SOME PROPERTIES IN MULTIDIMENSIONAL MULTIVALUED DISCRETE TORUS

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Abstract: *Current research concerns the following issues: n-dimensional discrete torus generated by cycles of even length is considered; the concept of standard arrangement in the torus is defined and some basic properties of this arrangement are investigated. The issues considered are similar to discrete isoperimetry constructions, being related to concept of neighbourhood in terms of linear arrangements of vertices. Considered are the basic properties of solving the discrete isoperimetry problem on torus.*

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Introduction

Isoperimetric problems appeared in the variational calculus of Euler and Lagrange as a classic mathematical issue. The similar considerations in discrete spaces are completely different in research technologies which can be easily seen by early works on isoperimetry [1-6]. The difference and the problem novelty appear on the boundary of the subset considered to be isoperimetric. First results were delivered in terms of linearization of domain elements. After this studies appeared example problems that exempt this property. And appeared one more model, - with cylindrical coordinates like the torus [5-6]. In which extend the rules of [1, 3] are extendable to this domain? This is the main topic of study of current investigation.

Basic Definitions

For any integers $1 \le k_1 \le k_2 \le \cdots \le k_n < \infty$ the multivalued n-dimensional torus $T^n_{k_1 k_2 \cdots k_n}$ has been defined as the set of vertices: $T^n_{k_1 k_2 \cdots k_n} = \{(x_1, x_2, \cdots, x_n) / \ -k_i + 1 \le x_i \le k_i,\ x_i \in Z, 1 \le i \le n\}$, where two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ of $T^n_{k_1 k_2 \cdots k_n}$ are considered neighbours, if they differ by exactly one coordinate for which either $| x_i - y_i | = 1$; or the values equal $-k_i + 1$ and k_i respectively. The sum and difference of these vectors has been defined in the following way: $z = x \pm y = (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n) = (z_1, z_2, \cdots, z_n)$, where $-k_i + 1 \le z_i \le k_i$ and $z_i \equiv (x_i \pm y_i)(\mathrm{mod}\ 2k_i)$.

We will consider discrete isoperimetric problem for the torus. First let us define the concept of interior and boundary vertices for subsets of $T^n_{k_1k_2\cdots k_n}$.

Definition 1 For a given subset $A \subseteq T^n_{k_1k_2\cdots k_n}$ we say that a vertex $x \in A$ is an interior point of A, if all its neighbouring vertices belong to A. Otherwise $x \in A$ is called a boundary vertex of A. We denote by $B(A)$ and $\Gamma(A)$, respectively, the subset of all interior and boundary points of A.

Discrete isoperimetric problem

Given an integer a, $0 \le a \le |T^n_{k_1k_2\cdots k_n}|$. Determine a subset $A \subseteq T^n_{k_1k_2\cdots k_n}$, $|A| = a$, that have the largest number of interior points among all subsets of size a:

$$|B(A)| = \max_{\substack{A' \subseteq T_{k_1k_2\cdots k_n} \\ |A'|=a}} |B(A')|.$$

Sets, being the solution of the discrete isoperimetric problem, we call optimal.

In case when $k_1 = k_2 = \cdots = k_n = 1$, $T^n_{k_1k_2\cdots k_n}$ becomes n-dimensional unit cube E^n; the solution of the discrete isoperimetric problem in E^n is given in [1-4]. In case of $k_1 = k_2 = \cdots = k_n = \infty$, the solution is given in [5]. Notice that in [4,5] the subset of boundary vertices of $A \subseteq T^n_{k_1k_2\cdots k_n}$ is defined as the set of vertices from $T^n_{k_1k_2\cdots k_n} \setminus A$, that have at least one neighbouring vertex from A.

We denote by $\|x\|$ the norm of a vertex $x = (x_1, x_2, \ldots, x_n)$ where $\|x\| = \sum_{i=1}^{n} |x_i|$, and denote by $\rho(x,y)$ the distance between the vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ where $\rho(x,y) = \|x - y\|$.

Now we define the concepts of sphere and envelope with a given centre and radius.

Definition 2 The set $S^n(x,k) = \{\, y \in T^n_{k_1k_2\cdots k_n} \,/\, \rho(x,y) \le k \,\}$ is called a sphere with the centre $x \in T^n_{k_1k_2\cdots k_n}$ and radius k, and the set $O^n(x,k) = \{\, y \in T^n_{k_1k_2\cdots k_n} \,/\, \rho(x,y) = k \,\}$ is the envelope with centre x and radius k.

Let $e_i = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the unit vector of i-th direction, where $\alpha_i = 1$ and $\alpha_j = 0$ for $j \ne i$, and let $\tilde{1}$ and $\tilde{0}$ be the vectors with all 1 and all 0 coordinates respectively: $\tilde{1} = (1,1,\ldots,1)$ and $\tilde{0} = (0,0,\ldots,0)$.

For any subset $A \subseteq T^n_{k_1k_2\cdots k_n}$ and any $i, 1 \le i \le n$ and $j, -k_i + 1 \le j \le k_i$ we make the following designation:

$$A + je_i = \{\, x + je_i \,/\, x \in A \,\}.$$

We will consider partition of $T^n_{k_1k_2\cdots k_n}$ (respectively partition of $A \subseteq T^n_{k_1k_2\cdots k_n}$) on i-th direction, $1 \le i \le n$ and j-th value, $-k_i + 1 \le j \le k_i$ and will denote by $T^n_i(j)$ (respectively by $A_i(j)$):

$$T_i^n(j) = \{x = (x_1, x_2, \ldots, x_n) \in T_{k_1k_2\cdots k_n}^n \ / \ x_i = j\},$$

$$A_i(j) = \{x = (x_1, x_2, \ldots, x_n) \in A \ / \ x_i = j\} = A \cap T_i^n(j).$$

Notice that the intersections of the sphere $S^n(x,k)$ and the envelope $O^n(x,k)$ with the (n-1)-dimensional torus $T_i^n(x_i + j)$, are respectively the sphere and envelope with the centre $x + je_i$ and radius $k - |j|$ in $T_i^n(x_i + j)$. We make the following designations:

$$S_i^n(x + je_i, k - |j|) = \{y \in S^n(x,k) \ / \ y_i = x_i + j\} = S^n(x,k) \cap T_i^n(x_i + j);$$

$$O_i^n(x + je_i, k - |j|) = \{y \in O^n(x,k) \ / \ y_i = x_i + j\} = O^n(x,k) \cap T_i^n(x_i + j),$$

where in case of $k - |j| < 0$ these sets are empty: $S_i^n(x + je_i, k - |j|) = O_i^n(x + je_i, k - |j|) = \varnothing$.

It is clear that $T_{k_1k_2\cdots k_n}^n = \bigcup_{j=-k_i+1}^{k_i} T_i^n(j)$, $A = \bigcup_{j=-k_i+1}^{k_i} A_i(j)$, $S^n(x,k) = \bigcup_{j=-k_i+1}^{k_i} S_i^n(x + je_i, k - |j|)$,

$O^n(x,k) = \bigcup_{j=-k_i+1}^{k_i} O_i^n(x + je_i, k - |j|)$,

for each i, $1 \le i \le n$.

For each $A_i(j)$ in the partition of $A = \bigcup_{j=-k_i+1}^{k_i} A_i(j)$ we denote by $B(A_i(j))$ and $\Gamma(A_i(j))$, respectively, the subsets of its interior and boundary vertices in $(n-1)$ -dimensional torus $T_i^n(j)$.

For any vertex $x = (x_1, x_2, \ldots, x_n)$ of $T_{k_1k_2\cdots k_n}^n$, we denote by $|x|$ and $\delta(x)$ the vectors $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$ and $\delta(x) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i = 1$ for $x_{n-i+1} > 0$ and $\alpha_i = 0$ for $x_{n-i+1} \le 0$.

In general, for n -dimensional vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with nonnegative integer coordinates, we say that the vector x lexicographically precedes y (written by $x \prec y$), if there is a number r, $1 \le r \le n$, such that $x_i = y_i$ for $1 \le i < r$ and $x_r < y_r$.

Now we order the vertices of the torus $T_{k_1k_2\cdots k_n}^n$ as follows:

vertex x precedes vertex y written by $x \Leftarrow y$), if and only if

1. $\| x \| < \| y \|$, or

2. $\| x \| = \| y \|$ and $\delta(y)$ lexicographically precedes $\delta(x)$, or

3. $\| x \| = \| y \|$, $\delta(x) = \delta(y)$ and $|y|$ lexicographically precedes $|x|$.

It is easy to check that this ordering between the vertices of the torus $T^n_{k_1k_2\cdots k_n}$ is a linear order.

The first a vertices of the torus $T^n_{k_1k_2\cdots k_n}$ by the above determined liner order we call standard arrangement of cardinality a, $0 \le a \le |T^n_{k_1k_2\cdots k_n}|$.

Basic properties of the standard arrangement

In this section we investigate the basic properties of the standard arrangement.

Theorem 1. If a set A is the standard arrangement in $T^n_{k_1k_2\cdots k_n}$, then its interior vertices precede the boundary vertices.

Proof. Prove that if $(x_1, x_2, \ldots, x_n) = x \in B(A)$ and $y \Leftarrow x$, then $(y_1, y_2, \ldots, y_n) = y \in B(A)$. It suffices to show that $y \pm e_i \in A$ for $i, 1 \le i \le n$. Clearly, $y + e_i \in A$ for $y_i < 0$ or $y_i = k_i$ and $y - e_j \in A$ for $y_j > 0$, such that $\| y + e_i \| < \| y \|$ and $\| y - e_j \| < \| y \|$, i.e. $y + e_i \Leftarrow y$ and $y - e_j \Leftarrow y$. On the other hand, it is clear that $y + e_i \Leftarrow y - e_i$, if $y_i = 0$. Consequently, to complete the proof, it suffices to show that $y \oplus e_i \in A$, for $y_i \ne k_i$, where $y \oplus e_i = \begin{cases} y + e_i, \text{ if } y_i > 0, \\ y - e_i, \text{ if } y_i \le 0 \end{cases}$. Starting from the definition of the linear ordering $\Leftarrow$, consider the following three cases:

I. $\| y \| < \| x \|$.

If $\| x \| = \sum_{i=1}^{n} k_i$, then $A = B(A) = T^n_{k_1k_2\cdots k_n}$, and then the vertex y is also interior. And if $\| x \| \ne \sum_{i=1}^{n} k_i$, then there exists i_0, that $x_{i_0} \ne k_{i_0}$. Then $\| x \oplus e_{i_0} \| > \| x \| \ge \| y \oplus e_i \|$ for each $i, 1 \le i \le n$, that is $y \oplus e_i \Leftarrow x \oplus e_{i_0}$. Hence $y \oplus e_i \in A$, for $1 \le i \le n$.

II. $\| y \| = \| x \|$ and $\delta(x) \prec \delta(y)$.

Let $\delta(x) = (\alpha_1, \alpha_2, \cdots, \alpha_{n-r}, 0, \cdots)$ and $\delta(y) = (\alpha_1, \alpha_2, \cdots, \alpha_{n-r}, 1, \cdots)$. If there is a number i_0 that

- $x_{i_0} > 0$ and $x_{i_0} \ne k_{i_0}$, or
- $x_{i_0} \le 0$ and $i_0 < r$, or
- $x_{i_0} \le 0, x_{i_0} \ne -k_{i_0} + 1$ and $i_0 \ge r$, then

for every $y_i \ne k_i$, the vertex $y \oplus e_i$ precedes the vertex $x \oplus e_{i_0}$, as $\| x \oplus e_{i_0} \| = \| y \oplus e_{i_0} \|$ and $\delta(x \oplus e_{i_0}) \prec \delta(y \oplus e_{i_0})$. Hence $y \oplus e_i \in A$, when $y_i \ne k_i$.

Otherwise, if $x_i = k_i$ for $1 \le i < r$, $x_r = -k_r + 1$ and $x_i = k_i$ or $x_i = -k_i + 1$ for $r < i \le n$ then from the condition $\| x \| = \| y \|$ we find:

$$\sum_{i=1}^{r-1}(k_i - | y_i |) + (k_r - 1 - y_r) + \sum_{i=r+1}^{n}(| x_i | - | y_i |) = 0 \qquad (1)$$

Since $k_i - | y_i | \ge 0$ for $1 \le i < r$, $k_r - 1 - y_r \ge -1$ and $| x_i | - | y_i | \ge 0$, for $r < i \le n$ (according to condition $\delta(x) \prec \delta(y)$), then it follows from (1) that the following cases take place:

a) $y_i = k_i$ for $1 \le i < r$, $y_r = k_r - 1$ and $x_i = y_i$ for $r < i \le n$; then it is clear, that for $i > r$ vertex $y \oplus e_i$ precedes vertex $x \oplus e_i$, and $y + e_r = x - e_r$, therefore $y \oplus e_i \in A$ for any $y_i \ne k_i$;

b) $y_i = k_i$ for $1 \le i \le r$, and there is a unique number $i_0 > r$ such, that $| x_{i_0} | = | y_{i_0} | + 1$ and $x_i = y_i$, for $r < i \le n, i \ne i_0$, then the vertex $y \oplus e_i$ is preceded by the vertex $x \oplus e_i$, for $i > r, i \ne i_0$, and $y \oplus e_{i_0} = x - e_r$, so again $y \oplus e_i \in A$, for any $y_i \ne k_i$;

c) there is a unique number $i_1 < r$ such that $| y_{i_1} | = k_{i_1} - 1, y_i = k_i$ for $r \le i \le n$, $i \ne i_1$, and $x_i = y_i$ for $r < i \le n$. In this case $y \oplus e_{i_1} = x - e_r$, and vertex $y \oplus e_i$ is preceded by the vertex $x \oplus e_i$ for $r < i \le n$. Hence $y \oplus e_i \in A$, when $y_i \ne k_i$.

III. $\| y \| = \| x \|$, $\delta(y) = \delta(x)$ and $| x | \prec | y |$.

Suppose that $x_i = y_i$ for $1 \le i < r \le n$, and $| x_r | < | y_r |$.

In this case if $y_i = -k_i + 1$, then $y - e_i \in A$, since $y - e_i \Leftarrow x \oplus e_r$. When $y_i \ne k_i$ and $y_i \ne -k_i + 1$, if

- $1 \le i \le r$, or
- $i > r$ and $| y_r | - | x_r | > 1$, or
- $i > r$, $| y_r | - | x_r | = 1$ and there is a number $i_0 > r$, that is $x_{i_0} \ne -k_{i_0} + 1$ *or* $x_{i_0} \ne k_{i_0}$,

then $y \oplus e_i \in A$, since in the first case, the vertex $y \oplus e_i$ precedes the vertex $x \oplus e_i$, in the second case – precedes the vertex $x \oplus e_r$, and the third – precedes the vertex $x \oplus e_{i_0}$.

Otherwise, if $| y_r | - | x_r | = 1$, $x_i = -k_i + 1$ *or* $x_i = k_i$ for $i > r$, then from the condition $\| x \| = \| y \|$ we find

$$\sum_{i=r+1}^{n}(| x_i | - | y_i |) = 1. \qquad (2)$$

Since $\delta(x)=\delta(y)$ then $|x_i|-|y_i|\geq 0$, for any $i>r$. Then (2) implies that there exists a unique number $i_0>r$, such that $|x_{i_0}|=|y_{i_0}|+1$, and for $i>r, i\neq i_0$, $x_i=y_i=k_i$ or $x_i=y_i=-k_i+1$. It is clear that $y\oplus e_{i_0}=x\oplus e_r$.

Thus, we proved that the vertex $y\pm e_i$ belongs to A, for any i, $1\leq i\leq n$, i.e. vertex y is an interior vertex of the set A. The theorem is proved.

Corollary 1. If A and C are standard arrangements in the $T^n_{k_1k_2\cdots k_n}$ and $|A|\geq|C|$, then $B(A)\supseteq B(C)$ and $A_i(j)\supseteq C_i(j)$ for any i and j, $1\leq i\leq n$, $-k_i+1\leq j\leq k_i$.

For a subset A of $T^n_{k_1k_2\cdots k_n}$, we denote by $O(A)=\{x\in T^n_{k_1k_2\cdots k_n} / \rho(x,y)\leq 1 \quad \textit{for some} \quad y\in A\}$.

Then the following statement takes place:

Lemma 1. If A is a standard arrangement, then $O(A)$ is a standard arrangement too.

Proof. Let A is the standard arrangement. Suppose that $x\in O(A)$ and $y\Leftarrow x$. All we have to show is that y belongs to $O(A)$. In case when $x\in A$, the proof is obvious. Now assume that $x\notin A$. Then there exists a vertex $x^1\in A$ and a direction i_0, that $x=x^1+e_{i_0}$, where $0\leq x^1_{i_0}<k_{i_0}$, or $x=x^1-e_{i_0}$ where $x^1_{i_0}\leq 0$. It is clear, that in both cases, $\delta(x^1)=\delta(x)$ or $\delta(x^1)\prec\delta(x)$ and $O(A)\supseteq S^n(\tilde{0}, \|x^1\|)$. Next we will find such a vertex that belongs to A and is located at distance one from the vertex y.

Since $y\Leftarrow x$ then, according to the definition of ordering $\Leftarrow$, the following three cases are possible:

Case I. $\|y\|<\|x\|$;

Case II. $\|y\|=\|x\|$ and $\delta(x)\prec\delta(y)$, where $\delta(x)=(\alpha_1,\alpha_2,\ldots,\alpha_{n-r},0,\ldots)$, $\delta(y)=(\alpha_1,\alpha_2,\ldots,\alpha_{n-r},1,\ldots)$;

Case III. $\|y\|=\|x\|$, $\delta(y)=\delta(x)$ and $|x|\prec|y|$, where $x_i=y_i$ for $1\leq i<r\leq n$ and $|x_r|<|y_r|$.

In the first case, it is obvious that $y\in S^n(\tilde{0},\|x^1\|)$, therefore y belongs $O(A)$ too.

In case II, if there is a number i_1 that $y_{i_1}\neq 0$, for $i_1<r$, or $y_{i_1}\neq 0$ and $y_{i_1}\neq 1$, when $i_1\geq r$, then vertex $y-e_{i_1}$ precedes x^1 in case of $y_{i_1}>0$, and in case when $y_{i_1}<0$, vertex $y+e_{i_1}$ precedes x^1 since

- $\|y-e_{i_1}\|=\|x^1\|$ and $\delta(x^1)\prec\delta(y-e_{i_1})$ when $y_{i_1}>0$;
- $\|y+e_{i_1}\|=\|x^1\|$ and $\delta(x^1)\prec\delta(y+e_{i_1})$ when $y_{i_1}<0$.

Thus, in case of $y_{i_1} > 0$, the vertex $y - e_{i_1} \in A$ and in case of $y_{i_1} < 0$ the vertex $y + e_{i_1}$ belongs to A, hence $y \in O(A)$. Otherwise, if $y_i = 0$, $1 \le i < r$, $y_r = 1$ and $y_i = 0$ or $y_i = 1$ for $r < i \le n$, then the conditions $\| y \| = \| x \|$ and $\delta(x) \prec \delta(y)$ imply the existence of a unique number i_2 such that the following conditions hold:

a) $i_2 \ne r,\ x_{i_2} = -1,\quad x_r = 0 \quad and \quad x_i = y_i$ for any $i \ne i_2, r$, or

b) $i_2 = r,\ x_{i_2} = -1 \quad and \quad x_i = y_i$ for any $i \ne i_2$, or

c) $i_2 < r,\ x_{i_2} = 1,\quad x_r = 0 \quad and \quad x_i = y_i$ for any $i \ne i_2, r$, or

d) $i_2 > r,\ x_{i_2} = 2,\quad x_r = 0 \quad and \quad x_i = y_i$ for any $i \ne i_2, r$.

These conditions in their turn imply that if $i_0 \ne i_2$, then $y - e_{i_0} \Leftarrow x^1$, and if $i_0 = i_2$, then $y - e_r = x^1$. So the vertex $y - e_{i_0}$ or the vertex $y - e_r$ belongs to A, hence, $y \in O(A)$.

In case III, if $|y_r| - |x_r| > 1$, or $|y_r| - |x_r| = 1$ and there is such number $i_3 > r$ that $y_{i_3} \ne 0, 1$, then vertex x^1 is preceded by $y - e_{i_4}$ in case of $y_{i_4} > 0$, or is preceded by $y + e_{i_4}$ in case of $y_{i_1} < 0$, (where $i_4 = r$ for $|y_r| - |x_r| > 1$ and $i_4 = i_3$ for $|y_r| - |x_r| = 1$), as

$$\| y - e_{i_4} \| = \| x^1 \|,\ \delta(y - e_{i_4}) = \delta(x^1) \text{ and } |x^1| \prec |y - e_{i_4}|, \text{ for } y_{i_4} > 0,$$

$$\| y + e_{i_4} \| = \| x^1 \|,\ \delta(y + e_{i_4}) = \delta(x^1) \text{ and } |x^1| \prec |y + e_{i_4}|, \text{ for } y_{i_4} < 0.$$

Thus, either vertex $y - e_{i_4}$ or vertex $y + e_{i_4}$ belongs to A, and hence $y \in O(A)$.

Now consider the cases when $|y_r| - |x_r| = 1$ and $y_i = 0$ or $y_i = 1$, for any $i, r < i \le n$.

In both cases it follows from the conditions $\| y \| = \| x \|$ and $\delta(x) = \delta(y)$, that there is a unique number $i_5 > r$ such, that

- $x_{i_5} = -1,\ y_{i_5} = 0$ and $x_i = y_i$, for $i \ne i_5, r$, or
- $x_{i_5} = 2,\ y_{i_5} = 1$ and $x_i = y_i$, for $i \ne i_5, r$.

Now if $i_5 \ne i_0$, then either vertex $y - e_{i_0}$ (for $y_{i_0} > 0$) or vertex $y + e_{i_0}$ (for $y_{i_0} < 0$) precede the vertex x^1.

If $i_5 = i_0$, then vertex $y - e_r$ coincides with x^1 when $y_r > 0$, or vertex $y + e_r$ coincides with x^1 for $y_r < 0$. Therefore, again we get a vertex belonging to A and located at distance one from the vertex y. Hence $y \in O(A)$. Lemma is proved.

Now let us derive some properties of the standard arrangement in the torus.

Let A be a standard arrangement. Consider its partitions by i-th direction, $1 \le i \le n$: $A = \bigcup_{j=-k_i+1}^{k_i} A_i(j)$.

Further in this section we will prove the following properties:

1^0. each subset $A_i(j)$ is a standard arrangement of $T_i^n(j)$;

2^0. for any j, $1 \le j < k_n$, $A_n(-j) + e_n \subseteq B(A_n(-j+1))$;

3^0. for any j, $1 < j \le k_n$, $A_n(j) - e_n \subseteq B(A_n(j-1))$;

4^0. for any j, $0 \le j < k_n$, $A_n(-j) \supseteq A_n(j+1) - (2j+1)e_n \supseteq B(A_n(-j))$

or $A_n(-j) = T_n^n(-j)$ and $A_n(j+1) = T_n^n(j+1) \setminus \{(k_1, k_2, \cdots, k_{n-1}, j+1)\}$;

5^0. for any j, $0 < j < k_n$, $A_n(j) \supseteq A_n(-j) + 2je_n \supseteq B(A_n(j))$ or $A_n(j) = T_n^n(j)$ and

$A_n(-j) = T_n^n(-j) \setminus \{(k_1, k_2, \cdots, k_{n-1}, -j)\}$.

1^0. This property is obvious.

2^0. Let, $x = (x_1, x_2, \cdots, x_{n-1}, -j) \in A_n(-j)$, where $1 \le j < k_n$. Then obviously

- $\| x + e_n \| < \| x \|$,
- $\| x + e_n + e_i \| < \| x + e_n \|$, when $x_i < 0$ or $x_i = k_i$,
- $\| x + e_n - e_i \| < \| x + e_n \|$, when $x_i > 0$,
- $\| x + e_n - e_i \| = \| x \|$ and $\delta(x) \prec \delta(x + e_n - e_i)$ when $x_i = -k_i + 1$,
- $\| x + e_n \oplus e_i \| = \| x \|$, $\delta(x) \prec \delta(x + e_n \oplus e_i)$ and $| x | \prec | x + e_n \oplus e_i |$, when $x_i \ne -k_i + 1, k_i$,
- $\| x + e_n + e_i \| = \| x \|$ and $\delta(x) \prec \delta(x + e_n + e_i)$ when $x_i = 0$.

From these conditions follows that $x + e_n \Leftarrow x$ and $x + e_n \pm e_i \Leftarrow x$, for any $i, 1 \le i \le n-1$. As A is standard arrangement, then $x + e_n \in A$ and $x + e_n \pm e_i \in A$ for any $i, 1 \le i \le n-1$, i.e, $x + e_n$ is the interior vertex of the subset $A_n(-j+1)$.

3^0. Can be proved in similar way.

4^0. Let, $x = (x_1, x_2, \cdots, x_{n-1}, j+1) \in A_n(j+1)$, where $0 \le j < k_n$. Then $x - (2j+1)e_n \Leftarrow x$, as $\| x - (2j+1)e_n \| < \| x \|$. Therefore, the vertex $x - (2j+1)e_n$ belongs $A_n(-j)$, i.e., $A_n(-j) \supseteq A_n(j+1) - (2j+1)e_n$. On the other hand, if $y = (y_1, y_2, \cdots, y_{n-1}, -j) \in B(A_n(-j))$ and $A_n(-j) \ne T_n^n(-j)$, then there exist such number i_0, $1 \le i_0 \le n-1$, that $y_{i_0} \ne k_{i_0}$ and $y \oplus e_{i_0} \in A$. Then vertex $y+(2j+1)e_n$ precedes the vertex $y\oplus e_{i_0}$, since $\| y+(2j+1)e_n \|=\| y\oplus e_{i_0} \|$ and $\delta(y\oplus e_{i_0})\prec\delta(y+(2j+1)e_n)$. Therefore, the vertex $y+(2j+1)e_n$ belongs A, that is, $A_n(j+1)-(2j+1)e_n \supseteq B(A_n(-j))$, when $A_n(-j)\neq T_n^n(-j)$. And if $A_n(-j)=T_n^n(-j)$, then all the vertices $y=(y_1,y_2,\cdots,y_{n-1},j+1)$ with norm less than or equal to $\sum_{i=1}^{n-1}k_i+j$ precede the vertex $y=(k_1,k_2,\cdots,k_{n-1},-j)$, and only the vertex $y=(k_1,k_2,\cdots,k_{n-1},j+1)$ from $T_n^n(j+1)$ might not belong to $A_n(j+1)$.

5^0. The proof is similar.

As a corollary from the above properties we get the following lemma.

Lemma 2. If A is standard arrangement and $|A|<|T^n_{k_1k_2\cdots k_n}|-1$ then

$$|B(A)|=|A|-|A_n(0)|-|A_n(1)|+|B(A_n(-k_n+1))|+|B(A_n(k_n))|.$$

In general, a set $A\subseteq T^n_{k_1k_2\cdots k_n}$ possessing the above properties 1^0 – 5^0 of standard arrangement, might be itself non standard arrangement. However, we have

Theorem 2. Let the partition of some set $A\subseteq T^n_{k_1k_2\cdots k_n}$ satisfies the following conditions: $A_n(j+1)=O(A_n(j)+e_n)$, when $j<0$, and $A_n(j)=O(A_n(j+1)-e_n)$, for $j\geq 1$, then if there is a number j_1, $0\leq j_1\leq k_n-1$, that either

a) $A_n(-j_1)=S_n^n(-j_1e_n,r+1)$, $A_n(j_1+1)=S_n^n((j_1+1)e_n,r)\cup S_1$, $S_1\subseteq O_n^n((j_1+1)e_n,r+1)$ and $A_n(j_1+1)$ is standard arrangement in $T_n^n(j_1+1)$, and $r=0$, when $j_1<k_n-1$, or

b) $A_n(-j_1)=S_n^n(-j_1e_n,r)\cup S_0$, $A_n(j_1+1)=S_n^n((j_1+1)e_n,r)$, $S_0\subseteq O_n^n(-j_1e_n,r+1)$ and $A_n(-j_1)$ is standard arrangement in $T_n^n(-j_1)$, and $r=0$, when $j_1<k_n-1$,

then A is standard arrangement in $T^n_{k_1k_2\cdots k_n}$.

Proof.

Consider case a). According to the conditions of the theorem and by Lemma 1, we have:

1. for any j, $-k_n+1\leq j\leq k_n$, $A_n(j)$ is standard arrangement of $T_n^n(j)$, and $A_n(-j)=A_n(j)=\varnothing$, for

$$j>j_1+1 \quad if \quad j_1<k_n-1;$$

2. $A=S^n(\widetilde{0},r+j_1+1)\cup A''$, where A'' is a set of vertices $z=(z_1,z_2,\cdots,z_n)$, for which $\| z\|=r+j_1+2$ and $z_n>0$ (note that for $A''=\varnothing$ theorem is obvious, therefore further will assume that $A''\neq\varnothing$);

3. for any $j \le 0$ the latest vertex of the subset $A_n(j)$ precedes the latest vertex of the subsets $A_n(j-1)$ in $T^n_{k_1 k_2 \cdots k_n}$;

4. for any $j \ge 1$ the latest vertex of the subset $A_n(j)$ precedes the latest vertex of the subset $A_n(j+1)$ in $T^n_{k_1 k_2 \cdots k_n}$.

Let $x = (x_1, x_{2,} \cdots, x_n)$ is the latest vertex of the set A (clearly, that $x \in A''$, i.e. $x_n > 0$), and $y = (y_1, y_2, \cdots, y_n) \in T^n_{k_1 k_2 \cdots k_n}$ is an arbitrary preceding x vertex in the $T^n_{k_1 k_2 \cdots k_n}$. We must show, that $y \in A$. First we notice that if $y_n = x_n$, then the vertex y precedes x in $T^n_n(x_n)$ too. Then also, y belongs to $A_n(x_n)$, so as $x \in A_n(x_n)$ and $A_n(x_n)$ is standard arrangement in $T^n_n(x_n)$. Therefore, further, we assume, that $y_n \ne x_n$. Since the vertex y precedes vertex x, then the following three cases are possible:

Case I. $\| y \| < \| x \|$;

In this case $\| y \| \le r + j_1 + 1$, i.e. $y \in S^n(\widetilde{0}, r + j_1 + 1)$, it means that $y \in A$.

Case II. $\| y \| = \| x \|$ and $\delta(x) \prec \delta(y)$.

Let $\delta(x) = (\alpha_1, \alpha_2, \ldots, \alpha_{n-i_0}, 0, \ldots)$ and $\delta(y) = (\alpha_1, \alpha_2, \ldots, \alpha_{n-i_0}, 1, \ldots)$. It is clear, that $y_n > 0$, as $x_n > 0$.

If $1 \le y_n < x_n$, then from the condition $\| y \| = \| x \|$ there exists such i_1 that $| x_{i_1} | - | y_{i_1} | < 0$, and if $i_1 \ne i_0$, then $\| x - e_n \oplus e_{i_1} \| = \| y \|$ and $\delta(x - e_n \oplus e_{i_1}) \prec \delta(y)$, and if $i_1 = i_0$ is a unique number for which $| x_{i_1} | - | y_{i_1} | < 0$, then or $| x_{i_1} | - | y_{i_1} | < -1$ (that means that $x_{i_1} \ne -k_{i_1} + 1$ so again $\| x - e_n \oplus e_{i_1} \| = \| y \|$ and $\delta(x - e_n \oplus e_{i_1}) \prec \delta(y)$), or $| x_{i_1} | - | y_{i_1} | = -1$ and then $x - e_n \oplus e_{i_1} = y$.

Thus we can always find a vertex $x^1 = x - e_n \oplus e_{i_1}$, that either, $y \Leftarrow x^1$ or $y = x^1$ and $x^1 \in A$, as, according to the conditions of the theorem, $x - e_n \in B(A_n(x_n - 1))$. Repeating this process again at $k = x_n - y_n$, we find a vertex $x^k = x^{k-1} - e_n \oplus e_{i_k}$ such that $y \Leftarrow x^k$ or $y = x^k$, and $x^k \in A_n(y_n)$.

Since $A_n(y_n)$ is standard arrangement, then $y \in A_n(y_n)$.

If $y_n > x_n \ge 1$ and if we assume that $y \notin A_n(y_n)$, then $x \Leftarrow z \Leftarrow y$ (according to property 4), where z is the latest vertex of the set $A_n(y_n)$, which contradicts the supposition $y \Leftarrow x$. Therefore $y \in A_n(y_n)$.

Case III. $\| y \| = \| x \|$, $\delta(y) = \delta(x)$ and $|x| \prec |y|$;

Suppose that $x_i = y_i$ for $1 \le i < r_0 \le n$, and $|x_{r_0}| < |y_{r_0}|$.

If $1 \le y_n < x_n$, then the condition $\| y \| = \| x \|$ and $\delta(y) = \delta(x)$ imply

- $|y_{r_0}| - |x_{r_0}| \ge 2$, or
- $|y_{r_0}| - |x_{r_0}| = 1$ and there exists such number r_1, $r_1 > r_0$ and $r_1 \ne n$, that $|y_{r_1}| - |x_{r_1}| > 0$, or
- $|y_{r_0}| - |x_{r_0}| = 1$, $x_n - y_n = 1$ and $x_i = y_i$, if $i \ne r_0, n$.

Then it is clear that in the first case $y \Leftarrow x - e_n \oplus e_{r_0}$, as $\| y \| = \| x - e_n \oplus e_{r_0} \|$, $\delta(y) = \delta(x - e_n \oplus e_{r_0})$, $|x - e_n \oplus e_{r_0}| \prec |y|$, in the second case $y \Leftarrow x - e_n \oplus e_{r_1}$, so as well $\| y \| = \| x - e_n \oplus e_{r_1} \|$, $\delta(y) = \delta(x - e_n \oplus e_{r_1})$ and $|x - e_n \oplus e_{r_1}| \prec |y|$, and in the third case $y = x - e_n \oplus e_{r_0}$. Thus there is always such vertex $x^1 = x - e_n \oplus e_{i_1}$ that $y \Leftarrow x^1$ or $y = x^1$, where $i_1 = r_0$ or $i_1 = r_1$, and $x^1 \in A$, under the conditions of the theorem. Repeating this process again at $k = x_n - y_n$, we find such vertex $x^k = x^{k-1} - e_n \oplus e_{i_k}$, that $y \Leftarrow x^k$ or $y = x^k$, and $x^k \in A_n(y_n)$. Since $A_n(y_n)$ is the standard arrangement in the $T_n^n(y_n)$, then $y \in A_n(y_n)$.

If $y_n > x_n \ge 1$, then again $y \in A_n(y_n)$, since otherwise (as in the case II) we would get $x \Leftarrow y$, that would contradict the condition $y \Leftarrow x$.

The proof is completed for the case a). Case b) can be proved in similar way.

Consider a subset $A \subseteq T^n_{k_1 k_2 \cdots k_n}$ and its partitions: $A = \bigcup_{j=-k_i+1}^{k_i} A_i(j)$. We replace each $A_i(j)$ by standard arrangement in the $T_i^n(j)$ of the same cardinality, and this transformation is called N_i - normalization of the set A in respect to i - th direction. The resulting set we denote by $N_i(A)$.

Below we formulate one more property of the standard arrangement which is a generalization of Lemma 4 of [1] and further will be used proving the optimality of the standard arrangement of n - dimensional torus.

Lemma 3. If the standard arrangement is the optimal subset in the $(n-1)$ - dimensional torus and $A \subseteq T^n_{k_1 k_2 \cdots k_n}$ is an arbitrary set, then for any i, $1 \le i \le n$

$$|B(N_i(A))| \ge |B(A)|.$$

For a Boolean vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ the set $\alpha(T^n_{k_1 k_2 \cdots k_n}) = \{x \in T^n_{k_1 k_2 \cdots k_n} \ / \ \delta(x) = \alpha\}$ will be called α - part of the torus $T^n_{k_1 k_2 \cdots k_n}$. In general for an arbitrary subset $A \subseteq T^n_{k_1k_2\cdots k_n}$, $\alpha(A) = \{x \in A \,/\, \delta(x) = \alpha\}$. It is clear that $T^n_{k_1k_2\cdots k_n} = \bigcup_{\alpha \in E^n} \alpha(T^n_{k_1k_2\cdots k_n})$ and all α - parts the torus are isomorphic. Notice also that α - parts of $T^n_{k_1k_2\cdots k_n}$ are arranged according to order $\Leftarrow$.

For two vertices $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ of $\alpha(T^n_{k_1k_2\cdots k_n})$, we define their sum as follows:

$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) = (z_1, z_2, \cdots, z_n)$,

where $x_i + y_i \equiv z_i (\mathrm{mod}\, k_i)$, $1 \le z_i \le k_i$ for $\alpha_i = 1$ and $-k_i + 1 \le z_i \le 0$ for $\alpha_i = 0$, for any i, $1 \le i \le n$.

In the α - part of $T^n_{k_1k_2\cdots k_n}$ we define sphere and envelope with the centre $x \in \alpha(T^n_{k_1k_2\cdots k_n})$ and radius k in the following way: $S^n_\alpha(x,k) = \{y = x + \sum_{i=1}^{n} (-1)^{1+\alpha_i} r_i e_i \,/\, \sum_{i=1}^{n} r_i \le k\}$ and $O^n_\alpha(x,k) = S^n_\alpha(x,k) \setminus S^n_\alpha(x,k-1)$, where r_i are non-negative integers, for any i, $1 \le i \le n$.

For any subset of α - parts of $T^n_{k_1k_2\cdots k_n}$, $A \subseteq \alpha(T^n_{k_1k_2\cdots k_n})$, the subset of interior vertices is defined as follows:

$B_\alpha(A) = \{x \in A \,/\, S^n_\alpha(x,1) \subseteq A\}$.

It is easy to check that the linear order $\Leftarrow$ between the vertices in each α - part of $T^n_{k_1k_2\cdots k_n}$ coincides with a diagonal sequence defined in [6], and its each initial segment is again called a standard arrangement.

It is proven in [6] that if A is the standard arrangement in $\alpha(T^n_{k_1k_2\cdots k_n})$, and $C \subseteq \alpha(T^n_{k_1k_2\cdots k_n})$ is an arbitrary set of cardinality $|A|$, then $|B_\alpha(A)| \ge |B_\alpha(C)|$.

Now we prove a statement, referring to the standard arrangements in α - parts, which is a generalization of Lemma 3 in [1].

Lemma 4. If A, E, F *and* C are such standard arrangements in the α - part of $T^n_{k_1k_2\cdots k_n}$, that $|A| \ge |E| \ge |F| \ge |C|$, $|A| + |C| = |E| + |F|$ and either A or C are a sphere in α - part, then

$$|B_\alpha(A)| + |B_\alpha(C)| \ge |B_\alpha(E)| + |B_\alpha(F)|.$$

Since all α - parts of the torus $T^n_{k_1k_2\cdots k_n}$ are isomorphic, then without loss of generality we will consider only the first α - part, that is $\alpha = (1, 1, \cdots, 1) = \tilde{1}$. Then the partitions of sets $S^n_\alpha(x,k)$ and $O^n_\alpha(x,k)$ by i-th direction are:

$$S_\alpha^n(x,k)=\bigcup_{j=0}^{k_i-1} S_{\alpha,i}^n(x+je_i,k-j),$$

$$O_\alpha^n(x,k)=\bigcup_{j=0}^{k_i-1} O_{\alpha,i}^n(x+je_i,k-j),$$

where $S_{\alpha,i}^n(x+je_i,k-j)\subseteq \alpha(T_i^n(x_i+j))$, $O_{\alpha,i}^n(x+je_i,k-j)\subseteq \alpha(T_i^n(x_i+j))$ and $S_{\alpha,i}^n(x+je_i,k-j)=O_{\alpha,i}^n(x+je_i,k-j)=\varnothing$ for $k-j<0$.

It is easy to see that if A is the standard arrangement in α - part, then all $A_1(j)$, except maybe one, are spheres in the $(n-1)$-dimensional α - part and

$$B_\alpha(A)=\bigcup_{j=1}^{k_1} B_\alpha(A_1(j))$$

$$\text{or} \tag{3}$$

$$B_\alpha(A)=\bigcup_{j=1}^{k_1} B_\alpha(A_1(j))\setminus\{(j_0,k_2,k_3,\cdots,k_n)\}$$

when $A_1(j_0)=\alpha(T_1^n(j_0))$ and $A_1(j_0+1)\neq\alpha(T_1^n(j_0+1))$.

Indeed, if x is the latest vertex of the set A and $x\in A_1(j_1)$, then any vertex $y=(j,y_2,y_3,\cdots,y_n)$, such that $\|y\|<\|x\|$ or $\|y\|=\|x\|$ and $j>j_1$, precedes x, and when $\|y\|>\|x\|$ or $\|y\|=\|x\|$ and $j<j_1$, none of the vertices $y=(j,y_2,y_3,\cdots,y_n)$ precede the vertex x. Consequently,

$$A_1(j)=\begin{cases} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1,\|x\|-n-j) \quad , \quad if \quad 1\le j<j_1 \\ S_{\alpha,1}^n(\tilde{1}+(j-1)e_1,\|x\|-n-j+1), \quad if \quad j_1<j\le k_1 \\ S_{\alpha,1}^n(\tilde{1}+(j_1-1)e_1,\|x\|-n-j_1)\cup S, \quad if \quad j=j_1, \\ \quad where \;\; S\subseteq O_{\alpha,1}^n(\tilde{1}+(j_1-1)e_1,\|x\|-n-j_1+1) \end{cases}$$

It follows that $B_\alpha(A_1(j))+e_1\subseteq A_1(j+1)$ for any $j, 1\le j<k_1$, except perhaps the one j_0, for which $A_1(j_0)=\alpha(T_1^n(j_0))$ and $A_1(j_0+1)\neq\alpha(T_1^n(j_0+1))$.

Now we prove the Lemma 4 .

The proof is by induction on n. For $n=1$ the proof is obvious. Suppose two standard arrangements are given in α - part: $A=S_\alpha^n(\tilde{1},k)\cup A'$ and $C=S_\alpha^n(\tilde{1},r)\cup C'$, where . Consider the partition of these sets by the first direction:

$$A=\bigcup_{j=1}^{k_1}A_1(j)=(\bigcup_{j=1}^{j_0-1}S_{\alpha,1}^n(\tilde{1}+(j-1)e_i,k-j+1))\cup S_{\alpha,1}^n(\tilde{1}+(j_0-1)e_1,k-j_0+1)\cup$$

$$\cup A_1^1\cup(\bigcup_{j=j_0+1}^{k_1}S_{\alpha,1}^n(\tilde{1}+(j-1)e_1,k-j+2)),$$

$$C=\bigcup_{j=1}^{k_1}C_1(j)=(\bigcup_{j=1}^{j_1-1}S_{\alpha,1}^n(\tilde{1}+(j-1)e_i,r-j+1))\cup S_{\alpha,1}^n(\tilde{1}+(j_1-1)e_1,r-j_1+1)\cup$$

$$\cup C_1^1\cup(\bigcup_{j=j_1+1}^{k_1}S_{\alpha,1}^n(\tilde{1}+(j-1)e_1,r-j+2)),$$

where $A_1^1\subset O_{\alpha,1}^n(\tilde{1}+(j_0-1)e_i,k-j_0+2)$, $\varnothing\neq C_1^1\subseteq O_{\alpha,1}^n(\tilde{1}+(j_1-1)e_i,r-j_1+2)$, and $S_{\alpha,i}^n(\tilde{1}+je_1,a)=S_{\alpha,1}^n(\tilde{1}+(j-1)e_1,a+1)+e_1$ for $a=\sum_{i=2}^{n}k_i$.

Two cases are possible:

Case I. $k-j_0>r-j_1$ or $k-j_0=r-j_1$ and $|A_1^1|\geq|C_1^1|>0$.

In this case, if we remove some number of vertices from the subset C_1^1 of the set $C_1(j_1)$ and add the same number of new vertices to the set $A_1(j_0)$ so that the newly formed subsets $C_1^1(j_1)$ and $A_1^1(j_0)$ also are standard arrangements in $\alpha(T_1^n(j_1))$ and $\alpha(T_1^n(j_0))$, where at least one of them was a sphere, then by property (4.4) and the induction supposition, the total number of interior vertices of the obtained sets

$A^1=(A\setminus A_1(j_0))\cup A_1^1(j_0)$ and $C^1=(C\setminus C_1(j_1))\cup C_1^1(j_1)$

will not decrease.

In the next step, instead of subsets $C_1(j_1)$, and $A_1(j_0)$ we consider

- the subsets $C_1(j_1+1)$ and $A_1^1(j_0)$, where on the first step the $C_1^1(j_1)$ was a sphere, or
- the subsets $C_1^1(j_1)$ and $A_1(j_0-1)$, where on the first step the $A_1^1(j_0)$ was a sphere, or
- the subsets $C_1(j_1+1)$ and $A_1(j_0-1)$, where on the first step $C_1^1(j_1)$ and $A_1^1(j_0)$ were the spheres,

and apply the above transfer of the vertices. This process continues until at least one of the sets A and C becomes a sphere.

Case II. $k-j_0<r-j_1$ or $k-j_0=r-j_1$ and $|A_1^1|<|C_1^1|$.

In this case, first of all we remove from subset A' of set A a certain number of vertices and add the same number of new vertices to the set C, so that one of the sets A and C will be sphere, and at each step this transfer takes place between some of the subsets $A_1(j)$, $j\geq j_0$, and $C_1(j)$, $j\leq j_1$. Therefore, by the induction assumption, the total number of internal vertices can only increase. Received after this transformation sets A^1 and C^1 can only be of two kinds:

a) $C^1 = S_\alpha^n(\tilde{1}, r+1) = \bigcup_{j=1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+1)$,

$$A^1 = S_\alpha^n(\tilde{1}, k) \cup A'' = (\bigcup_{j=1}^{j_2-1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, k-j+1)) \cup$$

$$\cup(S_{\alpha,1}^n(\tilde{1}+(j_2-1)e_1, k-j_2+1) \cup A_1^2) \cup (\bigcup_{j=j_2+1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, k-j+2)),$$

b) $A^1 = S_\alpha^n(\tilde{1}, k) = \bigcup_{j=1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, k-j+1)$,

$$C^1 = S_\alpha^n(\tilde{1}, r) \cup C'' = (\bigcup_{j=1}^{j_3-1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+1)) \cup$$

$$\cup(S_{\alpha,1}^n(\tilde{1}+(j_3-1)e_1, r-j_3+1) \cup C_1^2) \cup (\bigcup_{j=j_3+1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+2)).$$

In case a) it is clear that $k-j_2+1 < r+1$. Hence, if instead of sets A^1 and C^1 we take the set

$$A^2 = (\bigcup_{j=1}^{k-r} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, k-j+1)) \cup (\bigcup_{j=k-r+1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, k-j+2)),$$

$$C^2 = \bigcup_{j=1}^{r-k+j_2-1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+1) \cup (S_{\alpha,1}^n(\tilde{1}+(r-k+j_2-1)e_1, k-j_2+1) \cup C_1^2)$$

$$\cup (\bigcup_{j=r-k+j_2+1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+2)),$$

where $|C_1^2| = |A_1^2|$ and $C_1^2(r-k+j_2)$ is the standard arrangement in the $\alpha(T_1^n(r-k+j_2))$, it is obvious that $|A^2| + |C^2| = |A^1| + |C^1|$, and by (3)

$$|B_\alpha(A^2)| + |B_\alpha(C^2)| = |B_\alpha(A^1)| + |B_\alpha(C^1)|.$$

So, if one of the sets A^2 and C^2 is sphere, then the lemma is proved; otherwise we come to the case I, since $r+1 > k-j_2+1$.

In case b), since $k-j_0 < r-j_1$, $j_0 \le k_1$, $j_3 \le j_1$, then $r-k_1+1 \le k-k_1+1 \le r-j_3+1$. Hence, if instead of sets A^1 and C^1 take the sets

$$C^2 = (\bigcup_{j=1}^{r-k+k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+1)) \cup (\bigcup_{j=r-k+k_1+1}^{k_1} S_{\alpha,1}^n(\tilde{1}+(j-1)e_1, r-j+2)),$$

$$A^2 = \bigcup_{j=1}^{k-r+j_3-1} S^n_{\alpha,1}(\tilde{1}+(j-1)e_1, k-j+1) \cup (S^n_{\alpha,1}(\tilde{1}+(k-r+j_3-1)e_1, r-j_3+1) \cup$$

$$\cup A_1^2) \cup (\bigcup_{j=k-r+j_3+1}^{k_1} S^n_{\alpha,1}(\tilde{1}+(j-1)e_1, k-j+2)),$$

where $|C_1^2| = |A_1^2|$ and $A_1^2(k-r+j_3)$ is the standard arrangement at the $\alpha(T_1^n(k-r+j_3))$, then $|A^2| + |C^2| = |A^1| + |C^1|$ and, by (3) ,

$$|B_\alpha(A^2)| + |B_\alpha(C^2)| = |B_\alpha(A^1)| + |B_\alpha(C^1)|.$$

So, if one of the sets A^2 and C^2 is sphere, then the lemma is proved; otherwise we come to the case I, since $k-k_1 < r-j_3$. The proof is completed.

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