are the degenerate hyper-random phenomena.

3. In addition to the bounds of the distribution function, the main characteristics describe hyper-random variables and functions are bound's crude and the central moments and also crude and the central moment's bounds. They are, in particular, bound's mean, bound's variance and also mean's bounds variance's bounds and so on.

4. Estimations of hyper-random variables and functions were researched. It was paid attention to all real statistical conditions were continuously changed. Therefore all real physical phenomena usually regard as random tapes, in really, are hyper-random tapes. This particularity occurs not only in case of finite but also in case of infinite interval observations. It is follows from this that all estimations of real variables and functions are not consistent and so it is impossible to achieve infinite physical measurement accuracy in any real conditions.

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