ON THE STRUCTURE OF MAXIMUM INDEPENDENT SETS IN BIPARTITE GRAPHS

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Abstract: *In this paper it is shown that for bipartite graphs the structure of the family of maximum independent sets can be described constructively, in the following sense. For a bipartite graph there are some "basic" maximum independent sets, in terms of which any maximum independent set can be described, in the sense that there is one-to-one correspondence between a maximum independent set and an irreducible combination of these "basic" maximum independent sets. König's theorem states that there is duality between the cardinalities of maximum matching and minimum vertex cover. Viewing the mentioned structure, in this paper it is shown that another duality holds, which is between the sets rather than their cardinalities. We believe that this duality is not just of theoretical interest, but it also can yield to a usable algorithm for finding a maximum matching of bipartite graph. In this paper we do not present such algorithm; instead we mention what approaches we plan to use in further works to obtain such algorithm.*

Keywords: *bipartite graph, maximum independent set, distributive lattice, duality.*

ACM Classification Keywords: *G.2.1 Discrete mathematics: Combinatorics*

Introduction

Let $G = (W, E)$ be a graph, where W is the set of vertices, and E is the set of edges. Two vertices $u, v \in W$ are said to be *adjacent* with each other, if $\{u, v\} \in E$; otherwise they said to be *independent*. A set of vertices is called an *independent set*, if any two vertices of it are independent. For instance,a set consisting of one vertex is an independent set. A*maximum independent set* (*MMIS*) is one with the maximum cardinality among all independent sets (don't be confused with the *maximal independent set*, which is an independent set, no proper superset of which is an independent set). The cardinality of MMIS-es of G is denoted by$\alpha(G)$. A set of vertices in G, such that each edge of G is adjacent with some vertex in that set, is called a *vertex cover*. For instance, the set of all vertices of G is a vertex cover. A *minimum vertex cover* is one with minimum cardinality among all vertex covers;that cardinality is denoted by $\beta(G)$. It is easy to see that each vertex cover is a complement of some independent set and vice-versa, so the complement of any MMIS is a minimum vertex cover and vice-versa. Thus we get$\alpha(G) + \beta(G) = |W|$. The concepts of independent set and vertex cover are related with the concept of *matching*, which is a set of edges, no distinct two of which share a common vertex. For instance, a set consisting of one edge is a matching. A *maximum matching* is one with maximum cardinality among all matchings; that cardinality is denoted by $\gamma(G)$. Note, that for a given matching and a given vertex cover, each edge of the matching is "covered" by some vertex of the vertex cover, and different edges are covered with different vertices.This means that the cardinality of matching doesn't exceed the cardinality of the vertex cover, so

$\gamma(G) \leq \beta(G)$. $G$is called *bipartite*, if its vertices can be decomposed into two independent sets U and V, such that $W = U \cup V$ and $U \cap V = \emptyset$. A bipartite graph is denoted as $G = (U, V, E)$, where U and V are said to be its *parts*. König's theorem [Harary, 1969] states, that for a bipartite graph it holds$\gamma(G) = \beta(G)$. For a general graph the problem of finding a MMIS is NP-hard [Karp, 1972], however for various specific classes of graphs, including bipartite graphs, there are polynomial-time algorithms [Harary, 1969].In some applications [Johnson, 1988] it is needed to deal with all MMIS-es of the graph, so it is of both theoretical and practical interest to describe the family of all MMIS-es and to show how to construct not just any, but some particular MMIS. Usually this is done simply by generating all MMIS-es of the graph, and for some specific classes of graphs there are algorithms which generate all MMIS-es with polynomial-time delay between two successive outputs [Kashiwabara, 1992] (note, that in some cases the number of MMIS-es to be generated is potentially exponential, and there are various notions on what to consider a "polynomial-time" algorithm for problems of this kind [Johnson, 1988]). In this paper we describe the family of MMIS-es of a bipartite graph by describing its structure, rather than generating all MMIS-es. We show that MMIS-es of a bipartite graph form a *distributive lattice* with respect to simple set operations (see the preliminaries regarding lattice theory bellow in this section), and we show how to obtain that lattice. After it is done, various queries can be performed, and generating all MMIS-es is one of them. The classic solution of the problem of finding just one MMIS of a bipartite graph is by Ford-Fulkerson algorithm [Ford, Fulkerson, 1962], which provides a MMIS of bipartite graph $G = (U, V, E)$, performing $O(|E||W|)$ operations in worst case, where $W = U \cup V$. There is an optimization of this approach, called Hopcroft-Karp algorithm [Hopcroft, Karp, 1973], which provides a MMIS performing $O\left(|E|\sqrt{|W|}\right)$ operations in worst case. In this paper we also discuss the problem of providing an algorithm, which obtains a MMIS of bipartite graph while sequentially handling its vertices. Here we show that to do this, it is preferable to obtain the *greatest* (in the sense of the lattice of MMIS-es) MMIS, rather than just any MMIS.In some sense, the greatest MMIS corresponds to the intersection of all MMIS-es. In this paper we prove that a sort of duality holds between the intersection of all MMIS-es and the union of all maximum matchings. Besides this duality is of theoretical interest, we also believe, that it can yield to a usable algorithm which provides a MMIS of bipartite graph. In this paper we do not provide such algorithm; instead we mention what approaches we plan to use in order to obtain that result in further works. Next in this section we give some preliminaries regarding distributive lattices.

There are two equivalent definitions for lattices [Birkhoff, 1948]. Let$L$ be a carrier set. In terms of partially ordered sets*lattice* is a pair $(L, \preccurlyeq)$, where $\preccurlyeq$ is such partial order on L, that every two elements have infimum and supremum in$L$. In terms of abstract algebra, *lattice* is a triple $(L, \vee, \wedge)$, where $\vee$ and $\wedge$ are such binary operations on L, that for all $a, b, c \in L$ it holds:

$a \vee a = a$ and $a \wedge a = a$,

$a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,

$a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

$a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$, *distributive lattice* is one where the following property also holds:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{and} a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Operations $\vee$ and $\wedge$ are called *join* and *meet* respectively. The equivalence of mentioned two definitions can be checked by showing that if a lattice is defined as a partially ordered set, than one can define join and meet operations on it as $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$, and if lattice is defined as an abstract algebra, then one can define a partial order on it as$a \preccurlyeq b$ if and only if $a \vee b = b$ (or $a \wedge b = a$). For instance, the family of all subsets of a set is a lattice, where join and meet operations are respectively the union and intersection of subsets; as these operations are distributive with respect to each other, then the mentioned lattice is distributive as well. Actually, not only the family of all subsets, but any ring of subsets (i.e. a family of subsets, which is closed with respect to the union and intersection operations) is a distributive lattice. Birkhoff's representation theorem [Birkhoff, 1948] states that the opposite claim is also true, i.e. each distributive lattice is isomorphic to some ring of subsets. Another example of a distributive lattice is the set of natural numbers with operations of taking the least common multiple and the greatest common divider as join and meet operations respectively. Note, that the corresponding partial order is the divisibility of the numbers, and as one divides any number, and any number divides zero, then they are the least and the greatest elements of the lattice respectively. It may not be the case for infinite lattices, but any finite lattice has the least and the greatest elements. For a lattice $(L, \vee, \wedge)$, an element $x \in L$ is called *join-irreducible*, if it is not the least element, and if for all $a, b \in L$, $a \vee b = x$ implies $a = x$ or $b = x$. It is known [Birkhoff, 1948], thateach element of a distributive lattice has only one irreducible representation as a join of join-irreducible elements of that distributive lattice. In this sense, a distributive lattice can be considered as given, if its join-irreducible elements are given.

In the next section we show that MMIS-es of a bipartite graph form a distributive lattice with respect to simple set operations and show how to find the lowest and the join-irreducible elements of that lattice. In the next section we discuss the problem of obtaining a MMIS of bipartite graph while sequentially handling its vertices and show that a sort of duality holds between the intersection of all MMIS-es and the union of all maximum matchings. Finally we provide a short conclusion of this paper and mention the further works.

The lattice of MMIS-es

Let $G = (U, V, E)$ be a bipartite graph. Here we will define join and meet operations on MMIS-es of G and will show that the family of all MMIS-es of G is a distributive lattice with respect to that operations. First we need some notations.We denote by $\mathcal{M}$ the family of all MMIS-es of G. For a set of vertices $S \subseteq U \cup V$ we denote $S_U = S \cap U$and $S_V = S \cap V$; we will call these sets projections of S on U and V respecvely. Also for any set of edges $X \subseteq E$ and set of vertices $S \subseteq U \cup V$, we will denote by $X(S)$ the set of vertices of G, where each vertex is adjacent with some vertex of S by an edge of X. Now we claim that ifM_1 and $M_2$are MMIS-es of G, then ${M_1}_U \subseteq {M_2}_U$implies ${M_2}_V \subseteq {M_1}_V$ and vice-versa:

Claim 1: for all $M_1, M_2 \in \mathcal{M}\, {M_1}_U \subseteq {M_2}_U$ if and only if ${M_2}_V \subseteq {M_1}_V$.

Indeed, as M_1 and M_2 are MMIS-es, then ${M_1}_V$ is the set of all vertices in V, which are independent with ${M_1}_U$, and ${M_2}_V$ is the set of all vertices in V, which are independent with ${M_2}_U$, thus if ${M_1}_U \subseteq {M_2}_U$, then only some vertices of ${M_1}_V$ are also independent with ${M_2}_U$, so we get ${M_2}_V \subseteq {M_1}_V$. The opposite direction of the claim can be proved identically. We define a partial ordered set $(\mathcal{M}, \preccurlyeq)$ as follows:

$$\text{for all } M_1, M_2 \in \mathcal{M} \text{ we define } M_1 \preccurlyeq M_2 \text{ if } {M_1}_U \subseteq {M_2}_U. \tag{1}$$

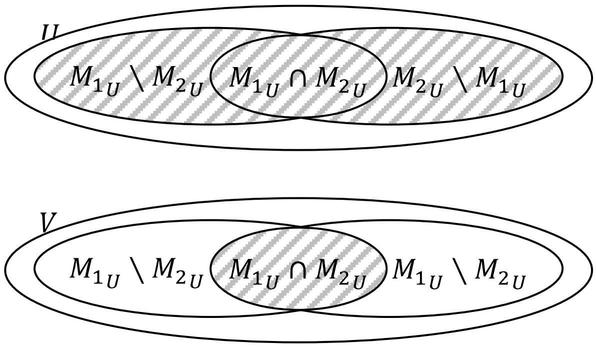

Figure 1a

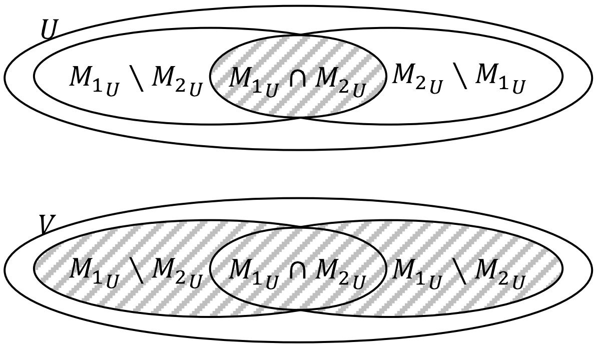

Figure 1b

So we have defined a partial order on MMIS-es according to their projections on U; from Claim 1 it follows, that we would get the dual partial order of one we got, if we define it according to the projections on V. In this sense, the partial ordered set $(\mathcal{M}, \preccurlyeq)$ is invariant with respect to the parts of G, though we define it with respect to U. Now let M_1 and M_2 be MMIS-es of G. We define join and meet operations for M_1 and M_2 as follows:

$$M_1 \vee M_2 = \left({M_1}_U \cup {M_2}_U\right) \cup \left({M_1}_V \cap {M_2}_V\right) \tag{2}$$

and

$$M_1 \wedge M_2 = \left({M_1}_U \cap {M_2}_U\right) \cup \left({M_1}_V \cup {M_2}_V\right): \tag{3}$$

At Figure 1a the shaded part corresponds to $M_1 \vee M_2$ and at Figure 1b the shaded part corresponds to $M_1 \wedge M_2$. We will show that $(\mathcal{M}, \preccurlyeq)$ is a distributive lattice with respect to these operations. First we show, that:

Claim 2: $\mathcal{M}$ is closed with respect to operations defined at (2) and (3); i.e. $M_1 \vee M_2$ and $M_1 \wedge M_2$ are MMIS-es.

Note, that $M_1 \vee M_2$ and $M_1 \wedge M_2$ are independent sets, as any vertex in $M_1 \cap M_2$ is independent with any vertex in $M_1 \cup M_2$, so to prove this claim we need to show that $|M \vee N| = |M \wedge N| = \alpha(G)$. From (2) and (3) it follows, that $(M_1 \vee M_2) \cup (M_1 \wedge M_2) = M_1 \cup M_2$ and $(M_1 \vee M_2) \cap (M_1 \wedge M_2) = M_1 \cap M_2$.

Note, that for any two sets A and B we have $|A| + |B| = |A \cup B| + |A \cap B|$. So we get $|M_1 \vee M_2| + |M_1 \wedge M_2| = |(M_1 \vee M_2) \cup (M_1 \wedge M_2)| + |(M_1 \vee M_2) \cap (M_1 \wedge M_2)| = |M_1 \cup M_2| + |M_1 \cap M_2| = |M_1| + |M_2| = 2\alpha(G)$ Thus, we got $|M_1 \vee M_2| + |M_1 \wedge M_2| = 2\alpha(G)$. As $M_1 \vee M_2$ and $M_1 \wedge M_2$ are independent sets, then we also have $|M \vee N| \leq \alpha(G)$ and $|M \wedge N| \leq \alpha(G)$, so we get $|M \vee N| = |M \wedge N| = \alpha(G)$, which proves the claim. Note that $(M_1 \vee M_2)_U = {M_1}_U \cup {M_2}_U$, so from (1) it follows, that $M_1 \vee M_2$ is the supremum of M_1 and M_2 in $(\mathcal{M}, \preccurlyeq)$. Similarly it can be show that $M_1 \wedge M_2$ is the infimum of M_1 and M_2 in $(\mathcal{M}, \preccurlyeq)$. Thus, we have shown that $(\mathcal{M}, \vee, \wedge)$ is a lattice; now we will show, that:

Claim 3: $(\mathcal{M}, \vee, \wedge)$ is distributive.

Indeed, from Claim 2 it follows that that the family of projections of all MMIS-es on U is closed with respect to the union and intersection operations, and thus, as it is mentioned above, forms a distributive lattice with respect to them (the same holds for the projections on V). Now note that the bijection $M_U \leftrightarrow M$ is an isomorphism between that lattice and $(\mathcal{M}, \vee, \wedge)$, so the last is also distributive. Also note, that the bijection $M_V \leftrightarrow M$ yields to the lattice $(\mathcal{M}, \wedge, \vee)$, which is the dual of $(\mathcal{M}, \vee, \wedge)$; this is because at (1) we have defined the partial order on $\mathcal{M}$ with respect to U. So we have proved that the family of all MMIS-es of a bipartite graph forms a distributive lattice with respect to join and meet operations defined by (2) and (3). As it is mentioned before, each distributive lattice is described by its least and join-irreducible elements. Next in this section we show how to find these elements for $(\mathcal{M}, \vee, \wedge)$.

Let $G = (U, V, E)$ be a bipartite graph and $(\mathcal{M}, \vee, \wedge)$ be the lattice of its MMIS-es. Let M be a MMIS of G and Z be a maximum matching of G. As it is mentioned before, the complement of M is a minimum vertex cover; we will denote by N. Also we will denote by K the set of vertices which are not adjacent with Z. We claim, that:

Claim 4: each edge of Z is adjacent with exactly one vertex of N, and each vertex of N is adjacent with some edge of Z (see Figure2).

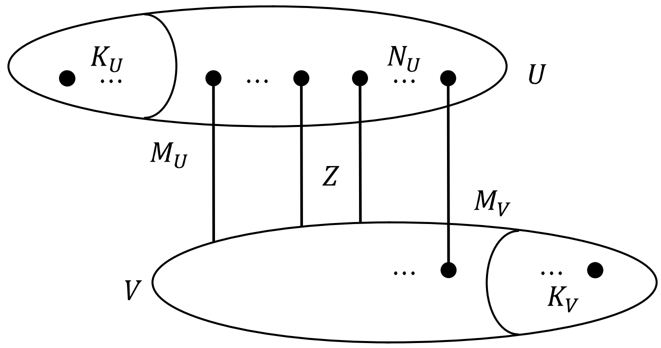

Figure 2

Indeed, as N is a vertex cover, then any edge of Z is adjacent with some vertex of N, and as Z is a matching, then different edges of Z are adjacent with different vertices of N. Note that Z is a maximum matching, and N is a minimum vertex cover, so by König's theorem we have $|Z| = |N|$. This means, that there are no vertices in N, which are not adjacent with an edge of Z, and that there are no edges of Z which are adjacent with two vertices of N (as otherwise there would be less edges in Z then there are vertices in N). So the claim is proved.

Note, that from this claim it follows, that for any MMIS M and for any maximum matching Z it holds $K \subseteq M$ (see Figure 2), where K is the set of vertices which are not adjacent with Z. Now let $S \subseteq U$be a set of vertices of G. If there are some MMIS-es containing S, then from the definition of lattice $(\mathcal{M},\vee,\wedge)$ it follows, that there is the least among them. Next we will describe how to find it.

From Claim 4 it follows, that:

if there is a MMIS which contains S, then it also contains $Z\big(E(S)\big)$; we will denote it by $(ZE)(S)$.

Indeed, if MMIS Mcontains S, then $S \subseteq M_U$ and obviously $E(S) \subseteq N_V$ (see Figure 2). From the other hand, Claim 4 states, that each edge of Z is adjacent with exactly one vertex of N, so no edge of Z connects a vertex of N_U with a vertex of N_U; this means, that if $T \subseteq N_V$ then $Z(T) \subseteq M_U$, which proves the claim (see Figure 2). From this claim it follows, that if there is a MMIS which contains S, then it also contains $(ZE)^i(S)$ for any $i \geq 0$. We will denote $\langle S\rangle = (ZE)^k(S)$, where k is the least integer, such that $(ZE)^k(S) = (ZE)^{k+1}(S)$ (obviously such k exists). We claim, that:

there exists a MMIS containing S if and only if for some maximum patching Z it holds $E(\langle S\rangle) \cap K_V = \emptyset$, and if it holds, then the least MMIS containing S is the following: $K_U \cup \langle S\rangle \cup \big(V \setminus E(\langle S\rangle)\big)$.

To be short, in the proof of this claim we will denote $A = K_U \cup \langle S\rangle$and $B = E(\langle S\rangle)$. From the denotation of $\langle S\rangle$ it follows, that there is no edge, which connects a vertex of A with a vertex of $V \setminus B$, and there is no edge of Z which connects a vertex of $U \setminus A$ with a vertex of B. Note, that if $B \cap K_V = \emptyset$, then all vertices in B are adjacent with Z, and as there is no edge of Z between B and $U \setminus A$, then $|B| + |U \setminus A| = |Z|$. From König's theorem it follows, that in this case $A \cup (V \setminus B)$ is a MMIS. Otherwise, if $B \cap K_V \neq \emptyset$, then $\langle S \rangle$ is adjacent with some vertex of K, and from Claim 4 it follows, that there is no MMIS containing S. So we have proved this claim. Based on this claim it is easy to describe an algorithm, which provides the least MMIS of G containing the given set of vertices $S \subseteq U$, if such MMIS exists. The algorithm takes as input the graph G, a maximum matching of it and a set of vertices $S \subseteq U$; if G has a MMIS containing S, then it provides the least of such MMIS-es, and otherwise it reports that no MMIS of G contains S:

Algorithm 1:

A1 denote by Z the given maximum matching of G,
denote by K the set of vertices which are note adjacent with Z,

A2 set $A = K_U \cup S$ and $B = \emptyset$,

A3 set $B = E(A)$,

A4 if $B \cap K_V \neq \emptyset$, then report, that G has no MMIS containing S and exit,

A5 if B didn't get greater, then provide $A \cup (V \setminus B)$as the least MMIS of G containing S and exit,

A6 set $A = Z(B)$,

A7 go to step A3.

Note that this algorithm performs $O(|E|)$ operations in the worst case. Also note that in order to find the least MMIS of G, we can find a maximum matching Z and call Algorithm 1 for set K_U, where K is the set of vertices which are not adjacent with Z, as any MMIS of G contains K. Obviously, this algorithm can be also used to find the greatest MMIS of G. As it is mentioned in the proof of Claim 3, the projections of all MMIS-es on U are closed with respect to the union and intersection operations, thus they form a ring of subsets, and the bijection $M_U \leftrightarrow M$ is an isomorphism between that ring and $(\mathcal{M},\vee,\wedge)$. This means, that the join-irreducible elements of $(\mathcal{M},\vee,\wedge)$ are the isomorphic images of the join-irreducible elements of the ring of projections, so for any $u \in U$, the least MMIS containing u is join-irreducible in $(\mathcal{M},\vee,\wedge)$. Thus, by Algorithm 1 one can find all join-irreducible elements of $(\mathcal{M},\vee,\wedge)$. For two partially ordered set X and $Y Y^X$ denotes the partial ordered set all isotonic functions from X to Y, where for two isotonic functions θ_1 and $\theta_2 \theta_1 \preccurlyeq \theta_2$ if for all $x \in X \theta_1(x) \preccurlyeq \theta_2(x)$ [Birkhoff, 1948]. Birkhoff's representation theorem states, that if L is a distributive lattice and A is the partially ordered set consisting of its join-irreducible elements, then $L \cong \mathbf{2}^{\breve{A}}$, where $\mathbf{2}$ denotes the chain with length 1 and $\breve{A}$ denotes the dual of A. From this theorem it follows, that each element of L has exactly one irreducible representation as join of join-irreducible elements of L. In this sense, the family of all MMIS-es of G can be considered as obtained, if for all $u \in U$ the least MMIS containing u, as well as the least MMIS of G are obtained.

Duality between the intersection of all MMIS-es and union of all maximum matchings

Let $G = (U, V, E)$ be a bipartite graph. In the previous section we have shown, that the MMIS-es of G form a distributive lattice with respect to join and meet operations defined at (2) and (3). This lattice has the greatest and the least elements, so in this sense there are the greatest and the least MMIS-es in G; we will denote them respectively by $\bigvee M$ and $\bigwedge M$. We will also denote the union and the intersection of all MMIS-es respectively by $\bigcup M$and $\bigcap M$. Note, that from (2) and (3) it follows, that:

$$\bigvee M = (\bigcup M_U) \cup (\bigcap M_V) \text{and} \bigwedge M = (\bigcap M_U) \cup (\bigcup M_V). \quad (4)$$

We will say, that “a new vertex u' is being added” to bipartite graph $G = (U, V, E)$, bearing in mind that we obtain a “new” bipartite graph G', which parts are $U \cup \{u'\}$ and V, and which edges are the edges of$E$ in addition with some “new” edges, which connect u' with some vertices of V. We claim, that:

Claim 5: while “adding a new vertex” u' to G, $\alpha(G)$ increments if and only if u' is independent with $\bigvee M$.

Obviously, either $\alpha(G') = \alpha(G) + 1$ or $\alpha(G') = \alpha(G)$. Note, that if u' is independent with $\bigcap M_V$, then $(\bigvee M) \cup \{u'\}$ is the greatest MMIS of G'. Otherwise, i.e. if u' is adjacent with some vertex of $\bigcap M_V$, then no MMIS of G is independent with u', so $\alpha(G') = \alpha(G)$. Thus the claim is proved. Next we will describe an algorithm, which obtains the greatest MMIS of G' based on the greatest MMIS of G. Note, that by so we will provide an algorithm, which sequentially handles vertices of a bipartite graph and provides a MMIS of it. As Claim 7 states, if $\alpha(G') = \alpha(G) + 1$, then the greatest MMIS of G' can be easily obtained. Otherwise, i.e. if $\alpha(G') = \alpha(G)$, then we have, that $\bigvee M$ is a MMIS of G', but it may not be the greatest one.If we obtain a maximum matching of G', then by Algorithm 1, we can obtain the greatest MMIS of G'.

We claim that:

each vertex of $\bigcap M_V$ is not adjacent with some maximum matching of G.

Let Z be a maximum matching of G, and $v \in \bigcap M_V$. If v is not adjacent with Z, then the claim is proved; otherwise, let $u \in \bigcap N_U$ be a vertex, such that $(u, v) \in Z$ (see Figure 3). As it is denoted at (4), $\bigvee M = (\bigcup M_U) \cup (\bigcap M_V)$ is the greatest MMIS of G, so no MMIS of G contains u. This means, that while calling Algorithm 1 for $\{u\}$, it stops at step A3 by finding a vertex $w \in K_V$ and a path from u to w, which has odd length, and even edges of which are edges of Z (see Figure 3).

Note that if we remove from Z the edge (u, v) and the even edges of the path found by Algorithm 1, then add to Z the odd edges of that path found by Algorithm 1, then we will get a maximum matching of G, which is not adjacent with v (see Figure 3). Thus the claim is proved. Note, that if $\alpha(G') = \alpha(G)$ (i.e. if u' is adjacent with some vertices in $\bigcap M_V$), then by König’s theorem we have $\gamma(G') = \gamma(G) + 1$, so in this case each “new” edge belongs to some maximum matching of G'.

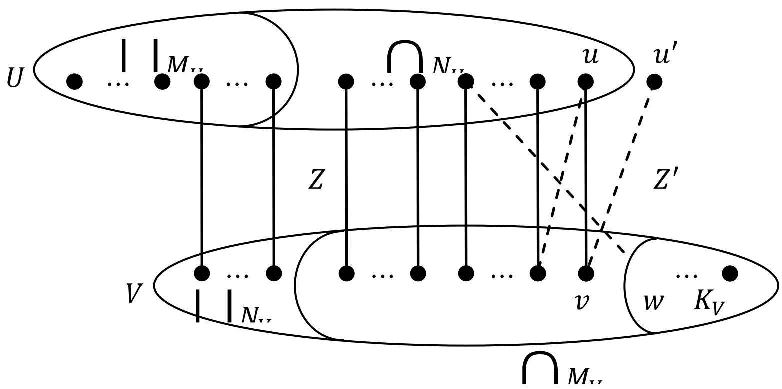

Figure 3

Now we will describe an algorithm, which provides the greatest MMIS of G' based on the greatest MMIS of G. The algorithm takes as input the graph G, the greatest MMIS of it, a maximum matching of it and the “new” vertex; it provides a maximum matching and the greatest MMIS of the “new” graph G':

denote by M the given greatest MMIS of G,

denote by Z the given maximum matching of G,

denote by u' the given “new” vertex,

denote by K the set of vertices which are note adjacent with Z,

if u' is independent with M, then set $M' = M \cup \{u'\}$, set $Z' = Z$ and go to step A8,

if u' is adjacent with some vertex $v \in K_V$, then set $Z' = Z \cup \{(u', v)\}$ and go to step A7,

pick a vertex $v \in M_V \setminus K_V$, which is adjacent with u' and denote by u the vertex for which $(u, v) \in Z$,

call Algorithm 1 for $\{u\}$, denote by p the path it finds to some vertex $w \in K_V$ and set $K' = K \setminus \{w\}$,

set $Z' = Z$, remove even edges of p from Z', add odd edges of p to Z' add (u', v) to Z',

call Algorithm 1 for K_V' and set M' to the set it provides,

provideM' as the MMIS of G' and Z' as a maximum matching of G'.

As Algorithm 1 performs $O(|E|)$ operations in the worst case, then Algorithm 2 also performs $O(|E|)$ operations in the worst case. This means, that the algorithm which sequentially handles vertices of a bipartite graph and for each vertex calls Algorithm 2, performs $O(|E||W|)$ operations in the worst case, where $W = U \cup V$. Next we show that a sort of duality holds between the intersection of all MMIS-es and the union of all maximum matchings. We believe that this duality can yield to a more efficient algorithm which provides the greatest MMIS of G' based on one of G, then Algorithm 2 is. However in this paper we do not provide such algorithm;instead we mention what approaches we plan to use in order to obtain that result in further works.

Let $G = (U, V, E)$ be a bipartite graph, M be a MMIS of G and Z be a maximum matching of G. As it is mentioned before, the complement of M is a minimum vertex cover; we will denote by N. Also we will denote by L the set of vertices which are adjacent with Z, and by K the set of vertices which are not adjacent with Z (i.e. K is the complement of L). As it is mentioned above, from Claim 5 it follows, that for each MMIS M and for each maximum matching Z it holds$K \subseteq M$ and $N \subseteq L$ (see Figure 2). This means that:

$$\cup K \subseteq \cap M \text{ and } \cup N \subseteq \cap L, \quad (5)$$

where the union and intersection operations are taken trough all MMIS-es and trough all maximum matchings. Next we will show that equality holds in (4). Note, that if $\{A\}$ is a family of subsets, and if the complement of subset A is denoted by B, then $\cap B$ is the complement of $\cup A$. Taking into account this and (5), on Figure 4 we schematically illustrate relations between sets K, L, M and N.

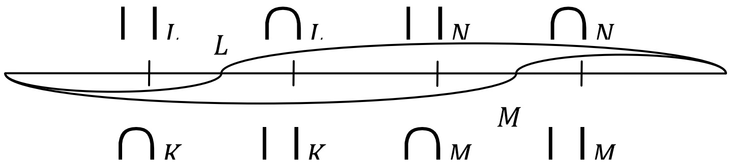

Figure 4

On Figure 4 the horizontal line corresponds to the set of vertices of graph G, sets noted in the same column are the complements of each other, sets noted at the bottom row are listed in increasing order (i.e. the right one is a superset of one on the left) and sets noted at the top row are listed in decreasing order (i.e. the right one is a subset of one on the left). Note that from Claim 8 it follows that $(\cap L) \cap (\cap M) = \emptyset$, so the following duality holds:

$$\cup K = \cap M \text{and } \cup N = \cap L. \quad (6)$$

(4) states that $\vee M = (\cup M_U) \cup (\cap M_V)$, where $\vee M$ is the greatest MMIS of G, so in some sense the duality (6) describes the relation between the greatest MMIS and all maximum matchings of G. Now let G' be a "new"

bipartite graph obtained by adding a new vertex $u'$to G, and $\vee M'$ be the greatest MMIS-es of G'. Next we will discuss the problem of providing $\vee M'$ based on $\vee M$. Analogically with the notations of G, we will denote by$\bigcup M'$ and $\bigcap M'$ respectively the union and the intersection of all MMIS-es of G'. We will also denote by K' a set of vertices of G', which are not adjacent with some maximum matching of G', and by $\bigcup K'$ we will denote the union of all such sets. (6) states that $\bigcap M' = \bigcup K'$. As it follows from Claim 7, if u' is independent with $\bigcap M_V$, then $\vee M' = \vee M \cup \{u\}$, so next we will discuss the case when u' is adjacent with some vertices $D \subseteq \bigcap M_V$. As it is mentioned before, in this case $\vee M$ is a MMIS of G', so $\bigcap M'_V \subseteq \bigcap M_V$. We claim that:

Claim 6: for$v \in V$ it holds$v \in \bigcap M'$ if and only if $D \cap (\bigcup_{v \in K} K) \neq \emptyset$.

Indeed, $\bigcap M' = \bigcup K'$, and v belongs to some K' if and only if there is a maximum matching in G' which is not adjacent with v. This holds if and only if there is a matching in G which is not adjacent with v and is adjacent with some vertex in D. It is easy to see, that this proves the claim. From this claim it follows, that for a vertex $v \in \bigcap M_V$ in order to find out whether $v \in \bigcap M'$ or not, we can check if $\bigcup_{v \in K} K$ has a common vertex with D or not. Thus based on Claim 9 we can provide an algorithm which obtains the greatest MMIS of G' based on the greatest MMIS of G, however performing the mentioned check for all $v \in \bigcap M_V$ cannot be performed efficiently, if we just roughly hold the family of subsets $\{K\}$ and generate $\{K'\}$. We believe that the family of subsets $\{K\}$ has some properties based on which $\{K'\}$ can be obtained efficiently and the check whether $\bigcup_{v \in K} K$ has a common vertex with D or not can be performed efficiently as well. In the next section we conclude this paper and mention about further works.

Conclusion and further works

In the second section of this paper we have shown that the family of MMIS-es of a bipartite graph forms a distributive lattice with respect to join and meet operations defined at (2) and (3). This result is not just of theoretical interest, as in applications where it is required to obtain all MMIS-es of a bipartite graph, the join-irreducible elements of the mentioned lattice can be obtained using Algorithm 1 in $O(|E||W|)$ time, and the obtained structure describes the family of all MMIS-es in the sense of Birkhoff's representation theorem. Algorithm 1 also can be used to obtain the greatest MMIS, as well as any MMIS containing the given set of vertices. In the third section of this paper we present Algorithm 2, which obtains the greatest MMIS of bipartite graph G' based on the greatest MMIS of G in $O(|E|)$ time, where G' is a bipartite graph obtained by “adding” a new vertex to bipartite graph G. Next in that section we prove that duality (6) holds between the MMIS-es and maximum matchings of a bipartite graph. We believe that this duality is not just of theoretical interest, and it can yield to a usable algorithm which sequentially handles vertices of a bipartite graph and maintains the greatest MMIS of it. Claim 9 shows how the duality (6) can be used in order to obtain the greatest MMIS of G' based on the greatest MMIS of G, however in this paper we do not provide an efficient technique of implementing the results of that claim. We believe, that such technique can be obtained using some properties of the family of subsets $\{K\}$, which is the family of sets of vertices which are not adjacent to some maximum matching. In further works we plan to obtain such technique.

Bibliography

[Birkhoff, 1948] G. Birkhoff. Lattice Theory: Revised Edition. American Mathematical Society. 1948.

[Harary, 1969] F. Harary. Graph Theory. Addison-Wesley, Reading. 1969.

[Karp, 1972] R. Karp.Reducibility Among Combinatorial Problems. Complexity of Computer Computations, pp. 85-103. 1972.

[Ford, Fulkerson, 1962] L. Ford, D. Fulkerson. Flows in Networks.Princeton University Press, 113. 1962.

[Hopcroft, Karp, 1973] J. Hopcroft,R. Karp. An n5/2algorithm for maximum matchings in bipartite graphs, SAIM Journal of Computing, 2 (4), pp. 225-231. 1973.

[Johnson, 1988] D. Johnson and M. Yannakanis.On Generating all Maximal Independent sets. Information Processing Letters, 27, pp. 119-123. 1998.

[Kashiwabara, 1992] T. Kashiwabara and S. Masuda, K. Nakajima and T. Fujisawa.Generation of Maximum Independents Sets of a Bipartite Graph and Maximum Cliques of a Circular-Arc Graph. Journal of Algorithms, 13, pp. 161-174. 1992.

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