ON RELIABILITY APPROACH TO MULTIPLE HYPOTHESES TESTING AND TO IDENTIFICATION OF PROBABILITY DISTRIBUTIONS OF TWO STOCHASTICALLY RELATED OBJECTS

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Abstract: *This paper is devoted to study of characteristics of logarithmically asymptotically optimal (LAO) hypotheses testing and identification for a model consisting of two related objects. In general case it is supposed that L_1 possible probability distributions of states constitute the family of possible hypotheses for the first object and the second object is distributed according to one of $L_1 \times L_2$ given conditional distributions depending on the distribution index and the current observed state of the first object. For the first testing procedure the matrix of interdependencies of all possible pairs of the error probability exponents (reliabilities) in asymptotically optimal tests of distributions of both objects is studied. The identification of the distributions of two objects gives an answer to the question whether r_1-th and r_2-th distributions occurred or not on the first and the second objects, correspondingly. Reliabilities for the LAO identification are determined for each pair of double hypotheses. By the second approach the optimal interdependencies of lower estimates of all possible pairs of corresponding reliabilities are found and lower estimates of reliabilities for the LAO identification are studied for each pair of hypotheses. The more complete results are presented for model of statistically dependent objects, when distributions of the objects are dependent, but its current states are independent. For an example of two statistically dependent objects optimal interdependencies of pairs of reliabilities are calculated and graphically presented.*

Keywords: *Multiple hypotheses testing, Identification of distribution, Inference of many objects, Error probability exponents, Reliabilities.*

1. Introduction

As a development of the results on two and on multiple hypotheses logarithmically asymptotically optimal (LAO) testing of probability distributions of one object [1] -- [3], in paper [4] Ahlswede and Haroutunian formulated a number of problems with respect to multiple hypotheses testing and identification for many objects. Haroutunian and Hakobyan solved in [5] the problem of many hypotheses testing for two independent objects and in [6] the problem of the identification of distributions being based on samples of independent observations. In prepublications [7] -- [10] Haroutunian and Yessayan studied many hypotheses LAO testing for two objects under different kinds of relation.

LAO tests of its distributions for two hypotheses were analyzed first by Hoeffding [1], later by Csiszár and Longo [2] and by other authors. Here we investigate characteristics of procedures of LAO testing and identification of probability distributions of two stochastically dependent objects.

Let X_1 and X_2 be random variables (RVs) taking values in the same finite set of states X and $P(X)$ be the space of all possible distributions on X. There are given L_1 probability distributions (PDs) $G_{l_1} = \{G_{l_1}(x^1), x^1 \in X\}$, $l_1 = \overline{1, L_1}$, from $P(X)$. The first object is characterized by RV X_1 which has one of these L_1 possible PDs and the second object is dependent on the first and is characterized by RV X_2 which can have one of $L_1 \times L_2$ conditional PDs $G_{l_2/l_1} = \{G_{l_2/l_1}(x^2 \mid x^1), x^1, x^2 \in X\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$. Joint PDs are $G_{l_1} o G_{l_2/l_1} = \{G_{l_1,l_2}(x^1, x^2), x^1, x^2 \in X\}$, $l_1 = \overline{1, L_1}, l_2 = \overline{1, L_2}$, where $G_{l_1,l_2}(x^1, x^2) = G_{l_1}(x^1) G_{l_2/l_1}(x^2 \mid x^1)$. Let $(x_1, x_2) = ((x_1^1, x_1^2), (x_2^1, x_2^2), ...(x_N^1, x_N^2))$ be a sequence of results of N independent observations of the pair of objects. The probability $G^N_{l_1,l_2}(x_1, x_2)$ of vector (x_1, x_2) is the following product:

$$G^N_{l_1,l_2}(x_1, x_2) = G^N_{l_1}(x_1) G^N_{l_2/l_1}(x_2 \mid x_1) = \prod_{n=1}^{N} G_{l_1}(x_n^1) G_{l_2/l_1}(x_n^2 \mid x_n^1),$$

with $G^N_{l_1}(x_1) = \prod_{n=1}^{N} G_{l_1}(x_n^1)$ and $G^N_{l_2/l_1}(x_2 \mid x_1) = \prod_{n=1}^{N} G_{l_2/l_1}(x_n^2 \mid x_n^1)$.

For the object characterized by X_1 the non-randomized test $\varphi_1^N(x_1)$ can be determined by partition of the sample space X^N on L_1 disjoint subsets $A^N_{l_1} = \{x_1 : \varphi_1^N(x_1) = l_1\}$, $l_1 = \overline{1, L_1}$, i.e. the set $A^N_{l_1}$ consists of vectors x_1 for which the PD G_{l_1} is adopted. The probability $\alpha^N_{l_1|m_1}(\varphi_1^N)$ of the erroneous acceptance of PD G_{l_1} provided that G_{m_1} is true, $l_1, m_1 = \overline{1, L_1}, m_1 \neq l_1$, is defined by the probability $G^N_{m_1}$ of the set $A^N_{l_1}$:

$$\alpha^N_{l_1|m_1}(\varphi_1^N) \stackrel{\Delta}{=} G^N_{m_1}(A^N_{l_1}). \tag{1}$$

We define the probability to reject G_{m_1}, when it is true, as follows

$$\alpha^N_{m_1|m_1}(\varphi^N_1) \overset{\Delta}{=} \sum_{l_1: l_1 \neq m_1} \alpha^N_{l_1|m_1}(\varphi^N_1) = G^N_{m_1}(\overline{A^N_{m_1}}). \tag{2}$$

Denote by φ_1 the infinite sequences of tests for the first object. Corresponding error probability exponents, which we call reliabilities $E_{l_1|m_1}(\varphi_1)$ for test φ_1 are defined as

$$E_{l_1|m_1}(\varphi_1) \overset{\Delta}{=} \overline{\lim_{N\to\infty}} \{-\frac{1}{N} \log \alpha^N_{l_1|m_1}(\varphi^N_1)\},\ m_1, l_1 = \overline{1, L_1}. \tag{3}$$

It follows from (2) and (3) that

$$E_{m_1|m_1}(\varphi_1) = \min_{l_1: l_1 \neq m_1} E_{l_1|m_1}(\varphi_1),\ l_1, m_1 = \overline{1, L_1},\quad l_1 \neq m_1. \tag{4}$$

We shall reformulate now the Theorem from [3] for the case of one object with L_1 hypotheses. This requires some additional notions and notations. For some PD $Q = \{Q(x^1), x^1 \in X\}$ the entropy $H_Q(X_1)$ and the informational divergence $D(Q \| G_{l_1}),\ l_1 = \overline{1, L_1},$ are defined as follows:

$$H_Q(X_1) \overset{\Delta}{=} -\sum_{x^1 \in X} Q(x^1) \log Q(x^1),$$

$$D(Q \| G_{l_1}) \overset{\Delta}{=} \sum_{x^1 \in X} Q(x^1) \log \frac{Q(x^1)}{G_{l_1}(x^1)}.$$

For given positive numbers $E_{1|1}, \ldots, E_{L-1|L-1}$, let us consider the following sets of PDs $Q = \{Q(x^1), x^1 \in X\}$:

$$R_{l_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) \le E_{l_1|l_1}\},\ l_1 = \overline{1, L_1 - 1}, \tag{5a}$$

$$R_{L_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) > E_{l_1|l_1},\ l_1 = \overline{1, L_1 - 1}\}, \tag{5b}$$

and the elements of the reliability matrix $E(\varphi_1^*)$ of the LAO test φ_1^*:

$$E^*_{l_1|l_1} = E^*_{l_1|l_1}(E_{l_1|l_1}) \stackrel{\Delta}{=} E_{l_1|l_1},\ l_1 = \overline{1, L_1 - 1}, \tag{6a}$$

$$E^*_{l_1|m_1} = E^*_{l_1|m_1}(E_{l_1|l_1}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} D(Q \,\|\, G_{m_1}),\ m_1 = \overline{1, L_1},\ m_1 \neq l_1,\ l_1 = \overline{1, L_1 - 1}, \tag{6b}$$

$$E^*_{L_1|m_1} = E^*_{L_1|m_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \stackrel{\Delta}{=} \inf_{Q \in R_{L_1}} D(Q \,\|\, G_{m_1}),\ m_1 = \overline{1, L_1 - 1}, \tag{6c}$$

$$E^*_{L_1|L_1} = E^*_{L_1|L_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \stackrel{\Delta}{=} \min_{l_1 = \overline{1, L_1 - 1}} E^*_{l_1|L_1}. \tag{6d}$$

Theorem 1 [3]: *If all distributions G_{l_1}, $l_1 = \overline{1, L_1}$, are different in the sense that $D(G_{l_1} \| G_{m_1}) > 0$, $l_1 \neq m_1$, and the positive numbers $E_{1|1}, E_{2|2}, ..., E_{L-1|L-1}$ are such that the following inequalities hold*

$$E_{1|1} < \min_{l_1 = \overline{2, L_1}} D(G_{l_1} \,\|\, G_1),$$

$$\text{---} \tag{7}$$

$$E_{m_1|m_1} < \min\left(\min_{l_1 = \overline{m_1 + 1, L_1}} D(G_{l_1} \,\|\, G_{m_1}), \min_{l_1 = \overline{1, m_1 - 1}} E^*_{l_1|m_1}(E_{l_1|l_1})\right),\ m_1 = \overline{2, L_1 - 1},$$

then there exists a LAO sequence of tests φ_1^*, the reliability matrix of which $E(\varphi_1^*) = \{E_{l_1|m_1}(\varphi_1^*)\}$ is defined in (6) and all elements of it are positive.Inequalities (7) are necessary for existence of tests sequence with reliability matrix having in diagonal given elements $E_{l_1|l_1}$, $l_1 = \overline{1, L_1 - 1}$, and all other elements positive .

Corollary 1 [3]: *If, in contradiction to condition of strict positivity, one, or several diagonal elements $E_{m_1|m_1}$, $m_1 = \overline{1, L_1 - 1}$, of the reliability matrix are equal to zero, then the elements of the matrix determined in functions of this $E_{m_1|m_1}$ will be given as in the case of Stein's lemma [11], [12]*

$$E_{l_1|m_1}(E_{m_1|m_1}) = D(G_{l_1} \,\|\, G_{m_1}),\ m_1 \neq l_1,$$

and the remaining elements of the matrix $E(\varphi_1^*)$ will be defined by $E_{l_1|l_1} > 0$, $l_1 \neq m_1$, $l_1 = \overline{1, L_1 - 1}$, as follows from Theorem 1.

Now we formulate the concept of LAO approach to the identification problem for one object, which was introduced in [4]. We have one object, and there are known $L_1 \geq 2$ possible PDs. Identification is the answer to the question whether r_1-th distribution is correct, or not. As in the testing problem, the answer must be given on the base of a sample x with the help of an appropriate test.

There are two error probabilities for each $r_1 \in [1, L_1]$: the probability $\alpha_{l_1 \neq r_1 | m_1 = r_1}(\varphi_N)$ to accept l-th PD different from r_1, when PD r_1 is correct, and the probability $\alpha_{l_1 = r_1 | m_1 \neq r_1}(\varphi_N)$ that r_1 is accepted, when it is not correct.

The probability $\alpha_{l_1 \neq r_1 | m_1 = r_1}(\varphi_N)$ coincides with the probability $\alpha_{r_1 | r_1}(\varphi_N)$ which is equal to $\sum\limits_{l_1 : l_1 \neq r_1} \alpha_{l_1 | r_1}(\varphi_N)$.

The corresponding reliability $E_{l_1 \neq r_1 | m_1 = r_1}(\varphi)$ is equal to $E_{r_1 | r_1}(\varphi)$ which satisfies equality (4).

The reliability approach to identification assumes determining the optimal dependence of $E^*_{l_1 = r_1 | m_1 \neq r_1}$ upon given $E^*_{l_1 \neq r_1 | m_1 = r_1} = E^*_{r_1 | r_1}$, which can be an assigned value satisfying conditions (7). The solution of this problem assumes knowledge of some a priori PDs of the hypotheses.

The result from paper [4] is valid for the first object.

Theorem 2 [4]: *In case of distinct hypothetical PDs $G_1, G_2, ..., G_{L_1}$, under condition that the probabilities of all L_1 hypotheses are strictly positive for given $E^*_{l_1 \neq r_1 | m_1 = r_1} = E^*_{r_1 | r_1}$ the reliability $E^*_{l_1 = r_1 | m_1 \neq r_1}$ is the following:*

$$E^*_{l_1 = r_1 | m_1 \neq r_1}(E^*_{r_1 | r_1}) = \min_{m_1 : m_1 \neq r_1} \inf_{Q : D(Q \| G_{r_1}) \leq E_{r_1 | r_1}} D(Q \| G_{m_1}),\ r_1 = \overline{1, L_1}.$$

In Section 2 we consider two related objects as one complex object and we obtain corresponding reliabilities for LAO testing and identification. In Section 3 we will obtain the lower estimates of the reliabilities for LAO testing and in Section 4 for identification for the dependent object. These estimates serve for deducing of lower estimates of the reliabilities for LAO testing (in Section 5) and identification (in Section 6) of distributions of two related objects. Results of certain calculations for an example will be graphically presented in Section 7.

2. LAO Testing and Identification of the Probability Distributions for Two Stochastically Coupled Objects

We expose the direct approach for LAO testing and identification of PDs for two related objects. It consists in considering the pair of objects as one composite object [10]. The test, which we denote by Φ^N , is a procedure of making decision about unknown indices of PDs on the base of results of N observations (x_1, x_2) . For the objects characterized by X_1, X_2 the non-randomized test $\Phi^N(x_1, x_2)$ can be determined by partition of the sample space $(X \times X)^N$ on $L_1 \times L_2$ disjoint subsets $A^N_{l_1,l_2} = \{(x_1, x_2) : \Phi^N(x_1, x_2) = (l_1, l_2)\},\ l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$, i.e. the set $A^N_{l_1,l_2}$ consists of vectors (x_1, x_2) for which the PD G_{l_1,l_2} must be adopted. The probability $\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N)$ of the erroneous acceptance of PD G_{l_1,l_2} provided that G_{m_1,m_2} is true, $l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2},\ (m_1, m_2) \neq (l_1, l_2)$ is defined by the set $A^N_{l_1,l_2}$

$$\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N) \overset{\Delta}{=} G^N_{m_1,m_2}(A^N_{l_1,l_2}). \tag{8}$$

We define the probability to reject G_{m_1,m_2} , when it is true, as follows

$$\alpha^N_{m_1,m_2|m_1,m_2}(\Phi^N) \overset{\Delta}{=} \sum_{(l_1,l_2)\neq(m_1,m_2)} \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N) = G^N_{m_1,m_2}(\overline{A^N_{m_1,m_2}}). \tag{9}$$

Our intention is to study the reliabilities of the infinite sequence of tests Φ

$$E_{l_1,l_2|m_1,m_2}(\Phi) \overset{\Delta}{=} \overline{\lim_{N\to\infty}}\{-\frac{1}{N}\log \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N)\},\ l_1, m_1 = \overline{1, L_1},\ l_2,\ m_2 = \overline{1, L_2}. \tag{10}$$

From (9) and (10) we deduce that

$$E_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2)\neq(m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \tag{11}$$

The matrix $E(\Phi)=\{E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1,m_1=\overline{1,L_1},\ l_2,m_2=\overline{1,L_2}\}$ is called the reliability matrix of the sequence of tests Φ. Our aim is to investigate the reliability matrix of optimal tests, and the conditions ensuring positivity of all its elements.

For given positive numbers $E_{1,1|1,1},...,E_{L_1,L_2-1|L_1,L_2-1}$, let us consider the following sets of PDs

$$QoV \overset{\Delta}{=} \{Q(x^1)V(x^2\mid x^1), x^1, x^2\in X\}:$$

$$R_{l_1,l_2} \overset{\Delta}{=} \{QoV : D(QoV\,\|\,G_{l_1,l_2}) \le E_{l_1,l_2|l_1,l_2}\},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{12a}$$

$$R_{L_1,L_2} \overset{\Delta}{=} \{QoV : D(QoV\,\|\,G_{l_1,l_2}) > E_{l_1,l_2|l_1,l_2},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}\}, \tag{12b}$$

and the elements of the reliability matrix E^* of the LAO test:

$$E^*_{l_1,l_2|l_1,l_2} = E^*_{l_1,l_2|l_1,l_2}(E_{l_1,l_2|l_1,l_2}) \overset{\Delta}{=} E_{l_1,l_2|l_1,l_2},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{13a}$$

$$E^*_{l_1,l_2|m_1,m_2} = E^*_{l_1,l_2|m_1,m_2}(E_{l_1,l_2|l_1,l_2}) \overset{\Delta}{=} \inf_{QoV\in R_{l_1,l_2}} D(QoV\,\|\,G_{m_1,m_2}),\ m_1=\overline{1,L_1},$$
$$m_2=\overline{1,L_2},\ (l_1,l_2)\ne(m_1,m_2),\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{13b}$$

$$E^*_{L_1,L_2|m_1,m_2} = E^*_{L_1,L_2|m_1,m_2}(E_{1,1|1,1},E_{1,2|1,2},E_{1,3|1,3}...,E_{L_1,L_2-1|L_1,L_2-1})$$
$$\overset{\Delta}{=} \inf_{QoV\in R_{L_1,L_2}} D(QoV\,\|\,G_{m_1,m_2}),\ m_1=\overline{1,L_1},\ m_2=\overline{1,L_2-1}, \tag{13c}$$

$$E^*_{L_1,L_2|L_1,L_2} = E^*_{L_1,L_2|L_1,L_2}(E_{1,1|1,1},E_{1,2|1,2},E_{1,3|1,3}...,E_{L_1,L_2-1|L_1,L_2-1}) \overset{\Delta}{=} \min_{l_1=\overline{1,L_1}}\ \min_{l_2=\overline{1,L_2-1}} E^*_{l_1,l_2|L_1,L_2}. \tag{13d}$$

For simplicity we can take $(X_1, X_2) = $ Y, $X \times X = Y$ and $y = (y_1, y_2,..., y_N) \in Y^N$, where $y_n = (x_n^1, x_n^2), n = \overline{1, N}$, then we will have $L_1 \times L_2 = L$ new hypotheses for one object

$$G_{l_1,l_2} = \{G_{l_1}(x^1)G_{l_2/l_1}(x^2 \mid x^1), x^1, x^2 \in X\},\ l_1 = \overline{1, L_1}, l_2 = \overline{1, L_2} \text{ , where } G_{1,1} = K_1 \text{ ,}$$

$$G_{1,2} = K_2 \text{ , } G_{1,3} = K_3 \text{ ,..., } G_{1,L_2} = K_{L_2} \text{ , } G_{2,1} = K_{L_2+1} \text{ ,..., } G_{l_1,l_2} = K_{(l_1-1)L_2+l_2} \text{ ,}$$

$$l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2} \text{ , } \alpha_{l_1,l_2|m_1,m_2} = \alpha_{(l_1-1)L_2+l_2|(m_1-1)L_2+m_2} \text{ , } l_1 = \overline{1, L_1},\ \ l_2 = \overline{1, L_2}$$

$$E_{l_1,l_2|m_1,m_2} = E_{(l_1-1)L_2+l_2|(m_1-1)L_2+m_2} \text{ , } l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2}$$

and thus we have brought the original problem to the case of one object with $L_1 \times L_2$ hypotheses.

So applying Theorem 1 we can deduce that there exists a LAO sequence of tests Φ^* , the reliability matrix of which $E^* = \{E_{l|m}(\Phi^*)\}$ is defined in (13) and all elements of it are positive.

Using Theorem 2 for this composite object we can deduce that identification reliabilities are connected with the following formula

$$E_{l=r|m\neq r}(E_{r|r}) = \min_{m:m\neq r} \inf_{QoV:D(QoV\|K_r)\leq E_{r|r}} D(QoV \| K_m),\ r \in [1, L]. \tag{14}$$

Now let us consider the more general particular model, when X_1 and X_2 are related statistically, in the following way $G_{l_1,l_2}(x^1, x^2) = G_{l_1}(x^1)G_{l_2/l_1}(x^2)$. The probability of vector (x_1, x_2) is defined by the following PD G_{l_1,l_2}

$$G^N_{l_1,l_2}(x_1, x_2) = G^N_{l_1}(x_1)G^N_{l_2/l_1}(x_2) = \prod_{n=1}^N G_{l_1}(x_n^1)G_{l_2/l_1}(x_n^2),$$

where $G^N_{l_1}(x_1) = \prod_{n=1}^N G_{l_1}(x_n^1)$ and $G^N_{l_2/l_1}(x_2) = \prod_{n=1}^N G_{l_2/l_1}(x_n^2).$

In this case we can analogously bring the problem to the problem on one object with $L_1 \times L_2 = L$ hypotheses, where $G_{l_1,l_2} = \{G_{l_1}(x^1)G_{l_2/l_1}(x^2), x^1, x^2 \in X\},\ l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2}$, and for the sets R_{l_1,l_2}, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$ of PDs $QoV \overset{\Delta}{=} \{Q(x^1)V(x^2), x^1, x^2 \in X\}$:

When the objects X_1 and X_2 can have only different distributions from same L given probability distributions (PD) G_1, G_2, G_3, ..., G_L from $P(X)$, [4], [7] we can reduce the problem to the problem of one object and $L\times(L-1)$ hypotheses, where $G_{l_1,l_2}=\{G_{l_1}(x^1)G_{l_2}(x^2),x^1,x^2\in X\}$, $l_1,l_2=\overline{1,L}$, $l_1\neq l_2$ (see [4], [7]).

3. An Approach to Multiple Hypotheses Testing for the Second (Dependent) Object

Let us remark that test Φ^N can be composed of a pair of tests φ_1^N and φ_2^N for the separate objects: $\Phi^N=(\varphi_1^N,\varphi_2^N)$. For the second object characterized by RV X_2 depending on X_1 the non-randomized test $\varphi_2^N(x_2,x_1,l_1)$ based on vectors x_1,x_2 and on the index of the hypothesis l_1 adopted for X_1, can be given for each l_1 and x_1 by division of the sample space X^N on L_2 disjoint subsets $A^N_{l_2/l_1}(x_1)=\{x_2:\varphi_2^N(x_2,x_1,l_1)=l_2\}$, $l_1=\overline{1,L_1}$, $l_2=\overline{1,L_2}$. We upper estimate the error probabilities for second object proceeding from definition (8).

$$G^N_{m_1,m_2}(A^N_{l_1,l_2})=\sum_{x_1\in A^N_{l_1}}G^N_{m_1}(x_1)G^N_{m_2/m_1}(A^N_{l_2/l_1}(x_1)\,|\,x_1)\leq\max_{x_1\in A^N_{l_1}}G^N_{m_2/m_1}(A^N_{l_2/l_1}(x_1)\,|\,x_1)\sum_{x_1\in A^N_{l_1}}G^N_{m_1}(x_1)$$

$$=G_{m_1}(A^N_{l_1})\max_{x_1\in A^N_{l_1}}G^N_{m_2/m_1}(A^N_{l_2/l_1}(x_1)\,|\,x_1)\stackrel{\Delta}{=}\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N),(l_2,l_1)\neq(m_1,m_2). \tag{15}$$

Consequently we can deduce that "reliabilities"

$$F_{l_1,l_2|m_1,m_2}(\Phi)\stackrel{\Delta}{=}\overline{\lim_{N\to\infty}}\{-\frac{1}{N}\log\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N)\},(l_2,l_1)\neq(m_1,m_2),$$

$$l_1,m_1=\overline{1,L_1},\ l_2,m_2=\overline{1,L_2},$$

and

$$F_{m_1,m_2|m_1,m_2}(\Phi)=\min_{(l_1,l_2)\neq(m_1,m_2)}F_{l_1,l_2|m_1,m_2}(\Phi) \tag{16}$$

are lower estimates for $E_{l_1,l_2|m_1,m_2}(\Phi)$.

We can also introduce

$$\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N) \stackrel{\Delta}{=} \max_{x_1 \in A^N_{l_1}} G^N_{m_2/m_1}(A^N_{l_2/l_1}(x_1) \mid x_1),\ l_2 \neq m_2,\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2},\ m_2 \neq l_2,$$

We define also

$$\beta^N_{m_2|l_1,m_1,m_2}(\varphi_2^N) \stackrel{\Delta}{=} \max_{x_1 \in A^N_{l_1}} G^N_{m_2/m_1}(\overline{A^N_{m_2/l_1}(x_1)} \mid x_1) = \sum_{l_2 \neq m_2} \beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N). \tag{17}$$

The corresponding estimates of the reliabilities of test φ_2^N , are the following

$$F_{l_2|l_1,m_1,m_2}(\varphi_2) \stackrel{\Delta}{=} \overline{\lim_{N\to\infty}}\{-\frac{1}{N}\log \beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N)\},\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2},\ m_2 \neq l_2. \tag{18}$$

It is clear from (17) that

$$F_{m_2|l_1,m_1,m_2}(\varphi_2) = \min_{l_2: l_2 \neq m_2} F_{l_2|l_1,m_1,m_2}(\varphi_2),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \tag{19}$$

We need some notions and estimates from the method of types [11], [12]. The type of a vector x_1 is a PD

$$Q_{x_1} = \{Q_{x_1}(x^1) = \frac{1}{N} N(x^1 \mid x_1), x^1 \in X\},$$

where $N(x^1 \mid x^1)$ is the number of repetitions of the symbol x^1 in vector x_1. The subset of $P(X)$ consisting of the possible types of sequences $x_1 \in X^N$ is denoted by $P_N(X)$. The set of all vectors x_1 of the type Q_{x_1}

is denoted by $T^N_{Q_{x_1}}(X_1)$, remark that $T^N_Q(X_1)=\varnothing$ for $Q \notin P_N(X)$. The following estimates for the set $T^N_{Q_{x_1}}(X_1)$ hold

$$(N+1)^{-|X|}\exp\{NH_{Q_{x_1}}(X_1)\} \le |T^N_{Q_{x_1}}(X_1)| \le \exp\{NH_{Q_{x_1}}(X_1)\}.$$

For a pair of sequences $(x_1,x_2)\in X^N \times X^N$ let $N(x^1,x^2 \mid x_1,x_2)$ be the number of occurrences of pair $(x^1,x^2)\in X\times X$ in the similar places in the pair of vectors (x_1,x_2). The joint type of the pair (x_1,x_2) is PD $Q_{x_1,x_2}=\{Q_{x_1,x_2}(x^1,x^2), x^1,x^2\in X\}$ defined by

$$Q_{x_1,x_2}(x^1,x^2) \overset{\Delta}{=} \frac{1}{N} N(x^1,x^2 \mid x_1,x_2), \quad x^1,x^2 \in X.$$

The conditional type of x_2 for given x_1 is the conditional distribution

$V_{x_1,x_2} \overset{\Delta}{=} \{V_{x_2,x_1}(x^2 \mid x^1), x^1,x^2\in X\}$ defined as follows:

$$V_{x_1,x_2}(x^2 \mid x^1) \overset{\Delta}{=} \frac{Q_{x_1,x_2}(x^1,x^2)}{Q_{x_1}(x^1)} = \frac{N(x^1,x^2 \mid x_1,x_2)}{N(x^1 \mid x_1)}, \quad x^1,x^2\in X.$$

The conditional entropy of RV X_2 for given X_1 is:

$$H_{Q_{x_1},V_{x_1,x_2}}(X_2 \mid X_1) = -\sum_{x^1,x^2} Q_{x_1}(x^1) V_{x_1,x_2}(x^2 \mid x^1)\log V_{x_1,x_2}(x^2 \mid x^1).$$

For some conditional PD $V=\{V(x^2 \mid x^1), x^1,x^2\in X\}$ the conditional divergences of PD $\{Q(x^1)V(x^2 \mid x^1), x^1,x^2\in X\}$ with respect to PD $\{Q(x^1)G_{l_2/l_1}(x^2 \mid x^1), x^1,x^2\in X\}$ for all l_1,l_2 are defined as follows

$$D(V \| G_{l_2/l_1} \mid Q) \stackrel{\Delta}{=} \sum_{x^1,x^2} Q(x^1)V(x^2 \mid x^1)\log\frac{V(x^2 \mid x^1)}{G_{l_2/l_1}(x^2 \mid x^1)},$$

also

$$D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q) \stackrel{\Delta}{=} \sum_{x^1,x^2} Q(x^1)G_{l_2/l_1}(x^2 \mid x^1)\log\frac{G_{l_2/l_1}(x^2 \mid x^1)}{G_{m_2/m_1}(x^2 \mid x^1)}.$$

The family of vectors x_2 of the conditional type V for given x_1 of the type Q_{x_1} is denoted by $T^N_{Q_{x_1},V}(X_2 \mid x_1)$ and called V-shell of x_1. The set of all possible V-shells for x_1 of type Q_{x_1} is denoted by $V_N(X, Q_{x_1})$. For any conditional type V and $x_1 \in T^N_{Q_{x_1}}(X_1)$ it is known that

$$(N+1)^{-|X|^2}\exp\{NH_{Q_{x_1},V}(X_2 \mid X_1)\} \le |T^N_{Q_{x_1},V}(X_2 \mid x_1)| \le \exp\{NH_{Q_{x_1},V}(X_2 \mid X_1)\}. \quad (20)$$

For given positive numbers $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$, for $Q \in R_{l_1}(5.a),(5.b)$ and for each pair $l_1, m_1 = \overline{1, L_1}$ let us define the following regions and values:

$$R_{l_2/l_1}(Q) \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1} \mid Q) \le F_{l_2|l_1,m_1,l_2}\},\ l_2 = \overline{1, L_2 - 1}, \quad (21a)$$

$$R_{L_2/l_1}(Q) \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1} \mid Q) > F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}\}, \quad (21b)$$

$$R^N_{l_2/l_1}(Q_{x_1}) = R_{l_2/l_1}(Q) \cap V_N(X, Q_{x_1})$$

$$F^*_{l_2|l_1,m_1,l_2} = F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,l_2}) \stackrel{\Delta}{=} F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}, \quad (22a)$$

$$F^*_{l_2|l_1,m_1,m_2} = F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V \in R_{l_2/l_1}(Q)} D(V \| G_{m_2/m_1} \mid Q),\ m_2 = \overline{1, L_2},\ m_2 \ne l_2,$$
$$l_2 = \overline{1, L_2 - 1}, \quad (22b)$$

$$F^*_{L_2|l_1,m_1,m_2} = F^*_{L_2|l_1,m_1,m_2}(F_{1|l_1,m_1,1}, E_{2|l_1,m_1,2}, \ldots, F_{L_2-1|l_1,m_1,L_2-1})$$
$$\stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V \in R_{L_2/l_1}(Q)} D(V \| G_{m_2/m_1} \mid Q),\ m_2 = \overline{1, L_2 - 1}, \quad (22c)$$

$$F^*_{L_2|l_1,m_1,L_2} = F^*_{L_2|l_1,m_1,L_2}(F_{1|l_1,m_1,1}, F_{2|l_1,m_1,2},\dots,F_{L_2-1|l_1,m_1,L_2-1}) \stackrel{\Delta}{=} \min_{\overline{l_2=1,L_2-1}} F^*_{l_2|l_1,m_1,L_2}. \quad (22d)$$

We denote by $F(\varphi_2)$ the matrix of lower estimates for $E(\varphi_2)$.

Theorem 3: *If for fixed* $m_1, l_1 = \overline{1, L_1}$ *all conditional PDs* G_{l_2/l_1}, $l_2 = \overline{1, L_2}$, *are different in the sense that* $D(G_{l_2/l_1} \| G_{m_2/m_1} | Q) > 0$, *for all* $Q \in R_{l_1}$, $l_2 \neq m_2$, $m_2 = \overline{1, L_2}$, *when the numbers* $F_{1|l_1,m_1,1}, F_{2|l_1,m_1,2},\dots,F_{L_2-1|l_1,m_1,L_2-1}$ *are such that the following inequalities hold*

$$0 < F_{1|l_1,m_1,1} < \min_{\overline{l_2=2,L_2}} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1} \| G_{1/m_1} | Q), \quad (23a)$$

$$0 < F_{m_2|l_1,m_1,m_2} < \min\Big(\min_{\overline{l_2=m_2+1,L_2}} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} | Q),\ \min_{\overline{l_2=1,m_2-1}} F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2})\Big),$$
$$for\ m_2 = \overline{2, L_2 - 1}, \quad (23b)$$

then there exists a LAO sequence of tests φ_2^*, the matrix of lower estimate of which $F(\varphi_2^*)$ is defined in (22) with all elements of it strictly positive.

Inequalities (23) are necessary for existence of test sequence with matrix of lower estimates $F(\varphi_2^*)$ having in diagonal given elements $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$, and other elements positive.

Proof: For $x_1 \in X^N$, $x_2 \in T^N_{Q_{x_1},V}(X_2 | x_1)$ the conditional probability $G^N_{m_2/m_1}(x_2 | x_1)$ can be presented as follows

$$G^N_{m_2/m_1}(x_2 | x_1) = \prod_{n=1}^{N} G_{m_2/m_1}(x_n^2 | x_n^1)$$
$$= \prod_{x^1,x^2} G_{m_2/m_1}(x^2 | x^1)^{N(x^1,x^2|x_1,x_2)} = \prod_{x^1,x^2} G_{m_2/m_1}(x^2 | x^1)^{NQ_{x_1}(x^1)V(x^2|x^1)}$$
$$= \exp\{N \sum_{x^1,x^2} [-Q_{x_1}(x^1)V(x^2 | x^1)\log\frac{V(x^2 | x^1)}{G_{m_2/m_1}(x^2 | x^1)} + Q_{x_1}(x^1)V(x^2 | x^1)\log V(x^2 | x^1)]\}$$
$$= \exp\{-N[D(V \| G_{m_2/m_1} | Q_{x_1}) + H_{Q_{x_1},V}(X_2 | X_1)]\}. \quad (24)$$

We shall prove that the sequence of tests φ_2^*, defined for each $x_1 \in B_{l_1}^N = \bigcup_{Q \in R_{l_1}} T_Q^N(X_1)$ by the following collection of sets constructed of conditional types

$$B_{l_2/l_1}^{(N)}(x_1) = \bigcup_{V \in R_{l_2/l_1}^N(Q_{x_1})} T_{Q_{x_1},V}^N(X_2 \mid x_1), l_2 = \overline{1, L_2}, \tag{25}$$

is optimal with respect to lower estimates of corresponding reliabilities and the lower estimate matrix $F(\varphi_2^*)$ is defined in (22). First we show that each N-vector x_2 is in one and only one of $B_{l_2/l_1}^{(N)}(x_1)$, that is

$$B_{l_2/l_1}^{(N)}(x_1) \bigcap B_{m_2/l_1}^{(N)}(x_1) = \varnothing, l_2 = \overline{1, L_2 - 1},\ m_2 = \overline{l_2 + 1, L_2},\ and\ \bigcup_{l_2=1}^{L_2} B_{l_2/l_1}^{(N)}(x_1) = X^N.$$

Really, (21.b) and (25) show that

$$B_{l_2/l_1}^{(N)}(x_1) \bigcap B_{L_2/l_1}^{(N)}(x_1) = \varnothing,\ l_2 = \overline{1, L_2 - 1}.$$

For $l_2 = \overline{1, L_2 - 2}$, $m_2 = \overline{l_2 + 1, L_2 - 1}$, for each $x_1 \in B_{l_1}^N$ let us consider arbitrary $x_2 \in B_{l_2/l_1}^{(N)}(x_1)$. It follows from (17.a) and (21) that if $Q_{x_1} \in P_N(X)$ there are $V \in V_N(X, Q_{x_1})$ such that $D(V \| G_{l_2/l_1} \mid Q_{x_1}) \le F_{l_2|l_1,m_1,l_2}$ and $x_2 \in T_{Q_{x_1},V}^N(X_2 \mid x_1)$. From (21) -- (23) we have $F_{m_2|l_1,m_1,m_2} < F_{l_2|l_1,m_1,m_2}^*(F_{l_2|l_1,m_1,l_2}) < D(V \| G_{m_2/l_1} \mid Q_{x_1})$. From definition (25) for each m_1 we see that $x_2 \notin B_{m_2/m_1}^{(N)}(x_1)$, that is $x_2 \notin B_{m_2/l_1}^{(N)}(x_1)$.

Now for $m_2 = \overline{1, L_2 - 1}$, $l_1 \ne m_1$ using (17), (20), (21), (23) -- (25) we can upper estimate $\beta_{m_2|l_1,m_1,m_2}^{*N}$ as follows:

$$\beta^{*N}_{m_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(\overline{B^{(N)}_{m_2/l_1}(x_1)} \mid x_1) \le \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}\left(\bigcup_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1 \right)$$

$$\le (N+1)^{|X|^2} \max_{x_1 \in B^N_{l_1}} \sup_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} G_{m_2/m_1}(T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1)$$

$$\le (N+1)^{|X|^2} \sup_{Q_{x_1} \in R^N_{l_1}} \sup_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} \exp\{-ND(V \| G_{m_2/m_1} \mid Q_{x_1})\}$$

$$\le \exp\{-N[\inf_{Q_{x_1} \in R^N_{l_1}} \inf_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} D(V \| G_{m_2/m_1} \mid Q_{x_1}) - o_N(1)]\} \le \exp\{-N[F_{m_2|l_1,m_1,m_2} - o_N(1)]\}$$

For $l_2 \ne m_2$ we estimate by analogy

$$\beta^{*N}_{l_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(B^{(N)}_{l_2/l_1}(x_1) \mid x_1) = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(\bigcup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1)$$

$$\le (N+1)^{|X|^2} \max_{x_1 \in B^N_{l_1}} \sup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} G^N_{m_2/m_1}(T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1) \tag{26}$$

$$\le (N+1)^{|X|^2} \sup_{Q_{x_1} \in R^N_{l_1}} \sup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} \exp\{-ND(V \| G_{m_2/m_1} \mid Q_{x_1})\}$$

$$\le \exp\{-N[\inf_{Q_{x_1} \in R^N_{l_1}} \inf_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} D(V \| G_{m_2/m_1} \mid Q_{x_1}) - o_N(1))]\}.$$

Now we want to deduce the lower estimate

$$\beta^{*N}_{l_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(B^{(N)}_{l_2/l_1}(x_1) \mid x_1) = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(\bigcup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1)$$

$$\ge \max_{x_1 \in B^N_{l_1}} \sup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} G^N_{m_2/l_1}(T_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1) \ge (N+1)^{-|X|^2} \sup_{Q_{x_1} \in R^N_{l_1}} \sup_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} \exp\{-ND(V \| G_{m_2/m_1} \mid Q_{x_1})\}. \tag{27}$$

$$\ge \exp\{-N[\inf_{Q_{x_1} \in R^N_{l_1}} \inf_{V:V \in R^N_{l_2/l_1}(Q_{x_1})} D(V \| G_{m_2/m_1} \mid Q_{x_1}) + o_N(1))]\}.$$

Taking into account (26), (27) and the continuity of the functional $D(V \| G_{m_2/l_1} | Q)$ we obtain that $\lim_{N\to\infty}\{-N^{-1}\log\beta^{*N}_{l_2|l_1,m_1,m_2}\}$ exists and in correspondence with (22.b) equals to $F^{*}_{l_2|l_1,m_1,m_2}$. Thus $F_{l_2|l_1,m_1,m_2}(\varphi_2^*) = F^{*}_{l_2|l_1,m_1,m_2}$, $m_2 = \overline{1, L_2}$, $l_2 = \overline{1, L_2}$.

The proof of the first part of the theorem will be accomplished if we show that the sequence of the tests φ_2^* for given $F_{1|l_1.m_1,1},..., F_{L_2-1|l_1.m_1,L_2-1}$ and for any sequence of tests φ_2^{**} is such that for all $m_2, l_2 = \overline{1, L_2}$, $F^{**}_{l_2|l_1,m_1,m_2} \le F^{*}_{l_2|l_1,m_1.m_2}$.

Consider sequence φ_2^{**} of tests, which is defined by the sets $D^{(N)}_{1/l_1}(x_1), D^{(N)}_{2/l_1}(x_1), ..., D^{(N)}_{L_2/l_1}(x_1)$ such that $F^{**}_{l_2|l_1,m_1,m_2} \ge F^{*}_{l_2|l_1,m_1,m_2}$ for some l_2, m_2. For a large enough N we can replace this condition by the following inequality

$$\beta^{**N}_{l_2|l_1,m_1,m_2} \le \beta^{*N}_{l_2|l_1,m_1,m_2}. \tag{28}$$

Examine the sets $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{l_2/l_1}(x_1)$, $l_2 = \overline{1, L_2 - 1}$. This intersection cannot be empty, because in that case

$$\beta^{**N}_{l_2|l_1,m_1,l_2} = \max_{x_1 \in B^N_{l_1}} G^N_{l_2/l_1}(\overline{D}^{(N)}_{l_2/l_1}(x_1) | x_1) \ge \max_{x_1 \in B^N_{l_1}} G^N_{l_2/l_1}(B^{(N)}_{l_2/l_1}(x_1) | x_1)$$

$$\ge \max_{x_1 \in B^N_{l_1}} \sup_{V: D(V \| G_{l/l_1} | Q_{x_1}) \le F_{l_2|l_1,m_1,l_2}} G^N_{l_2/l_1}(T^N_{Q_{x_1},V}(X_2 | x_1) | x_1) \ge \exp\{-N(F_{l_2|l_1,m_1,l_2} + o_N(1))\},$$

and we have a contradiction with (28). Let us show that $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{m_2/l_1}(x_1) = \varnothing, m_2, l_2 = \overline{1, L_2 - 1},$ $l_2 \ne m_2$. If there exists V such that $D(V \| G_{m_2/l_1} | Q) \le F_{m_2|l_1,m_1,m_2}$ and $T^N_{V,Q_{x_1}}(X_2 | x_1) \in D^{(N)}_{l_2/l_1}(x_1)$, then

$$\beta^{**N}_{l_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(D^{(N)}_{l_2/l_1}(x_1) | x_1) > \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(T^N_{V,Q_{x_1}}(X_2 | x_1) | x_1) \ge \exp\{-N[F_{m_2|l_1,m_1,m_2} + o_N(1)]\}.$$

When $\varnothing \neq D_{l_2/l_1}^{(N)} \bigcap T_{V,Q_{x_1}}^{N}(X_2 \mid x_1) \neq T_{V,Q_{x_1}}^{N}(X_2 \mid x_1)$, we also obtain that

$$\beta_{l_2|l_1,m_1,m_2}^{**N} = \max_{x_1 \in B_{l_1}^N} G_{m_2/l_1}^N (D_{l_2/l_1}^{(N)} \mid x_1) > \max_{x_1 \in B_{l_1}^N} G_{m_2/l_1}^N (D_{l_2/l_1}^{(N)} \bigcap T_{V,Q_{x_1}}^N (X_2 \mid x_1) \mid x_1) \geq \exp\{-N(F_{m_2|l_1,m_2,m_2} + O_N(1))\}.$$

Thus we conclude that $F_{l_2|l_1,m_1,m_2}^{**} < F_{m_2|l_1,m_1,m_2}$, which contradicts to (19). Hence we obtain that $D_{l_2/l_1}^{(N)}(x_1) \bigcap B_{l_2/l_1}^{(N)}(x_1) = B_{l_2/l_1}^{(N)}(x_1)$ for $l_2 = \overline{1, L_2 - 1}$.

The following intersection $D_{l_2/l_1}^{(N)}(x_1) \bigcap B_{L_2/l_1}^{(N)}(x_1)$ is empty too, because otherwise we arrive to

$$\beta_{L_2|l_1,m_1,m_2}^{**N} \geq \beta_{L_2|l_1,m_1,m_2}^{*N},$$

which contradicts to (28), it means that $D_{l_2/l_1}^{(N)}(x_1) = B_{l_2/l_1}^{(N)}(x_1)$, for all $l_2 = \overline{1, L_2}$.

The proof of the second part of the Theorem is simple. If one of the conditions (23) is violated, then from (21), (22) and (23) -- (26) it follows that at least one of the elements $F_{l_2|l_1,m_1,m_2}$ is equal to 0. For example, let $F_{m_2|l_1,m_1,m_2} \geq \min_{l_2 = \overline{m_2+1, L_2}} \min_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q)$, then there is $l_2^{'} \in \overline{m_2 + 1, L_2}$ such that $F_{m_2|l_1,m_1,m_2} \geq \min_{Q \in R_{l_1}} D(G_{l_2^{'}/l_1} \| G_{m_2/m_1} \mid Q)$. After using (22b) we obtain that $F_{m_2|m_1,l_1,l_2^{'}}^{*} = 0$. From (19) we see that $F_{m_2|l_1,m_1,m_2} \leq \min_{l_2 = \overline{1, m_2 - 1}} F_{l_2|l_1,m_1,m_2}^{*}(F_{l_2|l_1,m_1,l_2})$. Theorem is proved.

Corollary 2: *If in contradiction to conditions (23) one, or several diagonal elements* $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$, *of the reliability matrix are equal to zero, then the elements of the matrix determined in functions of this* $F_{l_2|l_1,m_1,l_2}$ *are given as in the case of Stein's lemma [11], [12]*

$$F_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,l_2}) = \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q),\ m_1 = \overline{1, L_1},\ m_1 \neq l_1,$$

and the remaining elements of the matrix $F(\varphi_2^*)$ are defined in function of positive $F_{l_2|l_1,m_1,l_2} > 0$, $l_1 \neq m_1$, $l_1 = \overline{1, L_1 - 1}$, as follows from Theorem 3.

Proof: Really, if $F_{l_2|l_1,m_1,l_2} = 0$, then $\beta^N_{l_2|l_1,m_1,l_2}$ is not exponentially decreasing . Thus using Stein's lemma we have

$$\lim_{N\to\infty} \log \frac{1}{N} \beta^{(N)}_{l_2|l_1,m_1,m_2} (\beta^{(N)}_{l_2|l_1,m_1,l_2}(\varphi_2) = c) = -\inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \,\|\, G_{m_2/m_1} \,|\, Q), \, l_2 \neq m_2.$$

So the corollary is proved.

4. On Identification of the Probability Distribution of the Dependent Object

In this section we will obtain the lower estimates of the reliabilities of LAO identification for dependent object. Then we deduce the lower estimates of the reliabilities for LAO identification of two related objects.

There exist two error probabilities for each $r_2 = \overline{1, L_2}$: the probability $\alpha_{l_2 \neq r_2|l_1,m_1,m_2=r_2}(\varphi_N)$ to accept l_2 different from r_2, when r_2 is in reality, and the probability $\alpha_{l_2 = r_2|l_1,m_1,m_2 \neq r_2}(\varphi_N)$ to accept r_2, when it is not correct.

The upper estimate $\beta_{l_2 \neq r_2|l_1,m_1,m_2=r_2}(\varphi_2^N)$ of $\alpha_{l_2 \neq r_2|l_1,m_1,m_2=r_2}(\varphi_2^N)$ is already known, it coincides with the $\beta_{r_2|l_1,m_1,r_2}(\varphi_2^N)$ which is equal to $\sum\limits_{l_2 : l_2 \neq r_2} \beta_{l_2|l_1,m_1,r_2}(\varphi_2^N)$. The corresponding reliability $F_{l_2 \neq r_2|l_1,m_1,m_2=r_2}(\varphi_2)$ is equal to $F_{r_2|l_1,m_1,r_2}(\varphi_2)$ which satisfies the equality (19).

The reliability approach to identification of lower estimates assumes determining the optimal dependence of $F^*_{l_2 = r_2|l_1,m_1,m_2 \neq r_2}$ upon given $F^*_{l_2 \neq r_2|l_1,m_1,m_2=r_2} = F^*_{r_2|l_1,m_1,r_2}$, which can be an assigned values satisfying conditions (23).

Theorem 4: *In case of distinct PDs* $G_{1|l_1}, G_{2|l_1}, ..., G_{L_2|l_1}$, *for every* l_1 *under condition that the upper estimates of probabilities of all* L_2 *hypotheses are strictly positive the ``reliability"* $F_{l_2=r_2|l_1,m_1,m_2\neq r_2}$ *for given* $F_{l_2\neq r_2|l_1,m_1,m_2=r_2} = F_{r_2|l_1,m_1,r_2}$ *is the following*:

$$F_{l_2=r_2|l_1,m_1,m_2\neq r_2}(F_{r_2|l_1,m_1,r_2}) = \min_{m_2:m_2\neq r_2} \inf_{Q\in R_{l_1}} \inf_{V:D(V\|G_{r_2/l_1}|Q)\leq F_{r_2|l_1,m_1,r_2}} D(V\,\|\,G_{m_2|m_1}\,|\,Q),\ r_2=\overline{1,L_2}.$$

Proof: We have

$$\beta^N_{l_2=r_2|l_1,m_1,m_2\neq r_2} = \frac{Pr^N(m_2\neq r_2, l_2=r_2/l_1,m_1)}{Pr(m_2\neq r_2/m_1)} = \frac{\sum\limits_{m_2:m_2\neq r_2}\beta_{r_2|l_1,m_1,m_2} Pr(m_2/m_1)}{\sum\limits_{m_2\neq r_2} Pr(m_2/m_1)}.$$

Consequently, we obtain that

$$F_{l_2=r_2|l_1,m_1,m_2\neq r_2}(F_{r_2|l_1,m_1,r_2}) \overset{\Delta}{=} \overline{\lim_{N\to\infty}} -\frac{1}{N}\log\beta^N_{l_2=r_2|l_1,m_1,m_2\neq r_2}$$
$$= \overline{\lim_{N\to\infty}} -\frac{1}{N}(\log\sum_{m_2:m_2\neq r_2}\beta_{r_2|l_1,m_1,m_2}Pr(m_2/m_1) - \log\sum_{m_2\neq r_2}Pr(m_2/m_1))$$

$$= \overline{\lim_{N\to\infty}} -\frac{1}{N}(\log\max\beta_{r_2|l_1,m_1,m_2} + \log\sum_{m_2:m_2\neq r_2}\frac{\beta_{r_2|l_1,m_1,m_2}Pr(m_2/m_1)}{\max\beta_{r_2|l_1,m_1,m_2}} - \log\sum_{m_2\neq r_2}Pr(m_2/m_1)) = \min_{m_2:m_2\neq r_2} F_{r_2|l_1,m_1,m_2}.$$

And using $(22.b)$ we prove the theorem.

5. LAO Hypotheses Testing for Two Stochastically Dependent Objects

In this section we find the ``reliabilities" $F_{l_1,l_2|m_1,m_2}$ for LAO testing which will be lower bounds for corresponding $E_{l_1,l_2|m_1,m_2}$. Using (15) we can prove the following lemma

Lemma: If the elements $E_{l_1|m_1}(\varphi_1)\ and\ F_{l_2|l_1,m_1,m_2}(\varphi_2)$ are positive, then

$$F_{l_1,l_2|m_1,m_2}(\Phi)=E_{l_1|m_1}(\varphi_1)+F_{l_2|l_1,m_1,m_2}(\varphi_2),\ m_1\neq l_1,\ m_2\neq l_2, \tag{29a}$$

$$F_{l_1,l_2|m_1,m_2}(\Phi)=E_{l_1|m_1}(\varphi_1),\ m_1\neq l_1,\ m_2=l_2, \tag{29b}$$

$$F_{l_1,l_2|m_1,m_2}(\Phi)=F_{l_2|l_1,m_1,m_2}(\varphi_2),\ \ m_1=l_1,\ \ m_2\neq l_2. \tag{29c}$$

Proof: The following relations hold for upper estimates of error probabilities

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N)=\alpha^N_{l_1|m_1}(\varphi_1^N)\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N),\ m_1\neq l_1,\ \ m_2\neq l_2, \tag{30a}$$

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N)=\alpha^N_{l_1|m_1}(\varphi_1^N)(1-\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N)),\ m_1\neq l_1,\ m_2=l_2, \tag{30b}$$

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N)=(1-\alpha^N_{l_1|m_1}(\varphi_1^N))\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N),\ \ m_1=l_1,\ m_2\neq l_2. \tag{30c}$$

Thus, in light of (3) and (18), we obtain (29). The lemma is proved.

Let us define the following subsets of $P(X)$ for given strictly positive elements

$E_{L_1,l_2|l_1,l_2}$, $F_{l_1,L_2|l_1,l_2}$, $l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1}$:

$$R_{l_1}\overset{\Delta}{=}\{Q:D(Q\,\|\,G_{l_1})\leq E_{L_1,l_2|l_1,l_2}\},\ \ l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1},$$

$$R_{l_2/l_1}(Q)\overset{\Delta}{=}\{V:D(V\,\|\,G_{l_2/l_1}\,|\,Q)\leq F_{l_1,L_2|l_1,l_2}\},\ \ l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1},$$

$$R_{L_1}\overset{\Delta}{=}\{Q:D(Q\,\|\,G_{l_1})>E_{L_1,l_2|l_1,l_2},\ l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1}\},$$

$$R_{L_2/l_1}(Q)\overset{\Delta}{=}\{V:D(V\,\|\,G_{l_2/l_1}\,|\,Q)>F_{l_1,L_2|l_1,l_2},\ l_1=\overline{1,L_1-1},\ l_2=\overline{1,L_2-1}\}.$$

Assume also

$$F^{*}_{l_1,L_2|l_1,l_2} \stackrel{\Delta}{=} F_{l_1,L_2|l_1,l_2},\ E^{*}_{L_1,l_2|l_1,l_2} \stackrel{\Delta}{=} E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ \ l_2 = \overline{1, L_2 - 1}, \tag{31a}$$

$$E^{*}_{l_1,l_2|m_1,l_2} \stackrel{\Delta}{=} \inf_{Q:Q\in R_{l_1}} D(Q \,\|\, G_{m_1}),\ \ m_1 \neq l_1 \tag{31b}$$

$$F^{*}_{l_1,l_2|l_1,m_2} \stackrel{\Delta}{=} \inf_{Q\in R_{l_1}} \inf_{V:V\in R_{l_2/l_1}(Q)} D(V \,\|\, G_{m_2/m_1} \,|\, Q),\ m_2 \neq l_2 \tag{31c}$$

$$F^{*}_{l_1,l_2|m_1,m_2} \stackrel{\Delta}{=} F^{*}_{m_1,l_2|m_1,m_2} + E^{*}_{l_1,m_2|m_1,m_2},\ \ m_i \neq l_i,\ i = 1,2, \tag{31d}$$

$$F^{*}_{m_1,m_2|m_1,m_2} \stackrel{\Delta}{=} \min_{(l_1,l_2)\neq(m_1,m_2)} F^{*}_{l_1,l_2|m_1,m_2}. \tag{31e}$$

Theorem 5: *If all distributions G_{m_1}, $m_1 = \overline{1, L_1}$, are different, that is $D(G_{l_1} \| G_{m_1}) > 0$, $l_1 \neq m_1$, $l_1, m_1 = \overline{1, L_1}$, and all conditional distributions G_{l_2/l_1}, $l_2 = \overline{1, L_2}$, are also different for all $l_1 = \overline{1, L_1}$, in the sense that $D(G_{l_2/l_1} \| G_{m_2/l_1} \,|\, Q) > 0$, $l_2 \neq m_2$, then the following statements are valid.*

When given elements $E_{L_1,l_2|l_1,l_2}$ and $F_{l_1,L_2|l_1,l_2}$, $l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}$, meet the following conditions

$$0 < E_{L_1,l_2|1,l_2} < \min_{l_1 = \overline{2, L_1}} D(G_{l_1} \,\|\, G_1), \tag{32a}$$

$$0 < F_{l_1,L_2|l_1,1} < \min_{l_2 = \overline{2, L_2}} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1} \,\|\, G_{1/m_1} \,|\, Q), \tag{32b}$$

$$0 < E_{L_1,l_2|l_1,l_2} < \min[\min_{l_1 = \overline{1, m_1 - 1}} E^{*}_{l_1,l_2|m_1,l_2},\ \min_{l_1 = \overline{m_1 + 1, L_1}} D(G_{l_1} \,\|\, G_{m_1})],\ \ l_1 = \overline{2, L_1 - 1}, \tag{32c}$$

$$0 < F_{l_1,L_2|l_1,l_2} < \min[\min_{l_2 = \overline{1, m_2 - 1}} F^{*}_{l_1,l_2|l_1,m_2},\ \min_{l_2 = \overline{m_2 + 1, L_2}} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1} \,\|\, G_{m_2/m_1} \,|\, Q)],\ l_2 = \overline{2, L_2 - 1}, \tag{32d}$$

then there exists a LAO test sequence Φ^{*}, the lower estimate matrix of which $F(\Phi^{*}) = \{F_{l_1,l_2|m_1,m_2}(\Phi^{*})\}$ is defined in (31) and all elements of it are positive.

When even one of the inequalities (32) is violated, then at least one element of the lower estimate matrix $F(\Phi^*)$ is equal to 0.

Proof: It is proved in [7] that $E_{l_1|l_1} = E_{L_1|l_1}$, $l_1 = \overline{1, L_1 - 1}$. By analogy we can deduce that

$$F_{l_2|l_1,m_1,l_2} = F_{L_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}. \tag{33}$$

Applying the theorem of Kuhn-Tucker in (22.b) we can show that the elements $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$ can be determined by elements $F_{l_2|l_1,m_1,m_2}$, $m_2 \neq l_2$, $l_2 = \overline{1, L_2}$,

$$F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,m_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1}|Q) \leq F_{l_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q).$$

From (23) it is clear that $F_{m_2|l_1,m_1,m_2}$ can be equal only to one of $F_{l_2|l_1,m_1,m_2}$, $l_2 = \overline{m_2 + 1, L_2}$. Assume that (33) is not correct, that is $F_{m_2|l_1,m_1,m_2} = F_{l_2|l_1,m_1,m_2}$, $l_2 = \overline{m_2 + 1, L_2 - 1}$.

From (22.b) it follows that

$$F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,m_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1}|Q) \leq F_{l_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q)$$

$$= \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1}|Q) \leq F_{m_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q) = F_{m_2|l_1,m_1,l_2},\ m_2, l_2 = \overline{1, L_2 - 1},$$

$$m_2 < l_2,$$

but from conditions (23) it follows that $F_{l_2|l_1,m_1,l_2} < F_{m_2|l_1,m_1,l_2}$ for $m_2 = \overline{1, l_2 - 1}$. Our assumption is not true, thus (33) is valid.

Hence we can rewrite the inequalities (7) and (23) as follows:

$$0 < E_{L_1|1} < \min_{l_1=\overline{2,L_1}} D(G_{m_1} \| G_1), \tag{34a}$$

$$0 < F_{L_2|l_1,m_1,1} < \inf_{Q \in R_{l_1}} \min_{l_2=\overline{2,L_2}} D(G_{m_2/l_1} \| G_{1/l_1} | Q), \tag{34b}$$

$$0 < E_{L_1|l_1} < \min[\min_{l_1=\overline{1,l_1-1}} E^*_{l_1|m_1}, \min_{l_1=\overline{l_1+1,L_1}} D(G_{m_1} \| G_{l_1})], \ \ l_1 = \overline{2, L_1 - 1}, \tag{34c}$$

$$0 < F_{L_2|l_1,m_1,l_2} < \min[\min_{l_2=\overline{1,l_2-1}} F^*_{l_2|l_1,m_1,m_2}, \inf_{Q \in R_{l_1}} \min_{l_2=\overline{l_2+1,L_2}} D(G_{l_2/l_1} \| G_{m_2/l_1} | Q)], \ l_2 = \overline{2, L_2 - 1}. \tag{34d}$$

According to Theorem 1 and Theorem 2 there exist LAO sequences of tests φ_1^* and φ_2^*, for the first and second objects, such that the elements of the matrices $E(\varphi_1^*)$ are determined in (6) and the lower estimate matrix $F(\varphi_2^*)$ is determined in (22). The inequalities (34.a), (34.c) are equivalent to the inequalities (7) and (34.b), (34.d) are equivalent to the inequalities (23). Then using Lemma we deduce that the lower estimate matrix $F(\Phi^*)$ is determined in (31). The proof of the second assertion of the Theorem is obvious.

6 . On Identification of the Probability Distributions of Two Stochastically Dependent Objects

In this section we study an approach to deducing optimal interdependencies of lower estimates of corresponding reliabilities for LAO identification. The LAO test Φ^* is the compound test consisting of the pair of LAO tests φ_1^* and φ_2^* for respective separate objects, and for it the equalities (29) take place. The statistician has to answer to the question whether the pair of distributions (r_1, r_2) occurred or not. Let us consider two types of error probabilities for each pair (r_1, r_2) , $r_1 = \overline{1, L_1}, r_2 = \overline{1, L_2}$. We denote by $\alpha^N_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)}$ the probability, that pair (r_1, r_2) is true, but it is rejected. Note that this probability is equal to $\alpha_{r_1,r_2|r_1,r_2}(\Phi^N)$. Let $\alpha^N_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ be the probability that (r_1, r_2) is accepted, when it is not correct. The corresponding reliabilities are $E_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)} = E_{r_1,r_2|r_1,r_2}$ and $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$. Our aim is to determine the dependence of $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ on given $E_{r_1,r_2|r_1,r_2}(\Phi^N)$.

Now let us suppose that hypotheses $G_1, G_2, ..., G_{L_1}$ have a priori positive probabilities Pr (r_1), $r_1 = \overline{1, L_1}$ and $G_{1|l_1}, G_{2|l_1}, ..., G_{L_2|l_1}$ have a priori positive conditional probabilities Pr $(r_2 \mid l_1)$, $r_2 = \overline{1, L_2}$, and consider the probability, which we are interested

$$\beta^N_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \frac{Pr^N((m_1,m_2)\neq(r_1,r_2),(l_1,l_2)=(r_1,r_2))}{Pr((m_1,m_2)\neq(r_1,r_2))} = \frac{\sum\limits_{(m_1,m_2):(m_1,m_2)\neq(r_1,r_2)} \beta_{(r_1,r_2)|(m_1,m_2)} Pr((m_1,m_2))}{\sum\limits_{(m_1,m_2)\neq(r_1,r_2)} Pr(m_1,m_2)}$$

Consequently, we obtain that

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \min_{(m_1,m_2):(m_1,m_2)\neq(r_1,r_2)} F_{r_1,r_2|m_1,m_2}. \tag{35}$$

For every LAO test Φ^* from (11), (29) and (35) we obtain that

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \min_{m_1\neq r_1, m_2\neq r_2} \left(E_{r_1|m_1}(E_{r_1|r_1}), F_{r_2|l_1,m_1,m_2}(F_{r_2|l_1,m_1,r_2})\right), \tag{36}$$

where $E_{r_1|m_1}(E_{r_1|r_1}), F_{r_2|l_1,m_1,m_2}(F_{r_2|l_1,m_1,r_2})$ are determined by (6) and (22) for, correspondingly, the first and the second objects. For every LAO test Φ^* from (16) and (29) we deduce that

$$F_{r_1,r_2|r_1,r_2} = \min_{m_1\neq r_1, m_2\neq r_2} \left(E_{r_1|m_1}, F_{r_2|l_1,m_1,m_2}\right) = \min\left(E_{r_1|r_1}, F_{r_2|l_1,m_1,r_2}\right), \tag{37}$$

and each of $E_{r_1|r_1}, E_{r_2|l_1,m_1,r_2}$ satisfies the following conditions:

$$0 < E_{r_1|r_1} < \min\left[\min_{l_1=\overline{1,r_1-1}} E^*_{l_1|m_1}(E_{l_1|l_1}), \min_{l_1=\overline{r_1+1,L_1}} D(G_{l_1} \,\|\, G_{r_1})\right], \tag{38a}$$

$$0 < F_{r_2|l_1,m_1,r_2} < \min\left[\min_{l_2=\overline{1,r_2-1}} F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}), \inf_{Q\in R_{l_1}} \min_{l_2=\overline{r_2+1,L_2}} D(G_{l_2|l_1} \,\|\, G_{r_2|m_1} \mid Q)\right]. \tag{38b}$$

From (6.b) and (22.b) we see that the elements $E^*_{l_1|m_1}(E_{l_1|l_1}),\ l_1=\overline{1,r_1-1}$ and $E^*_{l_2|l_1,m_1,m_2}(E_{l_2|l_1,m_1,l_2}),$ $l_2=\overline{1,r_2-1}$ are determined only by $E_{l_1|l_1}$ and $F_{l_2|l_1,m_1,l_2}$. But we are considering only elements $E_{r_1|r_1}$ and $F_{r_2|l_1,m_1,r_2}$. We can use Corollary 1, Corollary 2 and upper estimates (38.a), (38.b) as follows:

$$0<E_{r_1|r_1}<\min\left[\min_{l=1,r_1-1} D(G_{r_1}\parallel G_{l_1}),\ \min_{l_1=r_1+1,L_1} D(G_{l_1}\parallel G_{r_1})\right], \tag{39a}$$

$$0<F_{r_2|l_1,m_1,r_2}<\min\left[\inf_{Q\in R_{l_1}}\min_{l_2=1,r_2-1} D(G_{r_2|m_1}\parallel G_{l_2|l_1}\mid Q),\ \inf_{Q\in R_{l_1}}\min_{l_2=r_2+1,L_2} D(G_{l_2|l_1}\parallel G_{r_2|m_1}\mid Q)\right]. \tag{39b}$$

From (37) we have that $F_{r_1,r_2|r_1,r_2}=E_{r_1|r_1}$, when $E_{r_1|r_1}\le F_{r_2|l_1,m_1,r_2}$, and when

$F_{r_1,r_2|r_1,r_2}=F_{r_2|l_1,m_1,r_2}$, then $F_{r_2|l_1,m_1,r_2}\le E_{r_1|r_1}$. Hence, it can be implied that given strictly positive element $F_{r_1,r_2|r_1,r_2}$ must meet both inequalities (39.a) and (39.b).

Using (37) we can determine reliability $F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ in function of $F_{r_1,r_2|r_1,r_2}$ as follows:

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}\left(F_{r_1,r_2|r_1,r_2}\right)=\min_{m_1\neq r_1,m_2\neq r_2}\left[E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2}),F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})\right], \tag{40}$$

where $E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2})$ and $F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})$ are determined respectively by (6.b) and by (22.b). Finally we obtained

Theorem 6: *If the distributions G_{m_1}, and $G_{m_2|m_1}$, $m_1=\overline{1,L_1}$, $m_2=\overline{1,L_2}$ are different and the given strictly positive number $F_{r_1,r_2|r_1,r_2}$ satisfies condition (39.a) or (39.b), then the lower estimate $F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ of $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ can be calculated by (40).*

In the particular case, when X_1 and X_2 are related statistically [8], [9] that is the second object depending on PD of the first is characterized by RV X_2 which can have one of $L_1\times L_2$ conditional PDs $G_{l_2/l_1}=\{G_{l_2/l_1}(x^2),x^2\in X\},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2}$, we will have $A^N_{l_2/l_1}=\{x_2:\varphi^N_2(x_2,l_1)=l_2\},$ $l_1=\overline{1,L_1},\ l_2=\overline{1,L_2}$, in place of the set $A^N_{l_2/l_1}(x_1)$ and in that case from [8] we have

$$G^N_{m_1,m_2}(A^N_{l_1,l_2}) = \sum_{(x_1,x_2)\in A^N_{l_1,l_2}} G^N_{m_1}(x_1)G^N_{m_2/m_1}(x_2) = \sum_{x_1\in A^N_{l_1}} G^N_{m_1}(x_1) \sum_{x_2\in A^N_{l_2/l_1}} G^N_{m_2/m_1}(x_2)$$

$$= G^N_{m_2/m_1}(A^N_{l_2/l_1})G_{m_1}(A^N_{l_1}), \quad (l_1,l_2)\neq(m_1,m_2).$$

The probabilities of the erroneous acceptance of PD G_{l_l/l_1} provided that G_{m_2/m_1} is true, $l_1, m_1 = \overline{1, L_1}$, are denoted by

$$\alpha^N_{l_2|l_1,m_1,m_2}(\varphi^N_2) = G^N_{m_2/m_1}(A^N_{l_2/l_1}),\ l_2 \neq m_2.$$

The probability to reject G_{m_2/m_1} , when it is true is denoted as follows

$$\alpha^N_{m_2|l_1,m_1,m_2}(\varphi^N_2) = G^N_{m_2/m_1}(\overline{A_{m_2/l_1}}) \overset{\Delta}{=} \sum_{l_2\neq m_2} \alpha^N_{l_2|l_1,m_1,m_2}(\varphi^N_2).$$

Thus in the conditions and in the results of Theorems 3-6, instead of conditional divergences

$\inf\limits_{Q\in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} | Q)$, $\inf\limits_{Q\in R_{l_1}} D(V \| G_{m_2/m_1} | Q)$ we will have just divergences

$D(G_{l_2/l_1} \| G_{m_2/m_1}), D(V \| G_{m_2/m_1})$ and in place of $F_{l_2|l_1,m_1,m_2}(\Phi), F_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},$

$l_2, m_2 = \overline{1, L_2}$, will be $E_{l_2|l_1,m_1,m_2}(\Phi), E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}$.

And in that case regions defined in (21) will be changed as follows:

$$R_{l_2/l_1} \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1}) \leq E_{l_2|l_1,m_1,l_2}\},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{L_2/l_1} \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1} | Q) > E_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}\},$$

$$R^N_{l_2/l_1} = R_{l_2/l_1} \cap P_N(X).$$

In case of two statistically dependent objects the corresponding regions will be

$$R_{l_1} \stackrel{\Delta}{=} \{Q : D(Q \,\|\, G_{l_1}) \le E_{L_1,l_2|l_1,l_2}\},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{l_2/l_1} \stackrel{\Delta}{=} \{V : D(V \,\|\, G_{l_2/l_1}) \le E_{l_1,L_2|l_1,l_2}\},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{L_1} \stackrel{\Delta}{=} \{Q : D(Q \,\|\, G_{l_1}) > E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}\},$$

$$R_{L_2/l_1} \stackrel{\Delta}{=} \{V : D(V \,\|\, G_{l_2/l_1}) > E_{l_1,L_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}\}.$$

So in this case we obtain the optimal interdependencies of reliabilities. The results were shown in [8] and in [9]. For this model in next section will present some results of calculations.

7. Example

. Let us consider the set of two elements $X = \{0,1\}$ and the following probability distributions given on X: $G_1 = \{0.84;0.16\}$, $G_2 = \{0.23;0.77\}$, $G_{1/1} = \{0.78;0.22\}$, $G_{2/1} = \{0.21;0.79\}$, $G_{1/2} = \{0.59;0.41\}$ $G_{2/2} = \{0.32;0.68\}$. In Fig.1 and Fig.2 the results of calculations of functions $E_{1,1|2,1}(E_{2,1|1,1})$ and $E_{1,2|2,1}(E_{2,1|1,1}, E_{1,2|1,1})$ are presented. For these distributions we have $D(G_2 \,\|\, G_1) \approx 1.3$ and $D(G_{2/1} \,\|\, G_{1/1}) \approx 1.06$. We see in Fig.1 that when an analog of the inequality (32.a) of Theorem 5 (for statistically dependent objects) is violated then $E_{1,1|2,1} = 0$ and in Fig.2 we see that when analogs of (32.a) and (32.b) equalities are violated then $E_{1,2|2,1} = 0$.

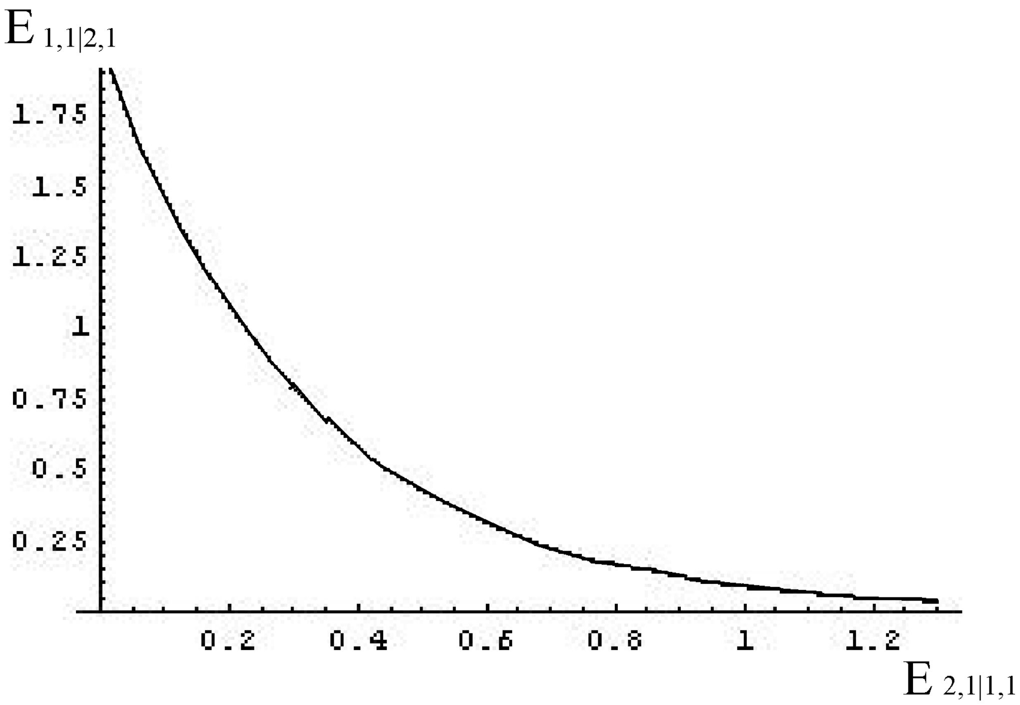
E 1,1|2,1
1.75
1.5
1.25
1
0.75
0.5
0.25
0.2
0.4
0.6
0.8
1
1.2
E 2,1|1,1

Fig. 1

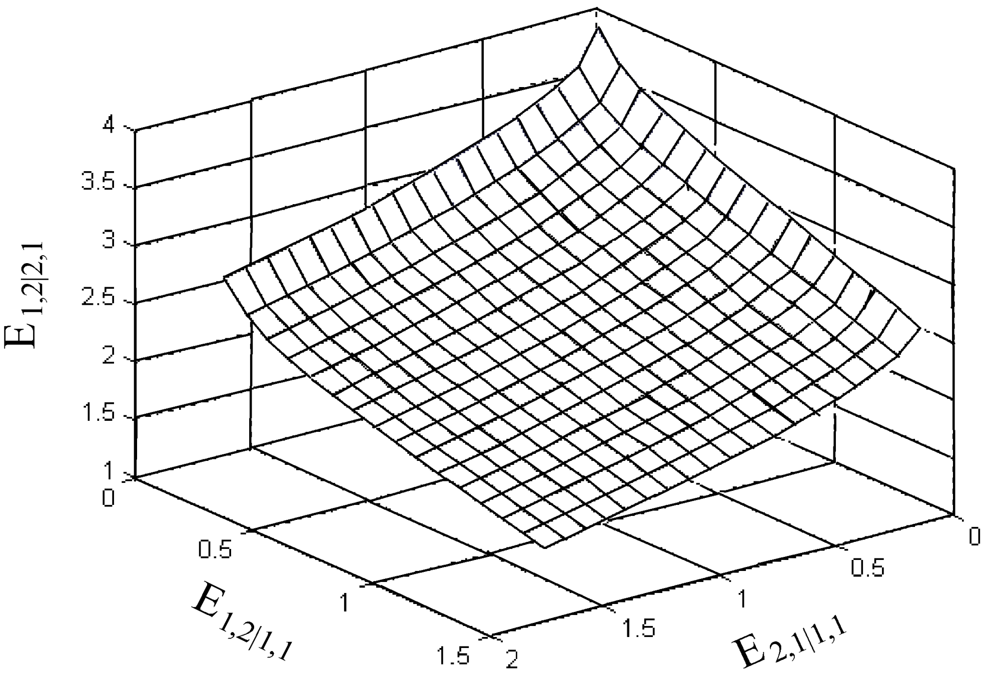
4
3.5
3
2.5
2
1.5
1
0
0.5
1
1.5
2
E1,2|2,1
E1,2|1,1
E2,1|1,1
0
0.5
1
1.5

Fig. 2

8. Conclusion

We studied the more general model of stochastically dependence of two discrete random variables. For this model reliability requirements to multiple hypotheses testing and identification are investigated. By the first approach optimal interdependencies of elements of reliability matrix of test Φ can be found when its L_1L_2-1 diagonal elements are given. But by this approach we do not have information about the reliabilities of the first and the second objects. By the second approach at first we find optimal interdependencies of reliabilities of the first object and then interdependencies of lower estimates of reliabilities of the second object. Similarly we also solve the identification problem for two objects. Results of the second approach are applied to finding the optimal interdependencies of lower estimates of reliabilities of two objects when L_1L_2-1 non diagonal elements of lower estimate matrix are given. If random variables X_1 and X_2 take values in different sets $\mathcal{X}_1$ and $\mathcal{X}_2$ only the notations become more complicated, so we omit this "generalization". The correspondence with other, less general, cases of objects relation is discussed in [5] -- [10].

Bibliography

[1] W. Hoeffding, "Asymptotically optimal tests for multinomial distributions", *The Annals of Mathematical Statistics*, vol. 36, pp. 369--401, 1965.

[2] I. Csiszár and G. Longo, "On the error exponent for source coding and for testing simple statistical hypotheses", *Studia Scientiarum Math. Hungarica*, vol. 6, pp. 181--191, 1971.

[3] E. A. Haroutunian, "Logarithmically asymptotically optimal testing of multiple statistical hypotheses" *Problems of Control and Information Theory*, vol. 19(5-6), pp. 413--421, 1990.

[4] R. F. Ahlswede and E. A. Haroutunian, "On logarithmically asymptotically optimal testing of hypotheses and identification". *Lecture Notes in Computer Science, vol. 4123, "General Theory of Information Transfer and Combinatorics"*, pp. 462-- 478, Springer, 2006.

[5] E. A. Haroutunian and P. M. Hakobyan, "Multiple hypotheses LAO testing for many independent object", *International journal ``Scholarly Research Exchange"*, vol. 2009, pp. 1-6, 2009.

[6] E. A. Haroutunian and P. M. Hakobyan, "On identification of distributions of two independent objects", *Mathematical Problems of Computer Science*, vol. XXVIII, pp. 114--119, 2007.

[7] E. A. Haroutunian and A. O. Yessayan, "On hypotheses testing for two differently distributed objects". *Mathematical Problems of Computer Science*, vol. XXVI, pp. 91--96, 2006.

[8] E. A. Haroutunian and A. O. Yessayan, "On logarithmically asymptotically optimal hypothesis testing for pair of statistically dependent objects", *Mathematical Problems of Computer Science*, vol. XXIX, pp. 97--103, 2007.

[9] A. O. Yessayan, "On reliability approach to identification of probability distributions of two statistically dependent objects", *Mathematical Problems of Computer Science*, vol. XXXII, pp. 65--69, 2009.

[10] E. A. Haroutunian and A. O. Yessayan, "On optimal hypotheses testing for pair of stochastically coupled objects", *Mathematical Problems of Computer Science*, vol. XXXI, pp. 49--59, 2008.

[11] E. A. Haroutunian, M. E. Haroutunian, and A. N. Harutyunyan, "Reliability criteria in information theory and in statistical hypotheses testing", *Foundations and Trends in Communications and Information Theory*, vol. 4, no. 2-3, 2008.

[12] I. Csiszár and J. Körner, *Information theory: Coding Theorems for Discrete Memoryless Systems.* New York: Academic Press, 1981.

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