ITHEA

International Journal INFORMATION THEORIES & APPLICATIONS

2010 Volume 17 Number 3

International Journal
INFORMATION THEORIES & APPLICATIONS
Volume 17 / 2010, Number 3

International Journal “INFORMATION THEORIES & APPLICATIONS” (IJ ITA)
is official publisher of the scientific papers of the members of
the ITHEA International Scientific Society

IJ ITA welcomes scientific papers connected with any information theory or its application.
IJ ITA rules for preparing the manuscripts are compulsory.
The **rules for the papers** for IJ ITA as well as the **subscription fees** are given on *www.foibg.com/ijita*.

Responsibility for papers published in IJ ITA belongs to authors.

General Sponsor of IJ ITA is the **Consortium FOI Bulgaria** (www.foibg.com).

International Journal “INFORMATION THEORIES & APPLICATIONS” Vol. 17, Number 3, 2010

Printed in Bulgaria

Edited by the **Institute of Information Theories and Applications FOI ITHEA**, Bulgaria, in collaboration with:
V.M.Glushkov Institute of Cybernetics of NAS, Ukraine,
Institute of Mathematics and Informatics, BAS, Bulgaria,
Universidad Politécnica de Madrid, Spain.

Publisher: **ITHEA**
Sofia, 1000, P.O.B. 775, Bulgaria. www.ithea.org, www.foibg.com, e-mail: info@foibg.com

ISSN 1310-0513 (printed) **ISSN 1313-0463 (online)** **ISSN 1313-0498 (CD/DVD)**

CONSTRAINT CONVEXITY TOMOGRAPHY AND LAGRANGIAN APPROXIMATIONS

Levon Aslanyan, Artyom Hovsepyan, Hasmik Sahakyan

***Abstract**: This paper considers one particular problem of general type of discrete tomography problems and introduces an approximate algorithm for its solution based on Lagrangian relaxation. A software implementation is given as well.*

Keywords: *discrete tomography, lagrangian relaxation.*

***ACM Classification Keywords**: F.2.2 Nonnumerical Algorithms and Problems: Computations on discrete structures.*

Introduction

Discrete tomography is a field which deals with problems of reconstructing objects from its projections. Usually in discrete tomography object T, represents a set of points in multidimensional lattice. Some measurements are performed on T, each of which contains projection, which calculates number of points of T along parallel directions. Given finite number of such measurements it is required to reconstruct object T, or if it is not possible to find unique reconstruction, construct an object which satisfies given projections. The object existence problem even by given 3 non-parallel projections is NP-complete [1].

In recent years discrete tomography draws huge attention because of the variety of mathematical formulations and applications. Theory of discrete tomography is widely used particularly in the field of medical image processing, which is based on so called computerized tomography.

Lets consider 2-dimensional lattice and horizontal and vertical projections only. Object T can be represented as a $m \times n$ $(0,1)$ matrix, where 1s corresponds to points in T. Vector of row sums corresponds to horizontal projection and vector of column sums to vertical projection. So the problem of reconstructing the object by given horizontal and vertical projections is equivalent to the $(0,1)$-matrix existence problem with given R and S row and column sums. The latter problem was solved independently by Gale and Ryser in 1957. They gave sufficient and necessary condition for such a matrix existence and also proposed an algorithm for the matrix construction. Same problem with condition of rows inequality was investigated in [6].

In many cases orthogonal projections does not contain enough information for the objects unique reconstruction.

That's why often we consider different classes of such problems, where we impose additional constraints, for instance of geometrical nature. Such constraints narrow the solutions set but at the same time could make the problem hard to solve. Typical examples of such constraints are convexity and connectivity.

We say that matrix has row (or horizontal) convexity feature if all ones in the row forms a continuous interval. Same way we define column (or vertical) convexity. Connectivity is the feature of moving between 1s in neighboring cells. In our case we consider only vertical and horizontal connectivity (not diagonal).

Existence problem for connected matrices is NP-complete [2]. Existence problems for horizontally or vertically convex, and for both horizontally and vertically convex matrices are also NP-complete [3].

Different authors proved that horizontally and vertically convex and connected matrices reconstruction problem can be solved in polynomial time. Given description shows how sensitive are this kind of problems to input conditions. We see that existence problem's complexity changes along with adding new constraints. At the same time there are a lot of other notations of the problem for those the complexity is not even known. Particularly that means that they also lacks easy solution algorithms.

So we consider several problems in the field of discrete tomography, propose ways for constructing such matrices that satisfy constraints (convex or nearly convex, satisfying given parameters or having values near to given parameters). Further we will formulate the problems as optimization problems and give ways for their approximation, based on the integer programming relaxation. The question is that integer programming model is known for being used to reformulate known NP complex optimization problems. This model's (precise or approximate algorithms construction) investigation is very important and often this model is used to approximate optimizations problems [4, 6]. Implemented algorithms and software package based on that algorithms give an ability to make calculations either for tomography problem or for similar problems, such that those calculations might guide us or give approximate or precise solutions.

In this paper we will consider one problem from the field of discrete tomography, horizontally convex matrix existence problem.

Horizontally convex matrix existence problem

Since 1's in the horizontally convex matrix are in neighboring position then if we count the number of 1's in the matrices rows, that number for convex matrices will be maximum for the ones with same parameters. That's why problems that are often considered are related to number of neighboring 1's, their constraints and optimization.

$R=(r_1,\cdots,r_m)$, $S=(s_1,\cdots,s_n)$, $R'=(r_1',\cdots,r_m')$ vectors are given. Is there a $m\times n$ $X=\{x_{i,j}\}$ matrix such that R is row sum vector for that matrix and S is column sums vector, and number of neighboring 1's in row i is equal to r_i'.

$$\begin{cases} \sum_{i=1}^{m} x_{i,j} = s_j, j = 1,\cdots,n \\ \sum_{j=1}^{n} x_{i,j} = r_i, i = 1,\cdots,m \\ \sum_{j=1}^{n-1} \min(x_{i,j}, x_{i,j+1}) = r_i', i = 1,\cdots,m \\ x_{i,j} \in \{0,1\} \end{cases}$$

In other words the problem is following, find the matrix with horizontal convexity in the class of $(0,1)$ matrices with given row and column sums. This problem is NP-complete, since for the case when $r_i' = r_i - 1, i = 1,\cdots,m$ it's equivalent to the horizontally convex matrix existence problem. Given particular case just require the matrix to be horizontally convex by neighboring 1s in the rows.

As we already mentioned lot of combinatorial problems are suitable to represent as integer linear optimization problems. Lets reformulate our problem as integer programming problem.

Lets define $y_{i,j} \in \{0,1\}$ variables the way that it provides neighboring 1's in row i.

$$(y_{i,j} = 1) \Leftrightarrow (x_{i,j} = 1) \,\&\, (x_{i,j+1} = 1), i = 1,\cdots,m;\ j = 1,\cdots,n-1$$

This can be done by satisfying conditions

$$\begin{cases} y_{i,j} \le x_{i,j} \\ y_{i,j} \le x_{i,j+1} \\ y_{i,j} \ge x_{i,j} + x_{i,j+1} - 1 \end{cases}$$

So we reformulate the problem in the following way.

$R = (r_1,\cdots,r_m)$, $S = (s_1,\cdots,s_n)$, $R' = (r_1',\cdots,r_m')$ vectors are given: Is there a $m \times n$ $X = \{x_{i,j}\}$ matrix such that

$$\begin{cases} (1)\sum_{i=1}^{m} x_{i,j} = s_j, j = 1,\cdots,n \\ (2)\sum_{j=1}^{n} x_{i,j} = r_i, i = 1,\cdots,m \\ (3)\begin{cases} y_{i,j} \le x_{i,j} \\ y_{i,j} \le x_{i,j+1} \\ y_{i,j} \ge x_{i,j} + x_{i,j+1} - 1 \end{cases} \quad i = 1,\cdots,m, j = 1,\cdots,n-1 \\ (4)\sum_{j=1}^{n-1} y_{i,j} = r_i^{'}, i = 1,\cdots,m \\ (5) x_{i,j} \in \{0,1\}, y_{i,j} \in \{0,1\} \end{cases}$$

Lagrangean relaxation and variable splitting

So we have horizontal row convex matrix existence problem, which is reformulated as linear integer programming problem I. We also know that problem I is NP-complete. To solve this problem we will use a method based on Lagrangian relaxation.

Obviously if we drop some of the constraints we will get problems relaxation. Assume that we can call one or several constraints hard in the since that by dropping those constraints we can solve resulted integer programming problem more easily. Constraints dropping could be embedded in more common method which is called Lagrangian relaxation. We can apply Lagrangian relaxation to given method in various ways. One of the ways, which we will use here is following, if the problem can be splitted to subproblems, which have common variables, first split those variables and then relax their equality constraint.

So, we take two set of variables $x_{i,j}^{h}$ and $x_{i,j}^{v}$ by duplicating $x_{i,j}$ variables, and reformulate our problem as

$$\begin{cases} (1)\sum_{i=1}^{m} x_{i,j}^{v} = s_j,\ j = 1,\cdots,n \\ (2)\sum_{j=1}^{n} x_{i,j}^{h} = r_i, i = 1,\cdots,m \\ (3)\begin{cases} y_{i,j} \le x_{i,j}^{h} \\ y_{i,j} \le x_{i,j+1}^{h} \\ y_{i,j} \ge x_{i,j}^{h} + x_{i,j+1}^{h} - 1 \end{cases} \quad i = 1,\cdots,m,\ j = 1,\cdots,n-1 \\ (4)\sum_{j=1}^{n-1} y_{i,j} = r_i^{'},\ i = 1,\cdots,m \\ (5) x_{i,j}^{h}, x_{i,j}^{v} \in \{0,1\},\ y_{i,j} \in \{0,1\} \\ (6) x_{i,j}^{h} = x_{i,j}^{v} \end{cases}$$

We split our original problem using variable splitting to two problems, each of which has its own variable set and which would be independent without constraint (6). From this point of view constraint (6) is the hardest one. We will relax constraint (6) using Lagrangian relaxation with coefficients $\lambda_{i,j}$.

We get following problem $VSI(\lambda)$, and its optimal value is $v^{VSI}(\lambda)$.

$$\begin{cases} \max\left(\sum_{i,j} \lambda_{i,j} (x_{i,j}^{h} - x_{i,j}^{v})\right) \\ (1)\sum_{i=1}^{m} x_{i,j}^{v} = s_j,\ j = 1,\cdots,n \\ (2)\sum_{j=1}^{n} x_{i,j}^{h} = r_i, i = 1,\cdots,m \\ (3)\begin{cases} y_{i,j} \le x_{i,j}^{h} \\ y_{i,j} \le x_{i,j+1}^{h} \\ y_{i,j} \ge x_{i,j}^{h} + x_{i,j+1}^{h} - 1 \end{cases} \quad i = 1,\cdots,m,\ j = 1,\cdots,n-1 \\ (4)\sum_{j=1}^{n-1} y_{i,j} = r_i^{'},\ i = 1,\cdots,m \\ (5) x_{i,j}^{h}, x_{i,j}^{v} \in \{0,1\},\ y_{i,j} \in \{0,1\} \end{cases}$$

Then using same method we can further split the problems into subproblems for rows and columns, which itself is reducing to the finding of simple path, with given number of edges and biggest weight on directed graph.

We can approach the problem in other way, by relaxing constraint (3) we would split the problem into two subproblems with $x_{i,j}$ and $y_{i,j}$ variables. But this paper is limited with first approach.

Obviously problem $VSI(\lambda)$ is relaxation of problem I, hence $v^{VSI}(\lambda)$ is upper limit for value of I. Find best upper limit means to solve Lagrangian dual problem which is

$$v^{VSD} = \min_{\lambda} v^{VSI}(\lambda)$$

This is convex non-differential optimization problem: There are different methods for solving this problem. One of them is subgradient optimization method. Subgradient optimization on each step calculates the value of $v^{VSI}(\lambda)$ for given $\lambda_{i,j}$, in this case that equals to solving following m independent problems

$$\begin{cases} \max(\sum_{j=1}^{n} c_j x_j) \\ \sum_{j=1}^{n} x_j = r \\ \begin{cases} y_j \le x_j \\ y_j \le x_{j+1} \qquad j = 1, \cdots, n-1 \\ y_j \ge x_j + x_{j+1} - 1 \end{cases} \\ \sum_{j=1}^{n-1} y_j = r' \\ x_j \in \{0,1\}, y_j \in \{0,1\} \end{cases} \qquad (*)$$

We will try to solve these problems using algorithm for finding simple path on acyclic directed graph with biggest cost and given number of edges.

Decomposed problem on graph and the solution

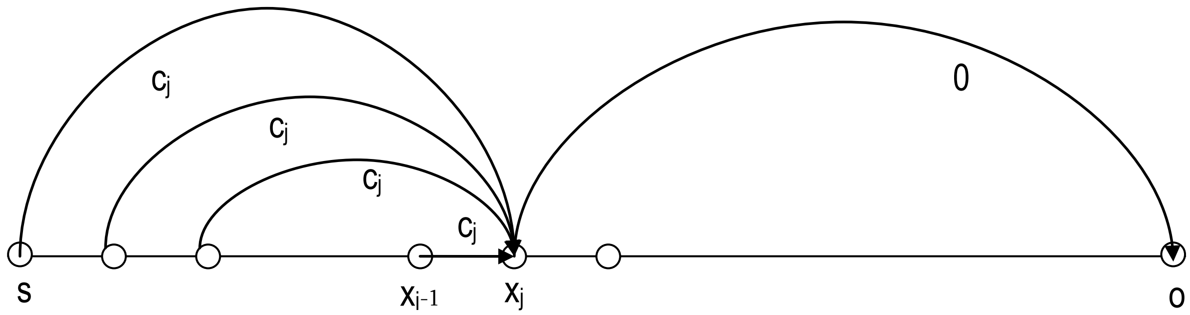
We consider directed graph $G=(V,E)$ which vertex set consists of vertexes for each x_j variable plus s source and o destination. We define edges in following way

(s,x_j) with weight c_j

$(x_i,x_j), i \le j-1$ with weight c_j

(x_j,o) with weight 0

Consider the paths form s to o. Only r variables corresponding to x_j vertexes, are 1's according to (*) and among them r' is neighboring 1's. Hence we are interested only in those paths from s to o that have only r vertexes and there are only r' with neighboring 1's. We need to find among those paths, the one that has maximum weight. Now by assigning 1's to variables corresponding to vertexes we will get solution to the problem (*).



Now lets give algorithmic description.

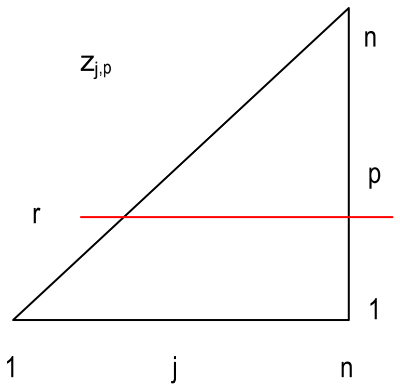
Let $z(j,p)$ is weight of the longest path from s to x_j vertex with p vertexes on it. Lets $w(j,p,q)$ is weight of the longest path from s to x_j which has p vertexes on it and there are q neighboring vertexes with corresponding variables equal to 1. In this case $z(j,p)$ and $w(j,p,q)$ can be calculated the following way. First of all consider $z(j,p)$

$$z(j,1) = c_j$$
$$z(j,p) = \max_{u<x_j}(z(u,p-1)+c_j)$$

And the optimal value we're looking for is $z(o,r) = \max_j z(x_j, r)$.

This part of the problem is solving in the following way.

For given n and r $Z_{j,p} = [z_{j,p}]$ array is constructed, where $j = 1,\ldots,n$ and for j $p = 0,\ldots, j-1$. In reality for fixed r its enough to consider $p = 1,\ldots,r$ layers, but $p = 1,\ldots,n$ will satisfy calculations needed for all r.



First of all $z_{1,1}$ value is calculated. That's equal to c_1. All values of row $p=1$ are calculated in the same way $z_{j,1} = c_j$. To calculate $z_{j,p}$ by our formula we need to know values for $p-1$ and for all $1,\ldots, j-1$ indexes. But in row $p-1$ first non-zero value is in $j = p-1$ position, which is on diagonal. So calculations can be done sequentially on $p = 1,\ldots,r,\ldots$ rows and in rows in order $j = p,\ldots,n$. This constructs are needed for software implementation and these give ability to measure number of operations in calculation. It doesn't exceed n^3, which means polynomial complexity.

Maximal weight paths can be stored in a separate array. They can be stored as 0,1 vectors or as indexes of non zero elements which however won't significantly decrease number of computations.

Now lets calculate values of $w(j,p,q)$. First of all lets consider edge values. From $w(j,p,q)$ we have maximal weight path from S to x_j which has p vertexes and there are q pairs with neighboring 1's. $q \le p-1$ and lets p's are decreased up to $q+1$. $w(j,q+1,q)$'s can be non-zero starting from $j \ge q+1$. For bigger q's and smaller j's $w(j,p,q)$'s are equal to 0.

Interestingly q can't be very small. If $p > \left]\frac{j+1}{2}\right[$ then q can't be 0 (at least 2 vertexes must have neighboring indexes).

Let $\tau = \left[\frac{j+1}{3}\right]$. In that case τ vertex pairs still might not be neighbors, which gives 2τ vertexes. After that any new vertex addition would add 2 new pairs.

Now lets consider common case. For calculating $w(j,p,q)$ lets consider class where for j $p \le j$ and for j,p pairs $q \le p-1$. This class is larger than needed but in reality it doesn't differ much from the minimal class which is necessary for calculations. For slight transition of edge values class is zeroed before performing calculations. Lets investigate value of $w(j,p,q)$. We do chain calculations and on each step consider 2 cases $x_{j-1} = 1$ and $x_{j-1} = 0$. So we get following values

$w(j-1,p-1,q-1)+c_j$ and $\max\limits_{u<x_j-1}(w(u,p-1,q)+c_j)$

We are interested in maximum of these values.

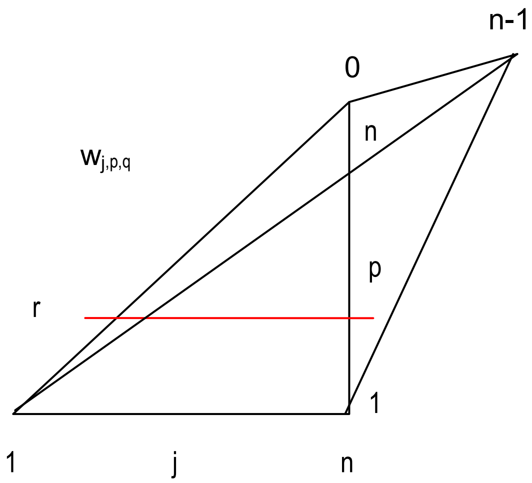
In $w(j-1,p-1,q-1)+c_j$ all indexes are less than preceding and we assume that this value is already calculated in previous steps. For calculating $\max\limits_{u<x_j-1}(w(u,p-1,q)+c_j)$ we do next step in chain calculation.

$$w(u-1,p-2,q-1)+c_j$$
$$\max_{v<u-1}(w(v,p-2,q)+c_j)$$

And the needed optimal value is $w(o,r,r') = \max_j w(x_j, r, r')$. This problem practically can be solved in following way.

For given n, r, r' we construct the class given above, array $W_{j,p,q} = [w_{j,p,q}]$, where for $j = 1,...,n$ and j $p = 1,...,j$ and for pair j, p $q = 0,..., p-1$.

In reality for fixed r it's enough to consider $p = 1,...,r$ layers and for q all values where $q \le r-1$. But calculations must be done in such sequence to be executable.



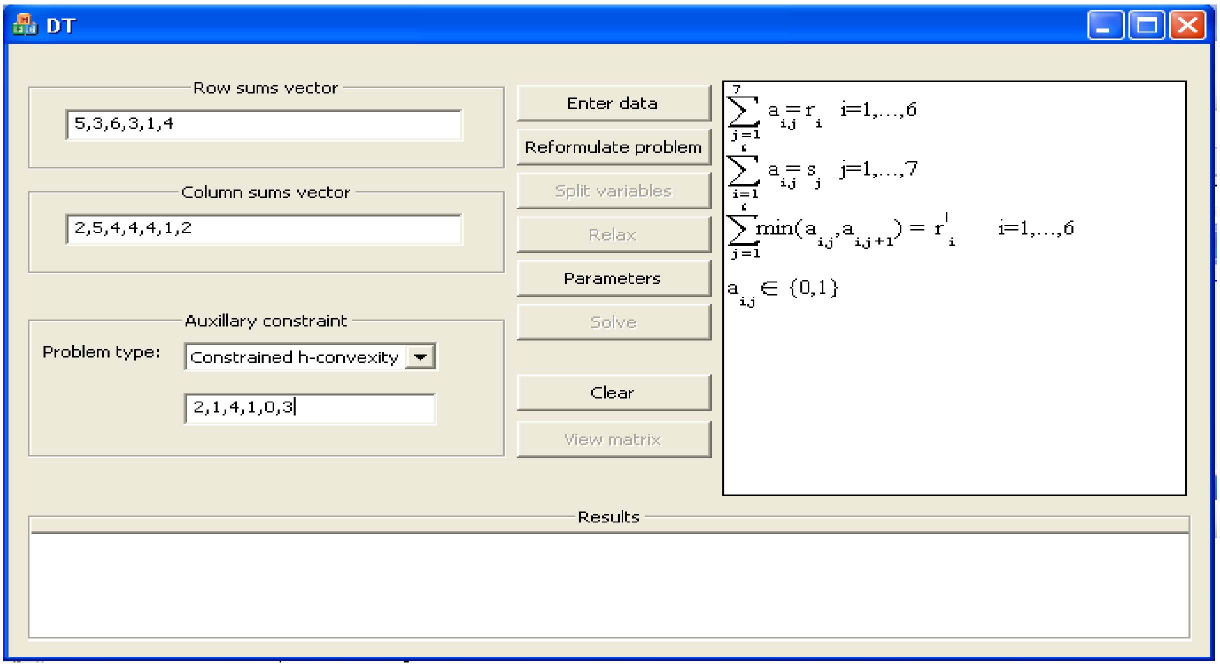
First $w_{1,1,0}$ values are calculated, $w_{1,1,0} = c_1$, all values in row $p = 1$ are calculated in the same way $w_{j,1,0} = c_j$. More, $q = 0$ values were already considered. To calculate $w_{j,p,q}$ based on our formula we need to know values for $p-1$ and all $1,..., j-1$. But in layer $p-1$ with current q value is either 0 or already calculated. Then calculations can be done in layers $p = 1,...,r,...$ sequentially and in layers in order of $j = p,...,n$. Given constructions are needed for software implementation and give ability to measure number of calculations. Those are not more than n^4 which means polynomial complexity.

Maximal weight paths that we're looking for could be stored in separate array as 0,1 vectors or as array of indexes with non-zero values, which however won't significantly lower number of calculations.

Software implementation

Based on given methods a software system with an UI was implemented, which can be used to solve some problems from the field of discrete tomography based on Lagrangian relaxation.

There are several fields which are used for data input. Since we are solving problems in the field of discrete tomography so input data are projections, in our case row sums and column sums. Also we are giving specific problem description by additional constraints. So we have special fields for that purpose. Then there is special control which can be used to reformulate given problem as mathematical programming problem. Then we can choose one or several constraints which we want to relax. Also we can do variable splitting etc. And there is an output window which is used for displaying results. For example value of Lagrangian Dual or variables difference as a result of splitting.



In given example as a problem is considered horizontal row convexity existence problem.

Now lets describe one of the main classes in the implementation, ProblemBase abstract class. This class is base for all problem. Class encapsulates problem data. It also has several virtual functions which are used for problem solution. For example function which reformulates the problem as mathematical programming problem, chooses constraints for relaxation. Important function in ProblemBase is Solve method, which invokes the method for specific problem. Since most of the problems are reducing to relatively easy problems on graphs and methods for those solutions can be used for different problems we put those methods in a separate library.

Bibliography

1. Gardner R.J., Gritzmann P., Prangenberg D., On the computational complexity of reconstructing lattice sets from their X-rays. Technical Report (970-05012), Techn. Univ. Munchen, fak. f. math, 1997.

2. G.J. Woeginger. The reconstruction of polyominoes from their orthogonal projections. Inform. Process. Lett., 77:225-229, 2001.

3. E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani. Reconstructing convex polyominoes from horizontal and vertical projections. Theoret. Comput. Sci., 155:321-347, 1996.

4. G. Dahl and T. Flatberg. Lagrangian decomposition for reconstructing hv-convex (0, 1) matrices, Report 303, University of Oslo, p. 1-13, 2002.

5. M. Guignard and S. Kim. Lagrangian decomposition: a model yielding stronger lagrangian bounds. Math. Prog., 39:215-228, 1987.

6. H.J. Ryser. Combinatorial properties of matrices of zeros and ones. Canad. J. Math., 9:371-377, 1957.

7. D. Gale. A theorem on flows in networks. Paci_c J. Math., 7:1073-1082, 1957.

Authors' Information

Levon Aslanyan – *Head of Department, Institute for Informatics and Automation Problems, P.Sevak St. 1, Yerevan 14, Armenia, e-mail: lasl@.sci.am*

Artyom Hovsepyan – *Researcher, Institute for Informatics and Automation Problems, P.Sevak St. 1, Yerevan 14, Armenia, e-mail:* artyom.hovsepyan@gmail.com

Hasmik Sahakyan – *Leading Researcher, Institute for Informatics and Automation Problems, NAS RA, P.Sevak St. 1, Yerevan 14, Armenia, e-mail: hasmik@ipia.sci.am*

ON HYPERSIMPLE wtt -MITOTIC SETS, WHICH ARE NOT tt -MITOTIC

Arsen H. Mokatsian

Abstract: *A T-complete wtt-mitotic set is composed, which is not tt-mitotic. A relation is found out between structure of computably enumerable sets and the density of their unsolvability degrees.*

Let us adduce some definitions:

A computably enumerable (c.e.) set is tt - mitotic (wtt - mitotic) set if it is the disjoint union of two c.e. sets both of the same tt -degree (wtt -degree) of unsolvability.

Let A be an infinite set. f majorize A if $(\forall n[f(n) \geq z_n])$, where $z_0, z_1, \ldots$ are the members of A in strictly increasing order.

A is hyperimmune (abbreviated h-immune) if A is infinite and ($\forall$ recursive f) [f does not majorize A].

A is hypersimple if A is c.e. and $\overline{A}$ is hyperimmune.

A is hyperhyperimmune if A is infinite and $\neg(\exists$ recursive $f)$ so that

[$(\forall u)[W_{f(u)}$ is finite & $W_{f(u)} \cap A \neq \varnothing]$ & $(\forall u)(\forall v)[u \neq v \Rightarrow W_{f(u)} \cap W_{f(v)} = \varnothing]$].

A is hyperhypersimple if A is computably enumerable and $\overline{A}$ is hyperhyperimmune.

We shall denote T-degrees by small bold Latin letters.

A degree $\mathbf{a} \leq \mathbf{0}'$ is low if $\mathbf{a}' = \mathbf{0}'$ (i.e. if the jump $\mathbf{a}'$ has the lowest degree possible).

*Theorem (Martin [6]). **a** is the degree of a maximal set $\Leftrightarrow$ **a** is the degree of a hypersimple set $\Leftrightarrow$ [**a** is c.e. and $\mathbf{a}' = \mathbf{0}''$].*

*Theorem (R. Robinson [8]). Let **b** and **c** be c.e. degrees such that $\mathbf{c} < \mathbf{b}$ and **c** is low. Then there exist incomparable low c.e. degrees $\mathbf{a_0}$ and $\mathbf{a}_1$, such that $\mathbf{b} = \mathbf{a_0} \cup \mathbf{a_1}$ and $\mathbf{a}_i > \mathbf{c}$, for $i < 2$.*

*Griffiths ([3]) proved that there is a low c.e. T-degree **u** such that if **v** is a c.e. T-degree and **u** $\leq$ **v** then **v** is not completely mitotic.*

In this article it is proved the following theorem:

Theorem. *There exists a low c.e. T-degree **u** such that if **v** is a c.e. T-degree and **u** $\leq$ **v** then **v** contains hypersimple wtt -mitotic set, which is not tt -mitotic.*

From the abovementioned theorems of Martin and R. Robinson follows that it is impossible to replace hypersimple by hyperhypersimple.

Keywords: *computably enumerable (c.e.) set, mitotic, wtt -reducibility, tt -reducibility, hypersimple set, low degree.*

ACM Classification Keywords: *F. Theory of Computation, F.1.3 Complexity Measures and Classes.*

Introduction

We shall use notions and terminology introduced in [9], [10].

The definitions of tt - and wtt - reducibilities are from [9].

$\varphi(x)\downarrow$ denotes, that $\varphi(x)$ is defined, and $\varphi(x)\uparrow$ denotes, that $\varphi(x)$ is undefined.

Definition. The order pair **<<** $x_1,\cdots,x_k$ **>,** α **>**, where **<** $x_1,\cdots,x_k$ **>** is a k -tuple of integers and α is a k -ary Boolean function ($k>0$) is called a *truth-table condition* (or tt *-condition*) *of norm* k . The set $\{x_1,\cdots,x_k\}$ is called the *associated set of the* tt *-condition*.

Definition. The tt -condition **<<** $x_1,\cdots,x_k$ **>,** α **>**, is *satisfied* by A if $\alpha(c_A(x_1),\cdots,c_A(x_k))=1$, where c_A is characteristic function for A .

Each tt -condition is a finite object; clearly an effective coding can be chosen which maps all tt -conditions (of varying norm) onto ω , on condition that

$$(\forall x)\left(\max\{z \mid z \text{ is the member of associated set of the } tt\text{ - condition } x\}\leq x\right).$$

Assume henceforth that a particular such coding has been chosen. Where we speak of “ tt -condition x ”, we shall mean the tt -condition with the code number x .

Definition. A is *truth-table reducible to* B (notation: $A\leq_{tt} B$) if there is a computable function f such that for all x , [$x\in A \Leftrightarrow tt$ -condition $f(x)$ is satisfied by B]. We also abbreviate “truth-table reducibility” as “ tt -reducibility”.

Definition. A is weak truth-table reducible to B (notation: $A\leq_{wtt} B$) if

($\exists z$)[$c_A=\varphi_z^B$ ($\exists$ computable f)

($\forall x$)[$D_{f(x)}$ contains all integers whose membership in B is used in the computation of $\varphi_z^B(x)$]].

Definition. A c.e. set is tt - mitotic (wtt - mitotic) set if it is the disjoint union of two c.e. sets both of the same tt -degree (wtt -degree) of unsolvability.

Let $A \leq_{tt} B$ and $(\forall x)$ [$x \in A \Leftrightarrow tt$ -condition $f(x)$ is satisfied by B] and $\varphi_n = f$. Then we say that $A \leq_{tt} B$ *by* φ_n.

Let us modify denotations defined in [4] with the purpose to adapt them to our theorem.

We say that $(A_o, A_1, \varphi_o, \varphi_1)$ is *tt -mitotic splitting of A* if A_o and A_1 are c.e., $A_o \bigcup A_1 = A$, $A_o \bigcap A_1 = \varnothing$, $A \leq_{tt} A_0$ by ψ_0 and $A \leq_{tt} A_0$ by ψ_0 $A \leq_{tt} A_1$ by ψ_1.

Let h be a recursive function from ω onto ω^4.

Define $(Y_i, Z_i, \vartheta_i, \psi_i)$ to be a quadruple $(W_{i_o}, W_{i_1}, \varphi_{i_2}, \varphi_{i_3})$, where $h(i) = (i_0, i_1, i_2, i_3)$. If A is c.e. then we say that the *non-tt -mitotic condition of i order is satisfied for A*, if it is not the case that $(Y_i, Z_i, \vartheta_i, \psi_i)$ is a *tt -mitotic splitting of A*.

Denotation. $u^{tt}(i,n,s) = \begin{cases} x^i_{k_n}, & if\ \varphi_{i,s},(n)\downarrow, \\ 0, & otherwise \end{cases}$,

where *tt*-condition $\varphi_i(n)$ = << $x_1^i, \cdots, x_{k_n}^i$ >, α_n^i > .

We define two computable functions that will be of use later.

1. $k(i,n,s) = \max\{n, \{u^{tt}(i_2, Y^s, m, s) : m \leq n\} \bigcup \{u^{tt}(i_3, Z^s, m, s) : m \leq n\}\}$,
2. $L(A,i,s) = \mu n[\neg(c_A(n) = 1 \Leftrightarrow tt\text{ - condition } \vartheta_i(n) \text{ satisfied by } Y_i) \vee$
   $\neg(c_A(n) = 1 \Leftrightarrow tt\text{ - condition } \psi_i(n) \text{ satisfied by } Z_i)]$,

where $h(i) = (i_0, i_1, i_2, i_3)$.

Adduce some information, concerning hypersimple sets.

Definitions.

Let *A* be an infinite set. *f majorizes A* if $(\forall n[f(n) \geq z_n])$, where $z_0, z_1, \ldots$ are the members of *A* in strictly increasing order.

A is *hyperimmune* (abbreviated *h-immune*) if *A* is infinite and ($\forall$ recursive f) [*f* does not majorize *A*].

A is *hypersimple* if A is computably enumerable and $\overline{A}$ is hyperimmune.

A useful characterization of hyperimmune sets is given in the following theorem.

Theorem (Kuznecov, Medvedev, Uspenskii [7]). A is hyperimmune $\Leftrightarrow$ A is infinite and

$\neg(\exists$ recursive $f)[(\forall u)[D_{f(u)} \cap A \neq \varnothing]\ \&(\forall u)(\forall v)[u \neq v \Rightarrow D_{f(u)} \cap D_{f(v)} = \varnothing]]$.

Definitions.

A is *hyperhyperrimmune* if A is infinite and $\neg(\exists$ recursive $f)$

$[(\forall u)[W_{f(u)}$ is finite $\&\ W_{f(u)} \cap A \neq \varnothing]\ \&(\forall u)(\forall v)[u \neq v \Rightarrow W_{f(u)} \cap W_{f(v)} = \varnothing]]$.

A is *hyperhypersimple* if A is computably enumarable and $\overline{A}$ is hyperhyperimmune.

A degree $\mathbf{a} \leq \mathbf{0}'$ is low if $\mathbf{a}' = \mathbf{0}'$ (i.e. if the jump $\mathbf{a}'$ has the lowest degree possible).

Theorem (Martin [6]). **a** is the degree of a maximal set $\Leftrightarrow$ **a** is the degree of a hypersimple set $\Leftrightarrow$ [**a** is c.e. and $\mathbf{a}' = \mathbf{0}''$].

Theorem (R. Robinson [8]). Let **b** and **c** be c.e. degrees such that $\mathbf{c} < \mathbf{b}$ and **c** is low. Then there exist incomparable low c.e. degrees $\mathbf{a_0}$ and $\mathbf{a}_1$, such that $\mathbf{b} = \mathbf{a_0} \cup \mathbf{a_1}$ and $\mathbf{a}_i > \mathbf{c}$, for $i < 2$.

Griffiths ([3]) proved that there is a low c.e. T-degree **u** such that if **v** is a c.e. T-degree and **u** $\leq$ **v** then **v** is not completely mitotic.

Let us prove the following theorem.

<u>**Theorem.**</u> There exists a low c.e. T-degree **u** such that if v is a c.e. T-degree and **u** $\leq$ **v** then **v** contains hypersimple wtt-mitotic set, which is not tt-mitotic.

From the abovementioned theorems of Martin and R. Robinson follows that it is impossible to replace hypersimple by hyperhypersimple.

Proof. This statement is proved using a finite injury priority argument. We construct a member U of $\mathbf{u}$ in stages

s, $U = \bigcup_{s\in\omega} U_s$. We also construct sets $\{V_e\}_{e\in\omega}$ to witness that each c.e. T-degree in upper cone of $\mathbf{u}$ contains a wtt-mitotic but non-tt-mitotic set.

Denote $\omega^0 = \{x : (\exists y)\,(2y = x)\}, \omega^1 = \omega \setminus \omega^0$.

Construct U, $\{V_e\}_{e\in\omega}$ to satisfy, for all $e \in \omega$, the requirements:

N_e: $\{e\}^U(e)\downarrow$ has a limit in s, the stage.

$R_{\langle e,i\rangle}$: The non-*tt*-mitotic condition of order i is satisfied for V_e.

$P_{<e,i>}$: $(\varphi_i$ is total computable & $(\forall u)[D_{f(u)} \cap A \neq \varnothing]$ &

$(\forall u)(\forall v)[u \neq v \Rightarrow D_{f(u)} \cap D_{f(v)} = \varnothing]) \Rightarrow (\exists z)(D_{\varphi_i(z)} \subset V_e)$.

$\widetilde{P}_e$: $W_e = \Lambda^{V_e}$ for some computable functional Λ.

We also ensure by permitting that $V_e \equiv_T U \oplus W_e$ and else $V_e^0 \equiv_{wtt} V_e^1$ (where $V_e^0 = V_e \cap \omega^0$ & $V_e^1 = V_e \cap \omega^1$).

If $U \leq_T W_e$ then the above ensure that $V_e \equiv_T U \oplus W_e \equiv_T W_e$ and V_e is not *tt*-mitotic. Hence, $deg(W_e)$ is not *tt*-mitotic but is wtt-mitotic, and $\mathbf{u} = deg(U)$ is the required degree.

Let $\langle\ ,\ \rangle$ be computable bijective pairing function increasing in both coordinates. At each stage s place markets $\lambda(e,x,s)$ on elements of $\overline{V}_{e,s}$. Values of λ will be used both as witnesses to prevent the *tt*-mitoticity of V_e sets (by corresponding $Y_i, Z_i, \vartheta_i, \psi_i$) and to ensure that W_e is T-reducible to V_e. Initially

$\lambda(e,x,0) = 4(\langle e,x\rangle + 1) - 2)$ for all $e, x \in \omega$.

Also define a function $\xi(e,i,s)$ for all $e,i \in \omega$ (at each stage s), $\xi(e,i,0) = i$ for all $e,i \in \omega$. We use ξ to ensure that only members of sufficiently large magnitude enter U at stage s, so we can satisfy the lowness requirements N_e.

According to the theorem (Kuznecov, Medvedev, Uspenskii) the satisfaction of $P_{<e,i>}$ (for all i) ensure the hypersimplicity of V_e.

Order the requirements in the following priority ranking:

$N_0, R_0, P_0, N_1, R_1, P_1, N_2, R_2, P_2, \ldots$

The $\{\widetilde{P}_e\}_{e\in\omega}$ do not appear in this ranking.

N_e *requires attention* if it is not *satisfied* and $\{e\}^U(e)[s]\downarrow$.

$R_{\langle e,i\rangle}$ *requires attention* if it is not *satisfied* and

$(\forall x_{\leq y})\left(\vartheta_i^s(x)\downarrow \;\&\; \psi_i^s(x)\downarrow\right)$, where $y=\lambda(e,\xi(e,i,s)s)$.

$(Y_i, Z_i, \vartheta_i, \psi_i)$ is *threatening* A *through* x *at stage* s if it is partially satisfied and all the following hold:

i) $i\leq s$,

ii) $x<L(A,i,s)$,

iii) $Y_i^s\bigcap Z_i^s=\varnothing$,

iv) $c_A^s(m)=(Y_i^s\bigcup Z_i^s)(m)$ for all $m\leq k(i,x,s)$.

(Note, that actually $R_{\langle e,i\rangle}$ is partially satisfied, if $R_{\langle e,i\rangle}$ requires attention (via some $y=\lambda(\ ,\ ,s)$) and corresponding $y-1,\ y-2$ belong to V_e, $y-1$ belongs to U_{s+1}. See Construction, Part A , a)).

We will build $U=\bigcup_s U_s$ and $V_s=\bigcup_s V_{e,s}$ for all $e\in\omega$. Initially all requirements N_e, $R_{\langle e,i\rangle}$ are declared *unsatisfied*.

Construction

Stage $s=0$. Let $U_0=\varnothing$, $V_{e,0}=\varnothing$ for all $e\in\omega$.

Stage $s+1$.

Part A. Act on the highest priority requirement which requires attention, if such a requirement exists.

a) If N_e *requires attention* then set $\xi\left(\hat{e},\hat{\imath},s+1\right)=\xi\left(\hat{e},\hat{\imath}+s,s\right)$ for each $\langle\hat{e},\hat{\imath}\rangle\geq e$. This action prevents injury to N_e by lower priority requirements as we assume that s bounds the use of the halting computation.

Define $\xi(\ ,\ ,s+1)$ not specified in Part A to be the same as $\xi(\ ,\ ,s)$.

Declare N_e satisfied; declare all lower priority R, N unsatisfied.

If $R_{\langle e,i\rangle}$ require attention via $y=\lambda\left(e,\xi(e,i,s+1),s\right)$ then set $\widetilde{V}_{e,s+1}=V_{e,s}\bigcup\{y-1,y-2\}$ and

$\widetilde{U}_{s+1} = U_s \bigcup \{y-1\}$. (Note that such $\langle e,i\rangle$ cannot be $\geq e$, so $\xi(e,i,s+1) = \xi(e,i,s)$). Declare $R_{\langle e,i\rangle}$ partially satisfied.

Define $\widetilde{V}_{e,s+1}, \widetilde{U}_{e,s+1}$ not specified in Part A, a) to be the same as $V_{e,s+1}, U_{e,s+1}$ respectively.

b) If ($Y_i, Z_i, \vartheta_i, \psi_i$) is threatening $\widetilde{V}_{e,s+1}$ through y at stage $s+1$ (so $R_{\langle e,i\rangle}$ is partially satisfied via $y = \lambda(e, \xi(e,i,s), s)$), then set $\widetilde{\widetilde{V}}_{e,s+1} = \widetilde{V}_{e,s+1} \bigcup \{y\}$ and $\widetilde{\widetilde{U}}_{s+1} = \widetilde{U}_{s+1} \bigcup \{y\}$.

If $R_{\langle e,i\rangle}$ is partially satisfied, via $y = \lambda(e, \xi(e,i,s), s)$, whether ($Y_i, Z_i, \vartheta_i, \psi_i$) is threatening $\widetilde{V}_{e,s+1}$ through y at stage $s+1$ or not define $\lambda^1(e, \xi(e,i,s), s+1) = \lambda(e, \xi(e, i+s, s), s)$.

Define $\widetilde{\widetilde{V}}_{e,s+1}, \widetilde{\widetilde{U}}_{e,s+1}, \lambda^1(\ ,\ , s+1)$ not specified in Part A, b) to be the same as $\widetilde{V}_{e,s+1}, \widetilde{U}_{e,s+1}, \lambda(\ ,\ , s)$ respectively.

Such definition of λ^1 allow us to satisfy $R_{\langle e,i\rangle}$ requirement (after Part A) whether ($Y_i, Z_i, \vartheta_i, \psi_i$) is threatening $\widetilde{V}_{e,s+1}$ through y or not (if don't take into consideration higher priority requirements).

Declare $R_{\langle e,i\rangle}$ satisfied; declare all lower priority R, N unsatisfied.

Part B. If $x \in W_{e,s+1} \setminus W_{e,s}$ then set

$$V^*_{e,s+1} = \widetilde{\widetilde{V}}_{e,s+1} \bigcup \left\{ \lambda^1(e, x, s+1) \right\} \quad \text{and}$$

$$\lambda^2(e, x+j, s+1) = \lambda^1(e, \xi(e, x+j+1, s+1), s+1) \text{ for all } j \in \omega.$$

Find all $\hat{i}$ such that $\lambda(e, \xi(e, \hat{i}, s+1), s) \geq \lambda^1(e, x, s+1)$ and declare $R_{\langle e,\hat{i}\rangle}$ *unsatisfied* for each such $\hat{i}$.

Define $V^*_{e,s+1}, \lambda^2(\ ,\ , s+1)$ not specified in Part B to be the same as $\widetilde{\widetilde{V}}_{e,s+1}, \lambda^1(\ ,\ , s+1)$ respectively.

Note that for all s, $\xi(e,i,s)$ is increasing in both e and i.

Part C. Let $m_{s+1} = \max\{ \{\xi(\hat{e}, \hat{i}, s+1) \mid \text{for all } \hat{e}, \hat{i} \text{ with } \langle \hat{e}, \hat{i}\rangle \leq \langle e,i\rangle\},$

$$\{\lambda^2(\hat{e}, \hat{i}, s+1) \mid \text{for all } \hat{e}, \hat{i} \text{ with } \langle \hat{e}, i\rangle \leq \langle e,i\rangle\} \}.$$

If $(\exists\, z)\left(\varphi_{i,s+1}(z)\downarrow\right)$, denote $z_0 = \mu y\left(y = D_{\varphi_i}(z)\right)$. Then, if $z_0 > m_0$ and $P_{\langle e,i\rangle}$ is not satisfied,

set $(\forall y)\left(y = D_{\varphi_i}(z)\right) \Rightarrow V_{e,s+1} = V^*_{e,s+1} \cup \{y, y+1\}\ \& U_{s+1} = \tilde{\tilde{U}}_{s+1} \cup \{z_0\}$.

Set $\lambda\,(e, \xi(e,\hat{i}, s+1), s+1) = \lambda^2(e, \xi(e, \hat{i}+s, s+1), s+1)$, for all $\hat{i} \geq i$.

Define $V_{e,s+1}, U_{e,s+1}, \lambda(\ ,\ , s+1)$ not specified in Part C, to be the same as $V^*_{e,s+1}, \tilde{\tilde{U}}_{e,s+1}, \lambda^2(\ ,\ , s+1)$ respectively.

Declare $P_{\langle e,i\rangle}$ *satisfied*, declare all lower priority R, N unsatisfied.

Verification

Lemma 1. *For all* e, i:

1. N_e is met, $\lim_s \xi(e,i,s) = \xi(e,i)$ exists.
2. $R_{\langle e,i\rangle}$ is met, $\lim_s \lambda\left(e, \xi(e,i,s), s\right)$ exists.

Proof. By induction on $j = \langle e,i\rangle$.

Suppose there exists a stage s_0 such that for all $\hat{e},\hat{i}$ with $\langle \hat{e},\hat{i}\rangle < j$:

1. $N_{\langle \hat{e},\hat{i}\rangle}$ is met and never acts after stage s_0, $\lim_s \xi(\hat{e},\hat{i},s) = \xi(\hat{e},\hat{i})$ exists and is attained by s_0.

2. $R_{\langle \hat{e},\hat{i}\rangle}$ is met and never acts after stage s_0, $\lim_s \lambda(\hat{e}, \xi(\hat{e},\hat{i},s), s)$ exists and is attained by s_0.

1). The proof of point 1 is similar to Lemma 1 of Theorem 2.2.2 [3].

After stage s_0 the requirements $N_{\langle \hat{e},\hat{i}\rangle}$, $R_{\langle \hat{e},\hat{i}\rangle}$ (for all $\hat{e},\hat{i}$ with $\langle \hat{e},\hat{i}\rangle < j$) do not injury N_j. Positive requirements $P_{\langle \hat{e},\hat{i}\rangle}$ for all $\hat{e},\hat{i}$ with $\langle \hat{e},\hat{i}\rangle < j$, can injury N_j only finitely. So there is stage s_1 after which if N_j receives attention, then it is met and never injured, so there is a $s_2 > s_1$ after which N_j does not receive attention. (Else set $s_2 = s_1$). Thus $\xi(e,i,s_2+1) = \xi(e,i)$, because $\xi(e,i,s_2+1)$ is not changed after.

2). Now consider point 2. Note, that positive requirements $P_{\langle e,i\rangle}$, $\tilde{P}_e$ injury each of the requirements with lower priority only finitely.

Let the stage s_1 is such stage, that $s_1 > s_0$ and $N_{\langle \hat{e},\hat{i} \rangle}$, $R_{\langle \hat{e},\hat{i} \rangle}$, $P_{\langle \hat{e},\hat{i} \rangle}$, $\widetilde{P}_{\langle \hat{e},\hat{i} \rangle}$ (for all $\hat{e},\hat{i}$ with $\left\langle \hat{e},\hat{i} \right\rangle < j$) are met and never acts after stage s_1 .

The following Lemma is used (in [4], [5]) for building the non-T-mitotic set :

Lemma. If $\left(Y_i, Z_{i,} \theta_i, \psi_i\right)$ is threating A through x at stage $s, x \in A - A_s$ and for all $m \neq x$ such that $m \leq k(i,x,s)$ we have $A_m - A_s(m)$, then the non-T-mitotic condition of order i is satisfied for A .

Similar lemma is thrue for tt-reducibility.

Let s_0 is such stage that $N_{<\hat{e},\hat{i}>}$, $R_{<\hat{e},\hat{i}>}$ $P_{<\hat{e},\hat{i}>}$ are met and never acts after stage s_1 .

If there isn't such $y = \lambda(e, \xi(e,i,s'+1), s')$ (where $s' > s_1$), that $R_{\langle e,i \rangle}$ is partially satisfied (via y), then $R_{\langle e,i \rangle}$ is met.

If there exists such y and $\left(Y_i, Z_{i,} \vartheta_i, \psi_i\right)$ never threatens V_e through y after stage s' , then certainly the condition is satisfied. On the other hand, if $\left(Y_i, Z_{i,} \vartheta_i, \psi_i\right)$ threatens V_e through y at time $t > s'$, then put y into V_e at time $t+1$, and never put any other number $\leq k(i,y,t)$ into V_e after stage t , so $R_{\langle e,i \rangle}$ is met.

Lemma 2. $P_{\langle e,i \rangle}$ is met.

According to Lemma1 $(\exists\, s_0)\, [\xi(\hat{e},\hat{i},s_0) = \xi(\hat{e},\hat{i})$, for all $\hat{e},\hat{i}$ with $(\hat{e},\hat{i}) \leq (e,i)$ &

$\& \lambda\left(\hat{e}, \xi(\hat{e},\hat{i},s_0), s_0\right) = \lambda\left(\hat{e}, \xi(\hat{e},\hat{i})\right)$ for all $\hat{e},\hat{i}$ with $(\hat{e},\hat{i}) \leq (e,i)]$.

Denote $m_0 = \max\left\{ \{\xi(\hat{e},\hat{i}) |\text{ for all } (\hat{e},\hat{i}) \leq (e,i)\}, \{\lambda\left(\hat{e}, \xi(\hat{e},\hat{i})\right) |\text{ for all } \hat{e},\hat{i} \text{ with } (\hat{e},\hat{i}) \leq (e,i)\} \right\}$.

Then if $(\varphi_i$ is total computable & $(\forall u)[D_{f(u)} \cap A \neq \varnothing]$ &

$(\forall u)(\forall v)[u \neq v \Rightarrow D_{f(u)} \cap D_{f(v)} = \varnothing]) \Rightarrow (\exists s_{>s_0})(\exists z)\left(\varphi_{i,s}(z)\downarrow \;\&\; \{0,1,\ldots,m_0\} \cap D_{\varphi_i(z)} = \varnothing\right)$.

Thus, according to constraction $P_{\langle e,i \rangle}$ is met, because $D_{\varphi_i(z)}$ enters V_e .

Lemma 3 . $V_e \leq_T U \oplus W_e$.

Proof. By permitting: in the construction a number k enters V_e only if a number less than or equal to k enters U or enters W_e.

Lemma 4. *For all* e, $\widetilde{P}_e$ *is satisfied, that is* $W_e = \Lambda^{V_e}$.

Proof. To determine whether $z \in W_e$ we need to find a stage such that $\lambda(e,z,s)$ has attained its limit. V_e computably determines $\lambda(e,0),\ldots,\lambda(e,z)$ (note that $\lambda(e,y,s)$ changes only if a number $\leq \lambda(e,y,s)$ enters V_e).

Find a stage s_z such that $V_{e,s_z} \upharpoonright \gamma_z + 1 = V_e \upharpoonright \gamma_z + 1,$ where $\gamma_z = \max\{\lambda(e,0),\ldots,\lambda(e,z)\}$. Then $z \in W_e$ iff $z \in W_{e,s_z}$.

Lemma 5. V_e *is* wtt *-mitotic.*

Proof. 1) Prove $V_e^0 \leq_{wtt} V_e^1$ (and hence $V_e \leq_{wtt} V_e^1$).

To determine whether $x \in V_e^0$ find such stage s, that $V_{e,s}^1 \upharpoonright x+2 = V_e^1 \upharpoonright x+2$. Then $x \in V_e^0 \Leftrightarrow x \in V_{e,s}^0$, because

i) if $\neg\exists i,\hat{s}$, such that $(Y_i, Z_i, \vartheta_i, \psi_i)$ is threatening $\widetilde{V}_{e,s+1}$ through x at stage $\hat{s}$, then $x \in V_e^0$, only if a number less than or equal to $x+1$ enters,

ii) otherwise, then find a stage s' (this stage s' obligatory is $\leq s$) such that $x-1 \in V_e^1$. If after the stage s' such changes happen in $V_{e,s}^1 \upharpoonright x+2$, which lead to displacement of marker $\lambda(e,x,s')$ and we have $x \notin V_{e,s}^0$, then $x \notin V_e^0$. Thus $x \in V_e^0 \Leftrightarrow x \in V_{e,s}^0$.

2) Prove $V_e^1 \leq_{wtt} V_e^0$.

The proof is similar to abovementioned in item 1), only without point *ii).*

Bibliography

1. R. G. Downey and T. A. Slaman. Completely mitotic r. e. degrees. In: Ann. Pure Appl. Logic, pp.119–152, 41, 1989.
2. R. G. Downey and M. Stob. Splitting theorems in recursion theory. In: Ann. Pure Appl. Logic, n. 1, v. 65, 106 p., 1993.
3. E. J. Griffiths. Completely Mitotic Turing degrees, Jump Classes and Enumeration Degrees. In: Ph. D. Thesis, University of Wisconsin-Madison, 1998.
4. H. Lachlan. The priority method. Zeitschrift für mathematische Logik und Grundlagen der Mathematik, vol.13, pp.1-10, 1967.
5. R. Ladner. Mitotic Enumerable Sets. In: The Journal of Symbolic Logic, pp. 199-211, v. 38, n. 2, June 1973.
6. D. A. Martin. A Theorem on Hyperhypersimple Sets. In: The journal of symbolic logic. vol.28, pp.273-278,1963.
7. Yu.T. Medvedev. On Nonisomorphic Recursively Enumerable Sets (Russian). In: Doklady Akademii Nauk SSSR, n.s., vol. 102, pp. 211-214, 1955.
8. R. W. Robinson. Interpolation and embedding in the recursively enumerable degrees. In: Ann. Of Math. (2), 93, pp. 285-314, 1971.
9. H. Rogers. Theory of Recursive Functions and Effective Computability. In: McGraw-Hill Book Company, 1967.
10. R. I. Soare. Recursively Enumerable Sets and Degrees. In: Springer-Verlag, 1987.
11.

Authors' Information

Arsen Mokatsian – Leading Researcher at the Institute for Informatics and Automation Problems of the National Academy of Sciences of Republic of Armenia, P.Sevak street, no. 1, 0014 Yerevan, Armenia; e-mail: arsenmokatsian@gmail.com

A NEW ALGORITM FOR THE LONGEST COMMON SUBSEQUENCE PROBLEM

Vahagn Minasyan

Abstract: *This paper discusses the problem of determining the longest common subsequence (LCS) of two sequences. Here we view this problem in the background of the well known algorithm for the longest increasing subsequence (LIS). This new approach leads us to a new online algorithm which runs in $O(d\ log\ n)$ time and in $O(n)$ space where n is the length of the input sequences and d is the number of minimal matches between them. Using an advanced technique of van Emde Boas trees the time complexity bound can be reduced to $O(d\ log\ log\ n)$ preserving the space bound of $O(n)$.*

Keywords: *longest common subsequence, longest increasing subsequence, online algorithm.*

ACM Classification Keywords:*G.2.1 Discrete mathematics: Combinatorics*

Introduction

Let $A = a_1 \cdots a_i \cdots a_m$and $B = b_1 \cdots b_j \cdots b_n$, $1 \leq m \leq n$, be two sequences over some alphabet Σ of size s, $s \geq 1$. A sequence $C = c_1 \cdots c_k \cdots c_l$, $1 \leq l$, over Σ is called a *subsequence* of A, if C can be obtained from A by deleting some of its elements, that is if exists a set of indices$\{i_1, \ldots, i_k, \ldots, i_l\}$ such that $1 \leq i_1 < \cdots < i_k < \cdots < i_l \leq m$ and $c_k = a_{i_k}$ for $1 \leq k \leq l$. C is said to be a *common subsequence* of A and B, if it is a subsequence of both sequences A and B;C is said to be a *longest common subsequence (LCS)* of A and B, if it has the maximum length among all common subsequences of A and B; that length is called the *LCS length* of A and B. In general the longest common subsequence is not unique.

The *Longest Common Subsequence Problem (LCS Problem)* is to determine a LCS of $A$and B. Oftenthe problem of determining theLCS length is also referred to as LCS Problem. This is due to the fact that most of algorithms intended to find the LCS length can easily be modified to determine a LCS [Bergroth, 2000]. In this paper we will concentrate on determining the LCS length rather thandetermining an actual LCS.The first known solution of the LCS Problem is based on dynamic programming [Cormen, 2009].For $1 \leq i \leq m$ and $1 \leq j \leq n$ denote by $l_{i,j}$ the LCS length of sequences $a_1 \cdots a_i$and $b_1 \cdots b_j$; thus $l_{m,n}$ is the LCS length of A and B. Note that thefollowing recursion holds for $l_{i,j}$:

$$l_{i,j} = \begin{cases} 0 \text{ if } i = 0 \text{ or } j = 0 \\ l_{i-1,j-1} + 1 \text{ if } a_i = b_j \\ \max\{l_{i-1,j}, l_{i,j-1}\} \text{ if } a_i \neq b_j \end{cases} \quad (1)$$

Based on this relation it is easy to construct an algorithm which fills an array of size $m \times n$, where (i, j)-th cell contains the value of $l_{i,j}$. As it follows form (1) such algorithm has to fill the rest of array before obtaining the value of (m, n)-th cell, so it will determine the LCS length of sequences A and Bin $\Theta(mn)$ time and $\Theta(mn)$ space ($\Theta(1)$ time for filling each cell and $\Theta(1)$ space for holding each cell).A simple trick can be used to make this algorithm require only $\Theta(m + n)$ space to obtain the value of the (m, n)-th cell [Cormen, 2009]. Here we give some definitions which will be used later in the paper. For $1 \leq i \leq m$ and $1 \leq j \leq n$ the pair (i, j) is called *matching* between sequences A and B if $a_i = b_j$; it is called *minimal (or dominant) matching* if for every other matching (i', j') such that $l_{i,j} = l_{i',j'}$ it holds $i' > i$ and $j' \leq j$ or $i' \leq i$ and $j' > j$.Note that if m' and n' are two integers such that $m \leq m'$and $n \leq n'$, then theLCS Problem for two sequences of size m and n is asymptotically not harder than the LCS Problem for two sequences of size m' and n'. Indeed, given two sequences of size m and n and an algorithm which solves the LCS Problem for two sequences of size m' and n', we can lengthen the given sequences (by appending to themsymbols which don’t occur in the initial sequences) up to size m' and n' respectively and pass the resulting two sequences to the given algorithm. It is easy to see that such algorithm will solve the LCS Problem for two sequences of size m and n in asymptotically the same time and space bounds as the given algorithm solves the LCS Problem for two sequences of size m' and n'. This means that each lower bound for the LCS Problem for two sequences of size m andeachupper bound for the LCS Problem for two sequences of size n are respectively lower and upper bounds for the LCS Problem for two sequences of size m and n (recall that $m \leq n$). At [Aho, 1976] the LCS Problem is examined using the decision tree model of computation where the decision tree vertices represent “equal-unequal” comparisons. There it is shown that each algorithm solving the LCS Problem and fitting this model has time complexity lower bound of $\Omega(ms)$, where s is the number of distinct symbols occurring in the sequences (i.e. the alphabet size). This means that the LCS Problem with unrestricted size of the alphabet has time complexity lower bound of $\Omega(mn)$, as such LCS Problem can be viewed as an LCS Problem with restricted alphabet of size $(m + n)$.In practice the underlying encoding scheme for the symbols of the alphabet implies a topological order between them. Algorithms which take into account this fact don’t fit the decision tree model with “equal-unequal” comparisons examined at [Aho, 1976]. At [Masek, 1980] it is presented an algorithm which applies the “Four Russians” trick to the dynamic programming approach, thusit doesn’t fit the model examined at [Aho, 1976] and has time complexitybound of$O(mn/\log m)$.This bound isasymptotically the best known for general case LCS Problem[Cormen, 2009].

Previous Results

Lot of algorithms have been developed for the LCS Problem that, although not improving the time complexity bound $O(mn)$, exhibit much better performance for some classes of sequences A and B [Bergroth, 2000]. Consider the special case when the alphabet Σ consists of first n integers, i.e. $\Sigma = \{1, \dots, j, \dots, n\}$, and the sequences A and B are two permutations of Σ. It is easy to check that this case can be reduced to the case where B is the identical permutation (by replacing b_j by j for $1 \le j \le n$ in both sequences A and B we will get two sequences which are equivalent to the initial ones with respect to the LCS Problem). In this case each LCS of A and B is an increasing sequence of some of first n integers and each such sequence is a LCS of $A$and B. Thusin the case when A and $B$are permutations the LCS Problem is reduced to the problem of determining a longest increasing subsequence of permutation B. The *Longest Increasing Subsequence (LIS) Problem* is to determine a non decreasing subsequence of maximum length in the given sequence of integers. The LIS Problemcan be solved in $O(n \log n)$time [Fredman, 1975], and using advanced data structures like van Emde Boas trees [Cormen, 2009]thistime bound can be reduced to $O(n \log \log n)$. Thus these bounds apply to the LCS Problem in the case of permutations.Also there are many algorithms for the general case LCS Problem which except$m$ and $n$are also sensitive for other parameters like the LCS length, the alphabet size, the number of matches and the number of minimal matches. A survey on such algorithms is given at [Bergroth, 2000].The table below gives a brief remark of some of known algorithms for the LCS Problem. There l denotes the LCS length, s denotes the alphabet size, r denotes the number of all matches and d denotes the number of minimal matches. It is known [Baeza-Yates, 1999] that for two random sequences of length n the expected LCS length is $O(n)$ and the expected number of minimal matches is $O(n^2)$ [Tronicek, 2002]. This means that (except the 5th) none of the algorithms mentioned in the table hastime complexity upper bound less than$O(mn)$not only in the worst case but also in the average case.

All these algorithms are developed in the background of building the $m \times n$ array mentioned in the dynamic programming approach, and they purport to perform fewer operations in order to obtain the (m, n)-th cell of that array. In this paper we view the LCS Problem inanother background, namely the background of the classical algorithm for the LIS Problem described at [Fredman, 1975]. For sure each term we deal with in this background has its direct analogue in the background of the $m \times n$ array; however our approach can be justified by the fact that it leadsus to simpler constructions and an $O(d \log n)$ algorithm for the LCS Problem which can be reduced to $O(d \log \log n)$ if using van Emde Boas trees (details are in the next section). Initially algorithms from 10th to 16th require $O(ns)$ space, but at [Apostolico, 1987] a trick is introduced which can be used to reduce the space complexity to $O(n)$, however in this case the time complexity bounds increase by a multiplicative factor of $\Theta(\log s)$. The 9th algorithm requires $O(ns)$ space but that trick cannot be used to reduce this space complexity bound[Apostolico, 1987]. Recall that d is the number of minimal matches. It can be checked that $d \le l(m - l)$ [Rick, 1994] and it is known that in average it holds $d = \Theta(mn)$ [Tronicek, 2002]. This means the 9th,10th and

14th algorithms mentioned in the table above have better time complexity bounds than the others mentioned there. The algorithm we present here has better time complexity bound than 10th and 14th in case when $s = \omega(\log n)$ (or $s = \omega(\log \log n)$ if the van Emde Boas trees are used), and it has better space complexity bound than 9th in cases when $s = \omega(1)$ (see [Cormen, 2009] for the ω-notation). Roughly speaking the algorithm we present here has better time and space complexity bounds than the ones mention in the table above when the alphabet size if relevantly larger. We present the algorithm in the next section.

No.	Year	Authors	Time Complexity	Ref.
1	1974	Wagner, Fischer	$O(mn)$	[Cormen, 2009]
2	1977	Hunt, Szymansky	$O(m + r \log l)$	[Hunt, 1977]
3	1977	Hirschberg	$O(ln)$	[Hirschberg, 1977]
4	1977	Hirschberg	$O(l(m-l) \log n)$	[Hirschberg, 1977]
5	1980	Masek, Paterson	$O(mn/\log m)$	[Masek, 1980]
6	1982	Nakatsu et al.	$O\big(n(m-l)\big)$	[Nakatsu, 1982]
7	1984	Hsu, Du	$O(lm \log(n/l))$	[Hsu, 1984]
8	1986	Myers	$O\big(n(n-l)\big)$	[Myers, 1980]
9	1987	Apostolico, Guerra	$O(m \log n + d \log(2mn/d))$	[Apostolico, 1987]
10	1987	Apostolico, Guerra	$O(lm \log(2n/m))$	[Apostolico, 1987]
11	1990	Chin, Poon	$O(ns + \min\{lm, ds\})$	[Chin, 1990]
12	1990	Wu, Manber, Myers	$O\big(n(m-l)\big)$	[Wu, 1990]
13	1992	Apostolico et al.	$O\big(n(m-l)\big)$	[Apostolico, 1992]
14	1994	Rick	$O(ns + \min\{lm, l(n-l)\})$	[Rick, 1994]
15	1994	Rick	$O(ns + \min\{lm, ds\})$	[Rick, 1994]
16	2002	Goeman, Clausen	$O(ns + \min\{lm, l(n-l)\})$	[Goeman, 2002]

The New Algorithm

First we will discuss the algorithm for the LIS Problem presented at [Fredman, 1975]. That algorithm is an *online algorithm* meaning that it sequentially handles the elements of the input sequence and determines the LIS length of the sequence handled so far.Online algorithms have advantage that they can run on dynamically changing input data. For instanceunlike the Selection Sort, the Insertion Sort algorithm can maintain the sorted list upon the appendingof the next element to the input list [Cormen, 2009]. Thus such algorithms are defined as update procedures which are to be performed upon the appending of the next element. Now back to the LIS Problem.Let $A = a_1 \cdots a_i \cdots a_m$ be a sequence of integers and let x be an integer which is being appended to A. We will describe an online algorithm which determines the LIS length of A' which is A appended by x. Denote by l the LIS length of A and by l' the LIS lengthof A'. Note that $l' = l$or $l' = l + 1$. For $1 \leq k \leq l$ there are increasing subsequences of length $k$in A. Let $x_k$be the minimumof their last elements. It is easy to check that

$$x_1 \leq \cdots \leq x_k \leq \cdots \leq x_l \tag{2}$$

We denote by x'_k the analogue of $x_k$inA': for $1 \leq k \leq l'$ let x'_k denote the minimum of the last elements of increasing subsequences of length k of A'. In order to obtain an online algorithm for the LIS Problem we will describe how to determine values $(x'_k)_{k=1}^{l'}$ based on values $(x_k)_{k=1}^{l}$. Firstly note that $l' = l + 1$ if and only if $x_l \leq x$, and if so then $x'_{l+1} = x$. It is easy to check that this claim can be generalized for any $1 \leq k \leq l$: let r denote $l + 1$ if $x_l \leq x$ and otherwise let r be the least index such that $x < x_r$. It is easy to check that for $k = r$ it holds $x'_k = x$and otherwise $x'_k = x_k$. Thus we have described a way how to obtain values $(x'_k)_{k=1}^{l'}$ based on values $(x_k)_{k=1}^{l}$. Next the online algorithm for the LIS Problem is described. The algorithm maintains the values $(x_k)_{k=1}^{l}$ in an array $\mathbf{endpoints}$. Upon the appending of the next element x to sequence A the algorithm just searches for the index r mentioned above and updates the value at that index.

LIS-update

Input: the next element $\mathbf{x}$ of sequence A

Output:the LIS length of the sequences$A$ handled so far

Method:

1. $\mathbf{if}\ (\ \mathbf{x}\ \mathit{has\ no\ upper\ bound\ in}\ \mathbf{endpoints})\ \mathbf{then\ do}$
2. $\mathit{append}\ (\ \mathbf{endpoints},\ \mathbf{x}\ )$
3. $\mathbf{done\ else\ do}$
4. $\mathbf{endpoints}[\mathit{upper\_bound}\ (\ \mathbf{endpoints},\ \mathbf{x}\ )] = \mathbf{x}$
5. $\mathbf{done}$
6. $\mathbf{output}\ \mathit{size}\ (\ \mathbf{endpoints}\ )$

Note that each call of this procedure requires $\Theta(\log l)$ time where l is the LIS length of the sequence handled so far. Thus we have described an online algorithm for the LIS Problem which runs in $\Theta(m \log l)$ time and in $\Theta(l)$ space where m is the length of the sequence handled so far and l is the LIS length of that sequence. Next we will present an online algorithm for the LCS Problem which determines the LCS length of two sequences of length $m$and n, $m \leq n$, in $O(d \log n)$ time where d is the number of minimal matches between the input sequences. As for the LCS Problem there are two input sequences some clarification is needed regarding the notion of online algorithms.By an online algorithm for the LCS Problem we mean an algorithm which can accept the next element of either of the two input sequences and provide the LCS length of the two sequences handled so far. Let $A = a_1 \cdots a_i \cdots a_m$ and $B = b_1 \cdots b_j \cdots b_n$ be two sequences over some alphabet Σ of size s and let $y \in \Sigma$ be a symbol being appendedto B. We will describe an online algorithm which determines the LCS length of $A$and B', where B' is Bappended by y. Denote by l the LCS length of A and B and by l' the LCS length of $A$and B'. Note that $l' = l$or $l' = l + 1$. For $1 \leq k \leq l$ there are subsequences of length $k$common to $A$and B. Let i_k be the minimum index such that there is a subsequence of length $k$common to$A$ and B ending at $i_k$in A.It is easy to check that

$$i_1 < \cdots < i_k < \cdots < i_l \tag{4}$$

Similarly for $1 \leq k \leq l$ we define j_k as the minimum index such that there is a subsequence of length k common to A and B ending at $j_k$in B, and we get

$$j_1 < \cdots < j_k < \cdots < j_l \tag{5}$$

We will call the indices at (4) *thresh indices* or *thresh values* of sequence$A$with respect to B and the indices at (5) thresh indices or thresh values of sequence B with respect to A. Let for $1 \leq k \leq l' i'_k$ be the thresh values of sequence$A$with respect to B' and j'_k be the thresh values of B' with respect to A. In order to obtain an online algorithm for the LCS Problem we will describe how to determine indices $(i'_k)_{k=1}^{l'}$ and $(j'_k)_{k=1}^{l'}$ based on indices $(i_k)_{k=1}^{l}$and $(j_k)_{k=1}^{l}$. Firstly note that $l' = l + 1$ if and only if there is some index r, $l < r \leq m$, such that $x_r = y$, and if so then i'_{l+1} is the minimum of such r-s. It is easy to check that this claim can be generalized for any $1 \leq k \leq l$: if r is the first occurrence of y in A after i_{k-1} and $r < i_k$ then $i'_k = r$ and otherwise $i'_k = i_k$ (see Figure 1).

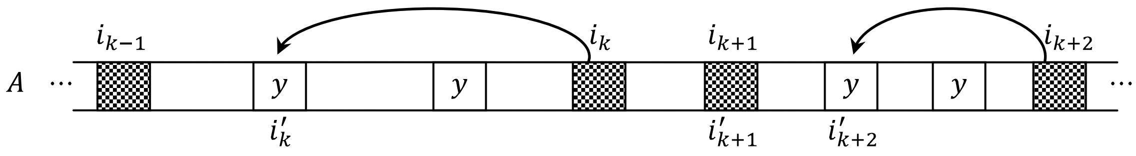


Figure 1

Thus we have described a way how to obtain sequences A and B' and their thresh indices based on sequences A and B and their thresh indices.So during this some thresh values are updated and the others are not.A trivial approach would be to handle all thresh values and update them if they has to be updated, however better would be to handle only those thresh values which has to be updated.Let r be the first occurrence of y in A aftersomei_k. Note that the least thresh value exceeding i_k which has to be updated is the first occurrence of thresh value after r. This means that while searching for the first occurrence of y (after some thresh value) the thresh values can be ignored.Also note that the thresh values of B' with respect to A, i.e. the $(j'_k)_{k=1}^{l'}$, can be obtained easily:there is a new thresh value there if and only if $l' = l + 1$ and if so then $j'_{l+1} = n + 1$. It can be checked that each update of a thresh value corresponds to a minimal match.Next the algorithm is presented. It consists of two update procedures: one for calling upon the appending the next element to sequence A and another upon the appending the next element tosequence B. We will restrict only on the second one as the first one can be obtained just by swapping symbols “A” and “B” in the text of the procedure. The algorithm maintains the sequences$A$ and B in arrays $\mathbf{sequenceA}$and $\mathbf{sequenceB}$ respectively and for each symbol$z$ of alphabet Σ it maintains the set of occurrences of$z$ in A and B in binary search trees $\mathbf{layersA}[z]$ and $\mathbf{layersB}[z]$ respectively.The algorithm also maintains the thresh indices $(i_k)_{k=1}^{l}$ and $(j_k)_{k=1}^{l}$ in binary search trees $\mathbf{threshA}$ and $\mathbf{threshB}$ respectively. Following is the update procedure which is to be called upon the appending the next element to sequence B.The procedure uses two temporary variables $\mathbf{p}$ and $\mathbf{q}$which correspondto the next and previous values of updating thresh indices.

Note that each iteration of the while loop at lines 3-12 updates a thresh value ($\mathbf{p} < \mathbf{q}$ at the end of each iteration) and the operations carried out during each iteration require $\Theta(\log n)$ time as they are performed on binary search trees. Recall that each update of the thresh value corresponds to a minimal match, so we have described an online algorithm for the LCS Problem which runs in $\Theta(d \log n)$ time and in $\Theta(m + n)$ space where$m$ and n are the lengths of the sequences handled so far and d is the number of minimal matches between that sequences. These bounds can be improved if using van Emde Boas trees [Cormen, 2009] instead of binary search trees.van Emde Boas tree is a data structure that for some a priory fixed integer$w$ can store some of first 2^w integers,itsupports operations of insertion deletion and search for the upper bound with worst case time complexity bound of $\Theta(\log \log n)$and it requires $\Theta(2^w)$ space regardless the number of integers stored in it. At [Cormen, 2009] it is shown how this data structure can be modified to require only $\Theta(n)$ space where n is the number of stored elements (there the modified data structure is called y-fast trie). In this case the operations of insertion and deletion do not have worst case time complexity bound of $\Theta(\log\log n)$ but this bound holds for the amortized time complexity. This fits with our needs as we perform $\Theta(d)$ insertions and deletions, thus we conclude that if using these modified van Emde Boas trees then the algorithm presented in this paper will run in $\Theta(d \log\log n)$ time and in $\Theta(n)$ space.

LCS-updateB

Input: the next element $\mathbf{y}$ of sequence B

Output: the LCS length of the sequences A and B handled so far

Method:

```
p = 0
q = 0
while ( true ) do
if ( q has no upper bound in layersA[y]) then break
p = upper_bound ( layersA[y], q )
erase ( layersA[sequenceA[p]], p )
if ( p has no upper bound in threshA) then break
q = upper_bound ( threshA, p )
insert ( layersA[sequenceA[q]], q )
erase ( threshA, p )
insert ( threshA, q )
done
if (p has no upper bound in threshA) then do
insert ( threshA, p )
insert ( threshB, size ( sequenceB ))
done else do
insert ( layersB[y], size ( sequenceB ))
done
append ( sequenceB, y )
output size ( threshA )
```

Bibliography

[Aho, 1976]A. Aho, C. Hirschberg, J. Ullman. Bounds on the Complexity of the Longest Common Subsequence Problem. Journal of the Association for Computing Machinery, Vol. 23, No. 1, 1976, pp. 1-12.

[Apostolico, 1987]A. Apostolico, G. Guerra. The longest common subsequence problem revisited. Algorithmica 2, 1987, pp. 315-336.

[Apostolico, 1992]. Apostolico, S. Browne, C. Guerra. Fast Linear-Space Computations of Longest Common Subsequences. Theoretical Computer Science, Vol. 92, 1992, pp. 3-17.

[Baeza-Yates, 1999] R. Baeza-Yates, R. Gavalada, G. Navarro, R. Scheihing.Bounding the Expected Length of Longest Common Subsequences and Forestes.Theory of Computing Systems, Vol. 32, 1999, pp. 435-452.

[Bergroth, 2000]L. Bergroth, H. Hakson, T. Ratia.A Survey of Longest Common Subsequence Algorithms.Proceedings of the Seventh International Symposium on String Processing Information Retrieval, 2000, pp. 39-48.

[Chin, 1990]. F. Chin, C. Poon. A Fast Algorithm for Computing Longest Common Subsequences of Small Alphabet Size.Journal of Information Processing, Vol.13, No. 4,1990, pp. 463-469.

[Cormen, 2009] T. Cormen, C. Leiserson, R. Rivest, C. Stein. Introduction to algorithms (Third Edition), pp. 43-65, pp. 390-397, pp.531-561, 2009.

[Fredman, 1975] M. Fredman. On Computing the of Longest Increasing Subsequences. Discrete Mathematics, Vol. 11, No. 1, 1975, pp. 29-35.

[Goeman, 2002]H. Goeman, M. Clausen. A new practical linear space algorithm for the longest common subsequence problem.Kybernetika, Vol. 38, No. 1, 2002, pp. 45-66.

[Hirschberg, 1977]D. Hirschberg.Algorithms for the Longest Common Subsequence Problem.Journal of Association for Computing Machinery, Vol. 24, No. 4, 1977, pp. 664-675.

[Hsu, 1984]W. Hsu, M. Du. New algorithms for the LCS problem. Journal of Computer and Syste Sciences, Vol. 29, 1984, pp. 133-152.

[Hunt, 1977]J. Hunt, T. Szymanski. A fast algorithm for computing longest common subsequences.Communications of Association for Computing Machinery, Vol. 20, No. 5, 1977, pp. 350-353.

[Masek, 1980] W. Masek, M. Paterson. A Faster Algorithm Computing String Edit Distances. Journal of Computer and System Sciences, Vol. 20, No. 1, 1980, pp. 18-31.

[Myers, 1986]E.Myers. An $O(ND)$ Difference Algorithm and its Variations. Algorithmica 1, 1986, pp. 251-266.

[Nakatsu, 1982]N. Nakatsu, Y. Kambayashi, S. Yajima. A Longest Common Subsequence Algorithm Suitable for Similar Text Strings. Acta Informatica Vol. 18, 1982, pp. 171-179.

[Rick, 1994] C. Rick. New Algorithms for the Longest Common Subsequence Problem.Research Report No. 85123-CS, Department of Computer Science, University of Bonn, 1994.

[Tronicek, 2002]Z. Tronicek.Common Subsequence Automatation.Research Report DC-2002-02, Department of Computer Science and Engineering, Czech Technical University, 2002.

[Wu, 1990] S. Wu, U. Manber, G. Myers, W. Miller.An $O(NP)$ Sequence Comparison Algorithm.Information Processing Letters, Vol. 35, 1990, pp. 317-323.

Authors' information

VahagnMinasyan *– Postgraduate student, Yerevan State University, faculty of Informatics and Applied Mathematics, department of Discrete Mathematics and Theoretical Informatics, Armenia, 0025, Yerevan, 1st Alex Manoogian; e-mail: vahagn.minasyan@gmail.com*

INTERFERENCE MINIMIZATION IN PHYSICAL MODEL OF WIRELESS NETWORKS

Hakob Aslanyan

Abstract: *Interference minimization problem in wireless sensor and ad-hoc networks is considered. That is to assign a transmission power to each node of a network such that the network is connected and at the same time the maximum of accumulated signal straight on network nodes is minimum. Previous works on interference minimization in wireless networks mainly consider the disk graph model of network. For disk graph model two approximation algorithms with* $O(\sqrt{n})$ *and* $O((opt \ln n)^2)$ *upper bounds of maximum interference are known, where* n *is the number of nodes and* opt *is the minimal interference of a given network. In current work we consider more general interference model, the physical interference model, where sender nodes' signal straight on a given node is a function of a sender/receiver node pair and sender nodes' transmission power. For this model we give a polynomial time approximation algorithm which finds a connected network with at most* $O((opt \ln n)^2/\beta)$ *interference, where* $\beta \geq 1$ *is the minimum signal straight necessary on receiver node for successfully receiving a message.*

Keywords: *interference, wireless networks, graph connectivity, set cover, randomized rounding.*

ACM Classification Keywords: *C.2.1 Network Architecture and Design - Network topology, G.2.2 Graph Theory - Network problems.*

Introduction

We consider interference minimization problem in energy limited wireless networks (wireless sensor and ad-hoc networks) where recharging or changing the energy sources of nodes is not feasible and sometimes due to environmental conditions not possible. In such networks it is important to consider the minimization of energy consumption of algorithms running on network nodes. By decreasing energy consumption we increase nodes operability time and as a result networks' lifetime. In different wireless sensor network (WSN) applications definition of networks' lifetime may be different (till all the nodes are alive, network is connected, given area is monitored by alive nodes, etc). In current work we tend to decrease energy consumption of nodes by decreasing the maximum interference of network algorithmically. Wireless communication of two nodes which is experiencing the third one is called interference. High interference on a receiver node (high value of accumulated signal straights on a node) makes difficulty to determine and accept the signals dedicated to it, this makes necessity for sender node to retransmit the signal until it is successfully accepted by receiver node, which is extra energy consumption and should be avoided.

Interference Minimization in Disk Graph Model of Wireless Networks

Consider a set of spatially distributed nodes, where each node equipped with radio transmitter/receiver and the power of nodes' transmitter is adjustable between zero and nodes' maximum transmission level. In disk graph model of network assumed that by fixing a transmission power for a node we define a transmission radius/disk of a node, i.e. the transmitted signal is reachable and uniform in any point of transmission disk of node and is zero outside of it. In this model two nodes considered connected if they are covered by each others transmission disks and interference on a given node defined as the number of transmission disks including that node. The overall interference of network is the maximum interference among all the nodes forming the network. The main weakness of disk graph model is the assumption that the radio coverage area is a perfect circle.

Assigning a transmission powers to a given set of spatially distributed nodes such that nodes form a connected network with assigned transmission powers while the interference of network is minimal called interference minimization problem in wireless networks.

One particular case of interference minimization problem described above is studied in [Rickenbach, 2005]. Authors considered the problem in one dimensional network, where all the nodes are distributed along the straight line, and named it a highway-model. For this model they showed that intuitive algorithm, which connects each node with its closest left and right nodes, can give a bad performance. An example of network where intuitive algorithm has worst performance is the exponential node chain, where distance between two consecutive nodes grows exponentially ($2^0, 2^1, \ldots, 2^{n-1}$). They also gave two algorithms for one dimensional case of interference minimization problem. The first algorithm, for a given set of distributed nodes, finds a connected network with at most $O(\sqrt{\Delta})$ interference where Δ is interference of uniform radius network under consideration and is $O(n)$ in some network instances. The second one is an approximation algorithm with $O(\sqrt[4]{\Delta})$ approximation ratio. By applying computational geometry and ε-net theory to ideas given in [Rickenbach, 2005], [Halldorsson, 2006] proposes a algorithm which gives $O(\sqrt{\Delta})$ interference bound for maximum interference in two and $O(\sqrt{\Delta log \Delta})$ for any constant dimensional network. Authors of [Aslanyan, 2010] give iterative algorithm based on linear program relaxation techniques which guaranties $O((opt \ln n)^2)$ interference bound for networks of n nodes, opt here is the optimal interference value for given instance of network. Logarithmic lower bound for interference minimization problem in disk graph model of networks under the general distance function is proven in [Bilo, 2006] by reducing minimum set cover to minimum interference problem.

Interference Minimization in Physical Model of Wireless Networks

Again, consider a set of spatially distributed wireless nodes, where each node has a radio transmitter/receiver with adjustable power level. In physical model of wireless networks we refuse the assumption that the signal coverage of a node is a perfect circle and assume that the signal straight on any given point (node) of network is a function of sender node, the node in question and the level of transmitted signal. In this model we are also given a constant β which is a signal acceptance threshold, i.e. it assumed that receiver node accepts the signal if it's straight is at least β. By this mean two nodes considered connected if their signals' straights are at least β on each other. Interference on a given node defined as a sum of signal straights on that node and interference of networks is the maximum interference among all the nodes forming the network.

The disk graph model can be deduced from physical model if we consider a signal straight function which for every node and its transmission level draws a disk and outputs a positive constant for every node within that disk and zero for the rest. Another example of signal straight function is $f(u,v,\xi)=\xi/d(u,v)^{\alpha}$ where u and v are sender and receiver nodes respectively, ξ is the transmission power of u, $\alpha\in[2,6]$ is the path lost exponent and $d(u,v)$ is the distance between nodes u and v [Pahlavan, 1995].

Interference minimization problem defined in a same way as for disk graph model.

Assign a transmission powers to a given set of spatially distributed wireless nodes such that nodes form a connected network with assigned transmission powers and the interference of network is minimal.

Our result is a deterministic polynomial time algorithm for interference minimization problem in wireless networks under the physical model of wireless networks in consideration, which for given network of n wireless nodes finds a connected network with at most $O((opt\ln n)^2/\beta)$ interference.

Formal Definitions

Consider a set V of n wireless nodes spatially distributed over a given area where nodes have adjustable transmission power and it can be fixed between zero and nodes' maximum transmission power. For any node $u\in V$ denote the range of feasible transmission powers by $R_u=[0,\xi_u^{max}]$, where ξ_u^{max} is the maximum transmission power for node u, and define a signal straight function $\phi_u: V\times R_u\to R^+$ where $\phi_u(v,\xi)$ is the signal straight of node u on node v when u uses the transmission power ξ. We assume that the signal straight function satisfies to following conditions

1. for any $\xi_1,\xi_2\in R_u$, from $\xi_1\geq\xi_2$ it follows that $\phi_u(v,\xi_1)\geq\phi_u(v,\xi_2)$
2. for given $\eta\in R^+$ it is easy to find a $\xi\in R_u$ (if exists) such that $\phi_u(v,\xi)=\eta$

Suppose that for any node u the suitable transmission power ξ_u is fixed, then any two nodes u and v

considered connected if $\phi_u(v,\xi_u) \geq \beta$ and $\phi_v(u,\xi_v) \geq \beta$ where $\beta \geq 1$ is the signal acceptance threshold of network. Interference on a given node u is the accumulated signal straight of all the nodes forming the network $I(u) = \sum_{v \in V \setminus \{u\}} \phi_v(u,\xi_v)$ and $I(V) = \max_{v \in V} I(V)$ is the overall network interference. At this point interference minimization problem can be formulated as follows:

Given a spatially distributed set of wireless nodes, assign a suitable transmission power to each node such that the network is connected and the interference of network is minimal.

This is the formulation of interference minimization problem by transmission power assignment.

Consider a network graph $G = (V,E)$ where $E = \{(u,v) \mid u,v \in V, \phi_u(v,\xi_u^{max}) \geq \beta, \phi_v(u,\xi_v^{max}) \geq \beta\}$ i.e. in graph G two vertexes/nodes are incident if their maximum transmission powers are enough for communicating with each other. By this mean interference minimization problem is formulated as follows.

For a given network graph $G = (V,E)$ find a connected spanning subgraph $H = (V,E')$ such that the interference of network computed by the selected set of edges is minimal.

Formally, having the subgraph $H = (V,E')$ it is correct to further extract transmission power for any node u as a minimum power such that u can communicate with all of its neighbors in H, $\xi_u = min_\xi \{\xi \mid \phi_u(v,\xi) \geq \beta \text{ for all } v \text{ that } (u,v) \in E'\}$, which avoids unnecessary interference.

Set Covering and Interference Minimization

In the classical set cover problem a set S and a collection C of subsets of S are given, it is required to find a minimum size sub collection C' of C such that the union of sets of C' is S. In a decision version of set cover problem a positive integer k is given and the question is if it is possible to choose at most k subsets from collection C such that the union of chosen sets is S. It is well known that decision version of set cover problem is NP-complete and in polynomial time the optimal solution can not be approximated closer than with a logarithmic factor [Johnson, 1974]. Several variants of set cover problem have been studied [Kuhn, 2005; Garg, 2006; Demaine, 2006; Guo, 2006; Mecke, 2004; Ruf, 2004; Aslanyan, 2003].

Being motivated by interference minimization problem in cellular networks the minimum membership set cover (MMSC) problem has been investigated in [Kuhn, 2005]. In MMSC a set S and a collection C of subsets of S are given, it is required to find a subset C' of C such that the union of sets in C' is S and the maximum covered element of S is covered by as few as possible subsets from C'. In a decision version of MMSC problem a positive integer k is given and the question is if it is possible to choose a sub collection of C such that the union of chosen sets is S and each element of S is covered by at most k different subsets. [Kuhn, 2005] Contains the proofs of NP-completeness of decision version of MMSC problem and non-approximability of MMSC optimization problem by factor closer than $O(\ln n)$ unless $NP \subset TIME(n^{O(\log\log n)})$. Also, by using the linear program relaxation and randomized rounding techniques, [Kuhn, 2005] gives a polynomial time algorithm, which approximates the optimal solution of MMSC with logarithmic factor $O(\ln n)$.

Minimum partial membership partial set cover (MPMPSC) problem has been proposed in [Aslanyan, 2010] and used for developing interference minimization algorithm for wireless networks (disk graph model under consideration). In MPMPSC a set $S = S_1 \cup S_2$, consisting of two disjoint sets S_1 and S_2, along with collection C of subsets of S are given, it is required to find a sub collection C' of C such that the union of sets in C' contains all the elements of S_1 and the maximum covered element of S_2 is covered by as few as possible subsets from C'. In a decision version of MPMPSC problem a positive integer k is given and the question is if it is possible to choose a sub collection of C such that the union of chosen sets contains all the elements of S_1 and each element of S_2 is covered by at most k different subsets. It is known that the decision version of MPMPSC problem is NP-Complete and that the deterministic polynomial time algorithm exists which approximates the optimal solution of optimization version of MPMPSC by logarithmic factor $O(log(max\{|S_1|,|S_2|\}))$ which asymptotically matches the lower bound [Aslanyan, 2010]. The approximation algorithm for MPMPSC is achieved by applying the same techniques which has been applied in [Kuhn, 2005] for solving the MMSC.

Being motivated by interference minimization problem in physical model of wireless networks we consider a weighted minimum partial membership partial set cover (WMPMPSC) problem which is a generalization of MPMPSC. In WMPMPSC a set $S = S_1 \cup S_2$, consisting of two disjoint sets S_1 and S_2, along with collection C of subsets of S are given. In each subset from C the elements of S_2 have weights in $[0,1]$. The same element of S_2 may have a different weights in different sets of C. It is required to find a sub collection C' of C such that the union of sets in C' contains all the elements of S_1 and the accumulated, among the subsets of C', weight of a node which has the maximum accumulated weight, is as small as possible. In a decision version of WMPMPSC problem a positive number k is given and the question is if it is possible to choose a sub collection of C such that the union of chosen sets contains all the elements of S_1 and the accumulated, among the chosen sets, weight of each node is at most k. It is easy to see that in WMPMPSC we get a instance of MPMPSC when each node has a weight 1 in all the sets of C. This last statement proves the NP-Completeness of the decision version of WMPMPSC and the logarithmic lower bound for optimization version of the problem.

LP Formulations

Let C' denote a subset of the collection C. To each subset $C_j \in C$ we assign a variable $x_j \in \{0,1\}$ such that $x_j = 1 \Leftrightarrow C_j \in C'$. For C' to be a set cover for S, it is required that for each element $u \in S$ at least one set C_j with $u \in C_j$ is in C'. Therefore, C' is a set cover for S if and only if for all $u \in S$ it holds that $\sum_{C_j \ni u} x_j \geq 1$. Let z is the maximum membership over all the elements caused by the sets in C'. Then for all $u \in S$ it follows that

$\sum_{C_j \ni u} x_j \le z$. Then the integer linear program IP_{MMSC} of MMSC problem can be formulated as:

minimize z

subject to

$$\sum_{C_j \ni u} x_j \ge 1, \qquad u \in S \tag{1}$$

$$\sum_{C_j \ni u} x_j \le z, \qquad u \in S \tag{2}$$

$$x_j \in \{0,1\}, \qquad C_j \in C \tag{3}$$

Integer linear program IP_{MPMPSC} of MPMPSC would be:

minimize z

subject to

$$\sum_{C_j \ni u} x_j \ge 1, \qquad u \in S_1 \tag{4}$$

$$\sum_{C_j \ni u} x_j \le z, \qquad u \in S_2 \tag{5}$$

$$x_j \in \{0,1\}, \qquad C_j \in C \tag{6}$$

After introducing the weight function $w: C \times S_2 \to [0,1]$, where $w(C_j, u)$ is the weight of u in subset C_j, the integer linear program $IP_{WMPMPSC}$ of WMPMPSC can be formulated as:

minimize z

subject to

$$\sum_{C_j \ni u} x_j \ge 1, \qquad u \in S_1 \tag{7}$$

$$\sum_{C_j \ni u} x_j w(C_j, u) \le z, \qquad u \in S_2 \tag{8}$$

$$x_j \in \{0,1\}, \qquad C_j \in C \tag{9}$$

By applying randomized rounding technique to IP_{MMSC} with relaxation of constraints (3), [Kuhn, 2005] gives a deterministic polynomial time approximation algorithm with $(1 + O(1/\sqrt{z'}))(\ln(n) + 1)$ approximation ratio for MMSC problem, where z' is the optimal solution for IP_{MMSC} relaxation. Later on [Aslanyan, 2010] states that by applying the same randomized rounding technique to IP_{MPMPSC} with relaxation of constraints (6) gives a deterministic polynomial time approximation algorithm with $(1 + O(1/\sqrt{z'}))(\ln(\max\{|S_1|, |S_2|\}) + 1)$ approximation ratio for MPMPSC problem, where z' is the optimal solution for IP_{MPMPSC} relaxation. In current work we state that the same randomized rounding technique can be applied to $IP_{WMPMPSC}$ with relaxation of constraints (9) to achieve a deterministic polynomial time approximation algorithm with $(1+O(1/\sqrt{z'}))(\ln(\max\{|S_1|,|S_2|\})+1)$ approximation ratio for WMPMPSC problem, where z' is the optimal solution for $IP_{WMPMPSC}$ relaxation. The proof of the last statement is presented in the Appendix of this work. To sum up, we have the following theorem.

Theorem 1. *For WMPMPSC problem, there exists a deterministic polynomial-time approximation algorithm with an approximation ratio of* $O(log(max\{|S_1|,|S_2|\}))$ [1]

Approximation Algorithm for Interference Minimization in Physical Model of Wireless Networks

Algorithm takes a network graph $G=(V,E)$ with n vertices as an input and after logarithmic number of $k\in O(\log n)$ iterations returns connected subgraph $G_k\subseteq G$ where interference of network corresponding to the graph G_k is bounded by $O((opt\cdot\ln n)^2/\beta)$, where $n=|V|$ is the number of network nodes and opt is the interference of minimum interference connected network.

Algorithm starts the work with the graph $G_0=(V,E_0)$ where $E_0=\varnothing$. On the l^{th} iteration, $l\geq 1$, algorithm chooses a subset $F_l\subseteq E\setminus E_{l-1}$ of new edges and adds them to already chosen edge set $E_{l-1}=\cup_{i=1}^{l-1}F_i$. As a consequence of such enlargement of edge set, interference on graph vertices may increase in some value depending on F_l. Algorithm finishes the work if the graph $G_l=(V,E_l)$ is connected otherwise goes for the next iteration. Below we present how algorithm chooses the set of edges $F_l\subseteq E\setminus E_{l-1}$ on the l^{th} iteration. Algorithms' quality, i.e the final maximal interference on nodes (its upper estimate) is bounded by the accumulated through the iterations interferences which we try to keep minimal. Let $G_{l-1}=(V,E_{l-1})$ is the graph obtained after the $(l-1)^{th}$ iteration, and has the set of connected components $C(G_{l-1})=\{C_{l-1}^1,\ldots,C_{l-1}^{k_{l-1}}\}$. Denote by $H_{l-1}\subseteq E\setminus E_{l-1}$ the set of all edges which have their endpoints in different connected components of G_{l-1}. On the l^{th} stage of algorithm a subset of H_{l-1} is selected to further reduce the number of connected components which finally brings us to a connected subgraph. In this way we build the collection $T(C(G_{l-1}),H_{l-1})$ of special sets as follows. Starting with H_{l-1} we add to the set $T(C(G_{l-1}),H_{l-1})$ of l^{th} stage specific weighted subsets

[1] See the Appendix A for the proof.

$T^l(u,v) = \{C_{l-1}^u, C_{l-1}^v\} \cup V$ defined by all $(u,v) \in H_{l-1}$, where u belongs to connected component C_{l-1}^u and v belongs to C_{l-1}^v. By selection of u and v we have that C_{l-1}^u and C_{l-1}^v are different. By definition of connectivity nodes u and v can communicate with each other if their signal transmission powers ξ_{uv} and ξ_{vu} satisfy to $\phi_u(v,\xi_{uv}) \geq \beta$ and $\phi_v(u,\xi_{vu}) \geq \beta$, where β is the signal acceptance threshold. To avoid unnecessary energy consumption and to reduce interference it would be right to adjust transmission powers ξ_{uv} and ξ_{vu} such that $\phi_u(v,\xi_{uv}) = \beta$ and $\phi_v(u,\xi_{vu}) = \beta$, this is possible to do because of the second property of the signal straight function ϕ. Then the noise of the link (u,v) on any node t can be calculated as $w((u,v),t) = \phi_u(t,\xi_{uv}) + \phi_v(t,\xi_{vu})$ which would be the weight $w(T^l(u,v),t)$ of node t in the subset $T^l(u,v)$. And so $T^l(u,v)$ is a composite set which includes two labels for components C_{l-1}^u and C_{l-1}^v and all the vertices in V along with the weights, which are the interference increase on nodes if the edge (u,v) is selected as a communication link. In terms of WMPMPSC the labels of connected components will compose the set S_1 and weighted V will be the set S_2.

Figure 1 demonstrates connected components that are input to the stage l, and the set H_{l-1} of all cross component edges.

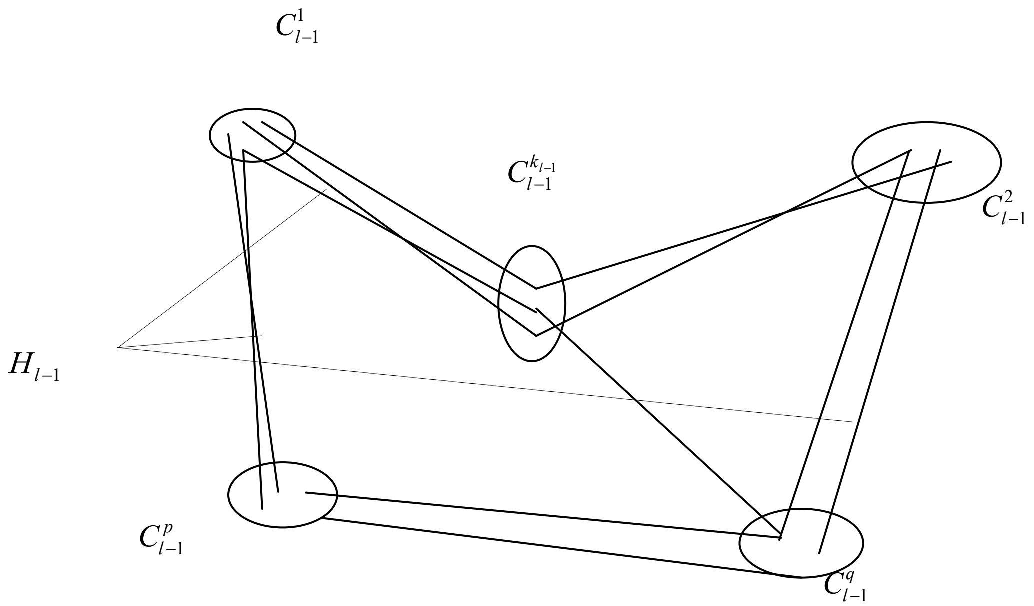


Figure 1: Connected components that are input to the l-th stage of the algorithm

After constructing $T(C(G_{l-1}), H_{l-1})$ we normalize the weights of elements by dividing all the weights by the maximum weight $w_{max} = \max_{t,(u,v)\in H_{l-1}} w((u,v),t)$ and solve the WMPMPSC on the set $C(G_{l-1}) \cup V$ and collection of subsets $T(C(G_{l-1}), H_{l-1})$, where condition for elements of $C(G_{l-1})$ is *to be covered* and for elements of V is *to have minimum accumulated weight*. Finally, based on the solution $W(C(G_{l-1}), H_{l-1}) \subseteq T(C(G_{l-1}), H_{l-1})$ of WMPMPSC we build the set F_l of network graph edges, selected at the l^{th} iteration of algorithm by adding to F_l all the edges $(u,v) \in H_{l-1}$ such that $T^l(u,v) \in W(C(G_{l-1}), H_{l-1})$ and multiply all the weights by w_{max} to receive the real interference increase.

Algorithm performance

Theorem 2. *On each iteration of algorithm the number of connected components is being reduced at least by factor of two, which bounds the total number of iterations by* $O(\log n)$.

Proof. For each connected component $C_{l-1}^u \in C(G_{l-1})$ of graph G_{l-1} the solution $W(C(G_{l-1}), H_{l-1})$ of WMPMPSC solved at l^{th} iteration contains at least one set $T^l(u,v) \in W(C(G_{l-1}), H_{l-1})$ such that $C_{l-1}^u \in T^l(u,v)$ (as $W(C(G_{l-1}), H_{l-1})$ is a cover for the set $C(G_{l-1})$). And as each set $T^l(u,v) \in W(C(G_{l-1}), H_{l-1})$ contains exactly two connected components, then by adding the edge (u,v) to our solution, we merge those two connected components into one (connecting by the edge (u,v)). So every connected component merges with at least one other component, which reduces the number of connected components at least by factor of 2.

Lemma 1. *Network corresponding to the graph* $G^l = (V, F_l)$, *where* F_l *is the edge set obtained on the* l^{th} *iteration of algorithm, has interference in* $O((opt^2 \cdot \ln n)/\beta)$.

Proof. Consider the set of connected components $C(G_{l-1}) = \{C_{l-1}^1, \dots, C_{l-1}^{k_{l-1}}\}$ of l^{th} iterative step of algorithm. Let E_{opt} is the set of the edges of some interference optimal connected network for our problem (edges of connected network with optimal interference opt). Then there is a subset $E_{opt}^l \subseteq E_{opt}$ which spans connected components $C(G_{l-1})$ and the network of the graph $G_{opt}^l = (V, E_{opt}^l)$ has interference not exceeding the opt.

Fact 1. *The maximal vertex interference due to the spanner* E_{opt}^l *of* $C(G_{l-1})$ *is at most* opt.

Now let us build the set collection $T_{opt}(C(G_{l-1}), E_{opt}^l) = \{T^l(u,v)/(u,v) \in E_{opt}^l\}$.

Fact 2. $T_{opt}(C(G_{l-1}), E_{opt}^l)$ *is a sub collection of* $T(C(G_{l-1}), H_{l-1})$ *built on the* l^{th} *iteration of algorithm and is a cover for* $C(G_{l-1})$, *i.e.* $T_{opt}(C(G_{l-1}), E_{opt}^l)$ *is a solution for the WMPMPSC problem, with some value* z^*, *solved* *on the l^{th} iteration of algorithm, not necessary optimal.* Now consider the matrix P_{opt}^{w} representing the transmission signals on some node w caused by communication links of E_{opt}^{l}.

$$P_{opt}^{w}=\begin{pmatrix} P_{u_1u_1}^{w} & P_{u_1u_2}^{w} & \cdots & P_{u_1u_j}^{w} & \cdots & P_{u_1u_n}^{w} \\ P_{u_2u_1}^{w} & P_{u_2u_2}^{w} & \cdots & P_{u_2u_j}^{w} & \cdots & P_{u_2u_n}^{w} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \\ P_{u_iu_1}^{w} & P_{u_iu_2}^{w} & \cdots & P_{u_iu_j}^{w} & \cdots & P_{u_iu_n}^{w} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \\ P_{u_nu_1}^{w} & P_{u_nu_2}^{w} & \cdots & P_{u_nu_j}^{w} & \cdots & P_{u_nu_n}^{w} \end{pmatrix}$$

where

$$P_{u_iu_j}^{w}=\begin{cases} 0, & \text{if } i=j \text{ or } (u_i,u_j)\notin E_{opt}^{l} \\ \phi_{u_i}(w,\xi_{u_iu_j}), & \text{otherwise} \end{cases}$$

is the signal straight of node u_i on node w when u_i uses the transmission power $\xi_{u_iu_j}$ (communicates with node u_j).

Fact 3. *and the sum of the matrix elements will give the interference increase we count (the real interference increase is the sum of the maximal elements from each row) on node w by edge set E_{opt}^{l}. Due to the Fact 1 and signal acceptance threshold β for any vertex u_i the number of sets $T^{l}(u_i,v)\in T_{opt}(C(G_{l-1}),E_{opt}^{l},w)$ will not exceed the $\lfloor opt/\beta \rfloor$, in other words the number of non zero elements on each row of matrix P_{opt}^{w} is bounded by $\lfloor opt/\beta \rfloor$.*

Fact 4. *The interference increase on node w by the edge set E_{opt}^{l} can be calculated as $\sum_{i=1}^{n}\max_j P_{u_iu_j}^{w}$ and due to the Fact 1 it doesn't exceed the opt.*

From facts 3 and 4 it follows that the sum of the matrix elements is bounded by opt^2/β, which means that the optimal value of WMPMPSC problem solved on the l^{th} iteration of algorithm is bounded by opt^2/β and therefor by Theorem 1 the interference increase by the edge set F_l is bounded by $O(opt^2 \cdot \ln n/\beta)$.

Theorem 3. *The network built by WMPMPSC relaxation algorithm has at most $O((opt^2 \cdot \ln^2 n)/\beta)$ interference.*

Proof. The proof is in combination of Theorem 2 and Lemma 1.

Conclusion and Future Work

In current work we considered the interference minimization problem in physical model of wireless networks and proposed a polynomial time approximation algorithm which for a given set of wireless nodes creates a connected network with at most $O((opt \cdot \ln n)^2 / \beta)$ interference. In some WSN applications network considered as functional while it is connected, therefore in future works on interference minimization the k-connectivity of network should be considered. Also considering the problem in Euclidean spaces, which is a realistic case for WSNs, may give a better approximation ratio.

References

[Shannon, 1949] C.E.Shannon. The Mathematical theory of communication. In: The Mathematical Theory of Communication. Ed. C.E.Shannon and W.Weaver. University of Illinois Press, Urbana, 1949.

[Aslanyan, 2003]H. Aslanyan. Greedy Set Cover Estimations. Computer Science and Information Technologies (CSIT), :143--144, 2003.

[Aslanyan, 2010] H. Aslanyan and J. Rolim. Interference Minimization in Wireless Networks. IEEE/IFIP International Conference on Embedded and Ubiquitous Computing (EUC 2010), :444--449, 2010.

[Bilo, 2006] D. Bilo and G. Proietti. On the complexity of minimizing interference in ad-hoc and sensor networks. 2nd International Workshop on Algorithmic Aspects of Wireless Sensor Networks, ALGOSENSORS, LNCS, 4240:13--24, 2006.

[Demaine, 2006] E.D. Demaine and U. Feige and M.T. Hajiaghayi and M. R. Salavatipour. Combination can be hard: Approximability of the unique coverage problem. Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), :162--171, 2006.

[Garg, 1997] N. Garg and V.V. Vazirani and M. Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees. Algorithmica 18:3-20, 1997.

[Guo, 2006] J. Guo and R. Niedermeier. Exact algorithms and applications for Tree-like Weighted Set Cover. Discrete Algorithms, 4(4):608--622, 2006.

[Halldorsson, 2006] M. M. Halldorsson and T. Tokuyama. Minimizing Interference of a Wireless Ad-Hoc Network in a Plane. Algorithmic Aspects of Wireless Sensor Networks, ALGOSENSORS, 4240/2006:71--82, 2006.

[Johnson, 1974] D. Johnson. Approximation algorithms for combinatorial problems. Journal of Computer and System Sciences, 9:256--278, 1974.

[Kuhn, 2005] F. Kuhn and P. von Rickenbach and R. Wattenhofer and E. Welzl and A. Zollinger. Interference in cellular networks: the minimum membership set cover proble. 11th International Computing and Combinatorics Conference, COCOON, LNCS, 3595:188--198, 2005.

[Mecke, 2004] S. Mecke and D. Wagner. Solving Geometric Covering Problems by Data Reduction. 12th ESA, LNCS, 3221:760--771, 2004.

[Pahlavan, 1995] K. Pahlavan and A. H. Levesque. Wireless information networks. Wiley-Interscience, 1995.

[Rickenbach, 2005] P. von Rickenbach and S. Schmid and R. Wattenhofer and A. Zollinger. A robust interference model for wireless ad-hoc networks. 5th International Workshop on Algorithms for Wireless, Mobile, Ad Hoc and Sensor Networks (WMAN), 2005.

[Ruf, 2004] N. Ruf and A. Schobel. Set covering with almost consecutive ones property. Discrete Optimization, 1(2):215--228, 2004.

Appendix A

Here we show how randomized rounding technique used in [Kuhn, 2005] for solving the IP_{MMSC} can be used for solving $IP_{WMPMPSC}$. This section mostly presents the work of [Kuhn, 2005].

Consider a instance $(S = S_1 \cup S_2, C, w)$ of $IP_{WMPMPSC}$ and the solution vector $\underline{x}'$ and $\underline{z}'$ of $LP_{WMPMPSC}$ relaxation of $IP_{WMPMPSC}$. Consider the following randomized rounding scheme, where an integer solution $\bar{x} \in 0,1^m$ is computed by setting

$$x_i = \begin{cases} 1, & \text{with probability } p_i := \min\{1, \alpha x_i'\} \\ 0, & \text{otherwise} \end{cases}$$

independently for each $i \in \{1,...,n\}$. Let A_i be the “bad” event that the i^{th} element is not covered.

Lemma A1. *The probability that the i^{th} element remains uncovered is*

$$P(A_i) = \prod_{C_j \ni u_i} (1 - p_j) < e^{-\alpha}$$

Proof. The proof is in Lemma 1 of [Kuhn, 2005].

Let B_i be the “bad” event that the weight of the i^{th} element is more than $\alpha\beta z'$ for some $\beta \geq 1$.

Lemma A2. *The probability that the weight of the i^{th} element is more than $\alpha\beta z'$ is*

$$P(B_i) < \frac{1}{\beta^{\alpha\beta z'}} \cdot \prod_{C_j \ni u_i} (1 + (\beta^{w(j,i)} - 1) p_j) \leq \left(\frac{e^{\beta - 1}}{\beta^{\beta}} \right)^{\alpha z'}$$

Proof. We use a Chernoff-type argument. For $t = \ln \beta > 0$, we have

$$P(B_i) = P\left(\sum_{C_j \ni u_i} x_j w(j,i) > \alpha\beta z'\right) = P\left(e^{t\cdot \sum_{C_j \ni u_i} x_j w(j,i)} > e^{t\cdot\alpha\beta z'}\right)$$

$$< \frac{E\left[e^{t\cdot\sum_{C_j \ni u_i} x_j w(j,i)}\right]}{e^{t\cdot\alpha\beta z'}} = \frac{1}{e^{t\cdot\alpha\beta z'}} \cdot \prod_{C_j \ni u_i}\left[p_j e^{t\cdot w(j,i)} + 1 - p_j\right]$$

$$= \frac{1}{\beta^{\alpha\beta z'}} \cdot \prod_{C_j \ni s_i}\left[1 + (\beta^{w(j,i)} - 1)p_j\right] \le \frac{1}{\beta^{\alpha\beta z'}} \cdot \prod_{C_j \ni s_i} e^{p_j(\beta^{w(j,i)}-1)}$$

$$\le \frac{1}{\beta^{\alpha\beta z'}} \cdot \prod_{C_j \ni s_i} e^{p_j(\beta-1)w(j,i)} = \frac{1}{\beta^{\alpha\beta z'}} \cdot e^{(\beta-1)\cdot\sum_{C_j \ni u_i} p_j w(j,i)} \le \left(\frac{e^{(\beta-1)}}{\beta^{\beta}}\right)^{\alpha z'}$$

The inequality and equality in the second line results by application of the Markov inequality and because of the independence of the x_j. The equality and inequality in the third line hold because $t = \ln\beta$ and $1 + x \le e^x$. For the inequalities in the last line we apply $\beta^x - 1 \le (\beta - 1)x$ for $\beta \ge 1$, $x \in [0,1]$ and $\sum_{C_j \ni u_i} p_j w(j,i) \le \alpha z'$.

Denote the probability upper bounds given by Lemmas $A1$ and $A2$ by $\overline{A}_i$ and $\overline{B}_i$:

$$\overline{A}_i := \prod_{C_j \ni s_i}(1 - p_j) \ \text{ and } \ \overline{B}_i := \frac{1}{\beta^{\alpha\beta z'}} \cdot \prod_{C_j \ni u_i}(1 + (\beta^{w(j,i)} - 1)p_j).$$

In order to bound the probability for any "bad" event to occur, we define a function P as follows

$$P(p_1,\ldots,p_m) := 2 - \prod_{i=1}^{n}(1 - \overline{A}_i) - \prod_{i=1}^{n}(1 - \overline{B}_i).$$

Lemma A3. *The probability that any element is not covered or has a weight more than $\alpha\beta z'$ is upper-bounded by $P(p_1,\ldots,p_m)$:*

$$P\left(\bigcup_{i=1}^{n} A_i \cup \bigcup_{i=1}^{n} B_i\right) < P(p_1,\ldots,p_m).$$

Proof. The proof is in Lemma 3 of [Kuhn, 2005].

The following shows that if α and β are chosen appropriately, $P(p_1,\ldots,p_m)$ is always less than 1.

Lemma A4. *When setting $\alpha = \ln(\max\{|S_1|,|S_2|\}) + 1$, then for $\beta = 1 + \max\{\sqrt{3/z'}, 3/z'\}$, we have $P(p_1,\ldots,p_m) < 4/5$.*

Proof. The proof is in Lemma 4 of [Kuhn, 2005].

Lemmas A1 – A4 lead to the following randomized algorithm for the WMPMPSC problem. As a first step, the linear program $LP_{WMPMPSC}$ has to be solved. Then, all x_i' are rounded to integer values $x_i \in \{0,1\}$ using the described randomized rounding scheme with $\alpha = \ln(\max\{|S_1|,|S_2|\}) + 1$. The rounding is repeated until the solution is feasible (all elements are covered) and the weight of the integer solution deviates from the fractional weight z' by at most a factor $\alpha\beta$ for $\beta = 1 + \max\{\sqrt{3/z'}, 3/z'\}$. Each time, the probability to be successful is at least $1/5$ and therefore, the probability of not being successful decreases exponentially in the number of trials.

We will now show that $P(p_1,\ldots,p_m)$ is a pessimistic estimator and that therefore, the algorithm described above can be derandomized. That is, P is an upper bound on the probability of obtaining a "bad" solution, $P < 1$ (P is a probabilistic proof that a "good" solution exists), and the p_i can be set to 0 or 1 without increasing P. The first two properties follow by Lemmas A3 and A4, the third property is shown by the following lemma.

Lemma A5.

For all i, either setting p_i to 0 or setting p_i to 1 does not increase P:

$$P(p_1,\ldots,p_m) \geq \min\{P(\ldots,p_{i-1},0,p_{i+1},\ldots), P(\ldots,p_{i-1},1,p_{i+1},\ldots)\}$$

Proof. The proof is in Lemma 5 of [Kuhn, 2005].

Lemmas A3, A4 and A5 lead to an efficient deterministic approximation algorithm for the WMPMPSC problem. First, the linear program $LP_{WMPMPSC}$ has to be solved. The probabilities p_i are determined as described above. For α and β as in Lemma A4, $P(p_1,\ldots,p_m) < 4/5$. The probabilities p_i are now set to 0 or 1 such that $P(p_1,\ldots,p_m)$ remains smaller than $4/5$. This is possible by Lemma A5. When all $p_i \in \{0,1\}$, we have an integer solution for $IP_{WMPMPSC}$. The probability that not all elements are covered or that the weight is larger than $\alpha\beta z'$ is smaller than $P < 4/5$. Because all p_i are 0 or 1, this probability must be 0. Hence, the computed $IP_{WMPMPSC}$-solution is an $\alpha\beta$-approximation for WMPMPSC.

Authors' Information

Hakob Aslanyan *– PhD student and research assistant; Computer Science Department, University of Geneva, Battelle Batiment A, route de Drize 7, 1227 Geneva, Switzerland; e-mail: hakob.aslanyan@unige.ch*

Major Fields of Scientific Research: combinatorial optimization, graph theory, approximation and exact algorithms, hardness of approximation, network design and connectivity

ON MEASURABLE MODELS OF PROMOTION OF NEGENTROPIC STRATEGIES BY COGNITION

Pogossian Edward

Abstract: *Could models of mind be independent from living realities but be classified as mind if the mind uses the same criteria to form the class mind? In the paper a constructive view on the models of mind, cognizers, is presented and the measurable criteria and schemes of experiments on mentality of cognizers are discussed.*

Keywords: *modeling, cognition, measures, negentropic, strategies.*

ACM Classification Keywords: *A.0 General Literature.*

1. Introduction

Due mind forms models of any realities including itself raises the question whether models of mind can be mental not being living realties (LR), assembled from LR or developed from the springs of LR?

In other words, whether are models of mind which do not depend from LR but are classified as mind possible if mind uses the same criteria when forms the class mind?

To answer the question constructive models of mind and criteria of measuring their mentality as well as the exhaustive experiments on revealing the truth are needed.

In what follows a measurable approach to the models of mind, cognizers, is presented and the criteria and experiments of testing of mentality of cognizers are questioned.

This approach to refining of cogs continues the approach started in [Pogossian,1983] and continued in [Pogossian,2005,2007] on interpretation of the recognized views on mind [Flavell,1962,Neuman,1966, Botvinnik,1984, Atkinson1993, Pylyshin,2004, Roy,2005, Winograd, 986,Mendler,2004,] by models having unanimous communalized meanings followed by experiments on validity of those models.

The paper describes the author's view on mental behavior and traditionally we should address to the readers by using words "our view" ,"we think», etc.

On the other hand, mental behavior, we assume, is identified with ourselves and we plan to discuss personalized and communalized constituents in communications.

That explains why we find possible in the paper to use the pronoun "I" for the mind along with "we" and "our" when they seem to be appropriate

2. A View on Mind

2.1. I am a *mind* and I am able to interpret, or *model* the *realities* I *perceive,* including myself, evaluate the quality, or *validity* of models and use those models to promote my *utilities*.

The models are composed from cause-effect relationships between realities, particularly between realities and utilities, and any composition of those relationships comprise the *meanings* of the realities.

The basic, or *nucleus* utilities and meanings are inborn while mind incrementally enriches them by assimilating and accommodating by Piaget [Flavel,1962, Mandler, 2004] cause-effect relationships between realities and already known utilities and meanings *solving* corresponding *tasks* and *problems* .

By Piaget "Mind neither starts with cognition of itself nor with cognition of the meanings of realities but cognizes their interactions and expanding to those two poles of interactions mind organizes itself organizing the world" [Flavell,1962].

As much coincide ontology, or *communalized* (vs. *personalized*) meanings of realities with meanings of their models and as much those meanings are *operational*, i.e. allow to reproduce realities having equal with the models meanings, so better is the validity of the models.

In what follows a personalized model of mind, a view W, and a communalized version of W , *cognizers,* are presented with discussion of the validity of cognizers and schemas to meet the requirements.

2.2.1. Minds are algorithms for promoting by certain *effectors* the utilities of *living realities* (LR) in their games against or with other *players* of those games.

The players can be LR, assembles of LR like communities of humans or populations of animals as well as can be some realities that become players because not voluntarily but they affect LR inducing games with environments or the units like programs or devices that have to be tested and response to the actions of engineers . To compare and discuss some hypothetic mental realities like Cosmic Mind by Buddhists and Solaris by Stanislaw Lem are considered as players as well. Note, that descriptions of religious spiritual creatures resemble algorithm ones.

2.2.2. A variety of economic, military, etc. games can be processed by players. But all LR in different ways play the main negentropic games against overall increase of the entropy in the universe [Shrodinger,1956].

In those negentropic games with the environments LR and their populations realize some versified *reproduction* and on-the-job selection *strategy elaboration algorithms* (r SEA).

The parent rSEA periodically generates springs of LR where each child of the springs realizes some particular strategy of survival of those children in on going environments. LR with successful survival strategies get the chance to give a new spring and continue the survival games realizing some versions of strategies of their parents while unsuccessful LR die.

2.2.3. The utilities of LR and their assembles initially are determined by their nucleus, basic interests in the games but can be expanded by new mental constructions promoting already known utilities. For example, the nucleus utilities of LR, in general, include the codes (genetic) of rSEA and algorithms for reconstructing rSEA using their genetic codes.

2.2.4. The periods of reproduction, the power of the springs and other characteristics of rSEA are kinds of means to enhance survival abilities of LR and vary for different LR depending, particularly, from the resources of energy available to LR and the velocity of changes of the environments of LR.

2.2.5. Minds can be interpreted as one of means to enhance the survival of LR. In fact, minds realize SEA but in contrast to on-the-job performance rSEA the strategies elaborated by minds are auxiliary relatively to rSEA and are selected by a priory modeling.

Correspondingly, the nucleus of mental LR in addition to rSEA codes include codes of mind developing algorithms like the adaptation algorithms by Piaget [Flavel,1962, Mandler, 2004].

2.3. Thus*, modeling* SEA, or mSEA, do, particularly, the following:

- form the models of games and their constituents
- classify models to form classes and other mental constructions
- use mental constructions for a priori selection the most prospective strategies for the players
- elaborate instructions for the effectors of players using the prospective strategies.

The effectors transform the instructions into external and internal *actions* and apply to the environments of mSEA and mSEA themselves, correspondingly, for developing the environments and mSEA and enhancing the success of the players.

2.4. Whether are the models of mind which are not dependent from LR but are classified as mind possible if mind uses the same criteria when forms the class *mind*?

To answer to the question constructive models of mind and criteria of measuring their mentality as well as the exhaustive experiments on revealing the truth are needed.

2.5. Let's name *cognizers* the models of mind not depending from LR while the models of mental constructions name *mentals*.

Apparently, this ongoing view W on mind is a kind of cognizers, say, for certainty, *1-cognizers, 1cogs* or *cogs* in this paper.

In what follows a constructive approach to cogs, the criteria and experiments of testing of mentality of cogs are presented.

3. Basic Approaches and Assumptions

3.1. Further refining of cogs extends the approach described above on interpretation of the recognized views on mind by models having unanimous communalized meanings followed by experiments on validity of those models to mind.

3.2.1. Later on it is assumed that cogs are object-oriented programs, say in Java.

All programs in Java are either classes or sets of classes.

Therefore, it is worth to accept that cogs and their constituents, mentals, are either Java classes or their compositions as well.

3.3. Accepting the above stated assumption the experiments on quality of cogs were run for SSRGT games.

Particularly, because chess represents the class and by variety of reasons is recognized as a regular environment to estimate models of mind [Botvinnik,1984, Pogossian,1983,2007, Atkinson,1993, Furnkranz, 2001] in what follows the constructions of mentals and experiments on mentality of cogs are accompanied, as a rule, by interpretations in chess.

3.4. Following to the view W cogs elaborate instructions for the effectors of players to promote their utilities. The effectors in turn transform instructions into *actions* applied to the players and their *environments.* They can be parts of the players or be constructed by cogs in their work.

It is assumed that certain *nucleus* mentals of cogs as well as the players and their effectors are predetermined and process in discrete time intervals while mentals of cogs can evolve in time.

The fundamental question on the origin of nucleus mentals and other structures needs further profound examination.

4. Refining Constituents of Cognizers

4.1.1. In general, *percepts* are the inputs of cogs and have the structure of bundles of instances of the classes of cogs composed in discrete time intervals.

The *realities* of cogs are refined as the causes of their *percepts*.

The *environments* and the *universe* of cogs are the sets and the totality of all *realities of cogs*, correspondingly.

More in details, the bundles of instances of attributes of a class X of cogs at time t are named *X percepts at t* and the causes of X/t percepts are named *X/t realities.*

It is worth to consider *t percepts* and *percepts* as the elements of the unions of X/t percepts and t percepts, correspondingly, and assume that there may be multiple causes for the same percept.

Analogically, *t realities* and *X/t realities* are defined.

In case percepts are bundles of instances of attributes of certain classes of cogs the realities causing them are the classes represented by those attributes.

Otherwise, cogs learn about the realities by means of the percepts corresponded to realities and by means of the responses of those percepts when cogs arrange actions by effectors.

Due cogs are continuously developed they start with percepts formed by nucleus classes followed by percepts formed by the union of new constructed and nucleus classes.

4.1.2. Cogs promote utilities by using links between utilities and percepts. They continuously memorize percepts, by certain criteria unite them in classes as *concepts* and distinguish realities to operate with them using *matching* methods associated with the concepts.

In addition some concepts are nominated by *communicators* to communicate about the realities of the domains of the concepts with other cogs or minds and enhance the effectiveness of operations of cogs in the environments.

4.2.1. The base criteria to unite percepts in concepts are *cause-effect relationships* (cers) between percepts, particularly, between percepts and utilities.

For revealing cers cogs **form and solve** *tasks* and *problems.*

Tasks are requirements to link given percepts (or realities) by certain cers and represent those cers in frame of certain classes.

4.2.2. The basic tasks are the *utility* tasks requiring for given percepts to find utilities that by some cers can be achieved from the percepts. In chess utility tasks require to search strategies for enhancing the chances to win from given positions.

The *generalization*, or *classification* tasks unite percepts (as well as some classes) with similar values into more advanced by some criteria classes and associate corresponding matching procedures with those classes to distinguish the percepts of the classes and causing them realities.

The *acquisition* tasks create new classes of cogs by transferring ready to use classes from other cogs or minds while the *inference* tasks infer by some general rules new classes as consequences of already known to cogs classes.

The *question* tasks can be considered as a kind of formation tasks inference tasks which induce new tasks applying syntax rules of question tags to the solutions of already solved tasks.

The *modeling* tasks require revealing or constructing realities having certain similarities in *meanings* with the given ones.

Before refining meanings of realities let’s note that to help to solve the original tasks some approximating them model tasks can be corresponded.

4.2.3. *Problems* are compositions of homogeneous tasks and *solutions of problems* are procedures composing the solutions of constituent tasks.

The problems can be with *given spaces* of possible *solutions* (GSS) or without GSS, or the *discovery* ones.

Tasks formation and *tasks solving procedures* form and solve tasks types.

4.3.1. To refine the meanings of realities and mentals it is convenient to interpret the percepts, uniting them concepts, nucleus classes and the constituents of those mentals as the nodes of the *graph of mentals* (GM) while the edges of GM are determined by utility, cers, attributive, part of and other relationships between those nodes.

Then the *meaning of a percept C* can be defined as the union of the totality of realities causing C and the connectivity sub graph of GM with root in C.

The *meaning of a concept X* is defined as the union of the meanings of the nodes of the connectivity sub graph of GM with the root in X.

The *meaning of realities R* causing the percept C is the union of the meanings of the nodes of the connectivity sub graph of GM with the root in the percept C.

4.3.2. Later on it is assumed that the *knowledge* of cogs unites, particularly, the cogs, GM and their constituents.

4.4.1. Processing of percepts and concepts is going either *consciously* or *unconsciously.* While unconsciousness, usually, addresses to the *intuition* and needs the long way of research efforts for its explanation, the consciousness is associated with the named concepts and percepts in languages and their usage for communications. Particularly, the vocabularies of languages provide names of variety of concepts and realities causing those percepts.

Mind operates with percepts, concepts and other mentals while names realities causing those mentals when it should communicate.

Particularly, this ongoing description of cogs follows to the rules for named realities while internally refers to corresponding mentals.

4.4.2. When mind operates *internally* with the representations of realities it is always able to address to their meanings or *to ground* those representations [8].

For *external* communications mind uses representations of realities, *communicators,* which can be separated from the original carriers of the meanings of those realities, i.e. from the percepts of those realities, and become *ungrounded*.

The role of communicators is to trigger [12] the desired meanings in the partners of communications. Therefore, if partners are deprived of appropriate grounding of the communicators special arrangements are needed like the ones provided by ontologies. If the communicators are not sufficiently grounded well known difficulties like the ones in human-computer communications can rise.

Note, that if the model R` is a grounded reality the meaning of R' can induce new unknown aspects of the meaning of the original ones.

4.5. Realities R` *represent* realities R, or R` is a *model* of R, if meanings of R` and R intersect.

Model R` is *equal* to R if R' and R have the same meanings. The more is the intersection of the meanings of R and R` relative to the meaning of R the greater is the *validity* of R`. For measuring the validity of models a variety of aspects of the meanings of original realities can be emphasized. Particularly, descriptive or behavioral aspects of the meanings can be considered, or be questioned whether the meanings are views only of the common use or they are specifications.

5. Questioning Validity of Mind

5.1. Modeling problems require constructing realities having certain similarities in meanings with the original ones.

When those realities are problems as well cogs correspond model problems to the original ones, run them to find model solutions and interpret them back to solve the original ones.

Apparently, solutions of problems are the most valid models of those problems but, unfortunately, not always can be found in frame of available search resources.

Valid models trade off between the approximations of the meanings of solutions of problems and between available resources to choose the best available approximations.

Due of that inevitable trade off the models are forced to focus on only the particular aspects of those solutions.

If communication aspects are emphasized the *descriptive* models and criteria of validity can be in use require the realities-models be equal only by communicative means of the communities.

On-the-job or *behavioral* criteria evaluate validity of models by comparing the performances of corresponding procedures.

The records of computer programs provide examples of descriptive models while when processed programs become the subject of behavioral validity. Sorts of behavioral validity provide functional testing and question-answer ones like Turing test.

Productive behavioral validity criteria compare the results of affection of the outputs of realities and their models on the environment. Fun Newman requirement on self-reproducibility of automata [Neuman, 1966] provides an example of productive validity. In its interpretation as *reflexive reproducibility* (RR) validity that criterion requires to construct 1-models of realities able to produce 2-models equal to the 1-models and able to chain the process.

5.2. To formulate criteria of validity of cogs it is worth to summarize the refined to this end views on mind as the following:

mind is an algorithm to solve problems on promotion of utilities of LR in their negentropic games

mind is composed from certain constituent algorithms for forming and solving tasks of certain classes including the utility, classification, modeling, questioning classes

mind uses solutions of problems to elaborate instructions for certain effectors to make the strategies of LR more effective and the environments of LR more favorable to enhance the success of LR in negentropic games.

5.3. Criteria of validity of cogs to mind have to answer whether cogs have meanings that minds have about themselves.

On the long way in approaching to valid cogs a chain of inductive inferences is expected aimed to converge eventually to target validity.

Inductive inferences unite science with arts and, unfortunately, the term of their stabilization can not be determined algorithmically. Nevertheless, what can be done is to arrange those inferences with the trend to converge to the target stabilization in limit [19].

To approach to valid cogs it is worth to order the requirements to the validity of cogs and try to achieve them incrementally, step by step.

The requirements v1- v4 to validity of cogs condition them to meet the following:

v1. be well positioned relatively to known psychological models of mind

v2. be able to form and solve the utility, classification, modeling and question tasks with acceptable quality of the solutions

v.3. be able to use the solutions of tasks and enhance the success of the players

v4. be able to form acceptable models of themselves, or be able to *self modeling*

The requirements v2 - v4 follow the basic views on mind while v1 requires positioning cogs relatively, at least, to the recognized psychological models of mind to compare and discuss their strengths and weaknesses.

Note, that parent minds of LR reproduce themselves in the children minds in indirect ways using certain forms of cloning, heritage and learning procedures.

Some constituents of reproduction of LR can already be processed artificially, i.e. by regular for the human community procedures.

The requirement v4 is questioning, in fact, whether completely artificial minds, cogs, can reproduce new cogs equal themselves and to the biological ones.

5.4. What are the validity criteria to make cogs equal by meaning to mind and whether cogs valid by those criteria can be constructed?

It is a long way journey to answer to these questions and elaborate some approaches to implement.

6. Conclusion

Valid cogs, if constructed, confirm the assertion that mind is a modeling based problem formation and solving procedure able to use knowledge gained from the solutions to promote the utilities of LR in their negentropic games.

Synchronously, mental cogs provide a constructive model of mind as the ultimate instrument for cognition. Knowledge on the nature of instruments for revealing new knowledge gives a new look on the knowledge already gained or expected and raise new consequent questions.

Therefore, revealing by cogs the new knowledge on the instruments of cognition it is worth to question the new aspects of relationships between mind and the overall knowledge mind creates and uses.

Ongoing experiments on study of cogs are based on the technique of evaluating adaptive programs and their parts by local tournaments and use the game solving package with its kernel Personalized Planning and Integrated Testing (PPIT) and Strategy Evaluation units [Pogossian,1983,2005,2007].

Bibliography

[Atkinson,1993] G. Atkinson Chess and Machine Intuition. Ablex Publishing Corporation, Norwood, New Jersey, 1993.

[Botvinnik,1984] M.Botvinnik Computers in Chess: Solving Inexact Search Problems. Springer Series in Symbolic Computation, with Appendixes, Springer-Verlag: NY, 1984.

[Flavell,1962] J. Flavell The Develometal Psycology of Jean Piaget, D.VanNostrand Company Inc., Princeton, New Jersey, 1962.

[Furnkranz, 2001] J.Furnkranz Machine Learning in Games: A Survey in "Machines that Learn to Play Games", Nova Scientific, 2001.

[Mandler,2004] Mandler J. The Foundations of Mind: Origins of Conceptual Thought. Oxford Univ. Press, 2004.

[Neuman, 1966] John von Neuman.Theory of Self-reproducing Automata. University of Illinois Press, 1966.

[Pogossian,2007] E.Pogossian. On Measures of Performance of Functions of Human Mind. 6th International Conference in Computer Science and Information Technologies, CSIT2007, Yerevan, 2007, 149-154

[Pogossian ,2006] E.Pogossian. Specifying Personalized Expertise. International Association for Development of the Information Society (IADIS): International Conference Cognition and Exploratory Learning in Digital Age (CELDA 2006), 8-10 Dec., Barcelona, Spain (2006) 151-159

[Pogossian,2005] E. Pogossian. Combinatorial Game Models For Security Systems. NATO ARW on "Security and Embedded Systems", Porto Rio, Patras, Greece, Aug. (2005) 8-18

[Pogossian,2007] E.Pogossian, V. Vahradyan, A. Grigoryan. On Competing Agents Consistent with Expert Knowledge", Lecture Notes in Computer Science, AIS-ADM-07: The Intern. Workshop on Autonomous Intelligent Systems - Agents and Data Mining, June 5 -7, 2007, St. Petersburg, 11pp.

[Pogossian,1983] E.Pogossian. Adaptation of Combinatorial Algorithms (a monograph in Russian), 293 pp. 1983. Yerevan.,

[Pylyshyn,2004] Z. Pylyshyn Seeing and Visualizing: It's Not What You Think, An Essay On Vision And Visual Imagination, http://ruccs.rutgers.edu/faculty/pylyshyn.htm,l2004.

[Roy,2005] D.Roy Grounding Language in the World: Signs, Schemas, and Meaning Cognitive Machines Group ,The Media Laboratory, MIT (http://www.media.mit.edu/cogmac/projects.html) 2005.

[Searle,1990] Searle J. Is the brain's mind a computer program? Scientific American 262, pp26-31, 1990.

[Shannon, 1949] C.E.Shannon. The Mathematical theory of communication. In: The Mathematical Theory of Communication. Ed. C.E.Shannon and W.Weaver. University of Illinois Press, Urbana, 1949.

[Shrodinger,1956] E.Shrodinger . Mind and Matter. Cambridge, 1956.

[Winograd,1986] T.Winograd, F.Flores. Understanding Computers and Cognition (A new foundation for design). Publishers, Chapter 2, pp. 11–59, Huntington, NY, 1986.

Authors' Information

Head of the Cognitive Algorithms and Models Laboratory at the Academy of Science of Armenia (IPIA) , professor at the State Engineering University of Armenia , Marshall Bagramyan Av.24, Academy of Sciences, Yerevan, 0019, epogossi@aua.am,

Major Fields of Scientific Research: models of cognition and knowledge processing, strategy elaboration, algorithms.

ON RELIABILITY APPROACH TO MULTIPLE HYPOTHESES TESTING AND TO IDENTIFICATION OF PROBABILITY DISTRIBUTIONS OF TWO STOCHASTICALLY RELATED OBJECTS

Evgueni Haroutunian, Aram Yessayan, Parandzem Hakobyan

Abstract: *This paper is devoted to study of characteristics of logarithmically asymptotically optimal (LAO) hypotheses testing and identification for a model consisting of two related objects. In general case it is supposed that L_1 possible probability distributions of states constitute the family of possible hypotheses for the first object and the second object is distributed according to one of $L_1 \times L_2$ given conditional distributions depending on the distribution index and the current observed state of the first object. For the first testing procedure the matrix of interdependencies of all possible pairs of the error probability exponents (reliabilities) in asymptotically optimal tests of distributions of both objects is studied. The identification of the distributions of two objects gives an answer to the question whether r_1-th and r_2-th distributions occurred or not on the first and the second objects, correspondingly. Reliabilities for the LAO identification are determined for each pair of double hypotheses. By the second approach the optimal interdependencies of lower estimates of all possible pairs of corresponding reliabilities are found and lower estimates of reliabilities for the LAO identification are studied for each pair of hypotheses. The more complete results are presented for model of statistically dependent objects, when distributions of the objects are dependent, but its current states are independent. For an example of two statistically dependent objects optimal interdependencies of pairs of reliabilities are calculated and graphically presented.*

Keywords: *Multiple hypotheses testing, Identification of distribution, Inference of many objects, Error probability exponents, Reliabilities.*

1. Introduction

As a development of the results on two and on multiple hypotheses logarithmically asymptotically optimal (LAO) testing of probability distributions of one object [1] -- [3], in paper [4] Ahlswede and Haroutunian formulated a number of problems with respect to multiple hypotheses testing and identification for many objects. Haroutunian and Hakobyan solved in [5] the problem of many hypotheses testing for two independent objects and in [6] the problem of the identification of distributions being based on samples of independent observations. In prepublications [7] -- [10] Haroutunian and Yessayan studied many hypotheses LAO testing for two objects under different kinds of relation.

LAO tests of its distributions for two hypotheses were analyzed first by Hoeffding [1], later by Csiszár and Longo [2] and by other authors. Here we investigate characteristics of procedures of LAO testing and identification of probability distributions of two stochastically dependent objects.

Let X_1 and X_2 be random variables (RVs) taking values in the same finite set of states X and $P(X)$ be the space of all possible distributions on X. There are given L_1 probability distributions (PDs) $G_{l_1}=\{G_{l_1}(x^1), x^1\in X\}$, $l_1=\overline{1,L_1}$, from $P(X)$. The first object is characterized by RV X_1 which has one of these L_1 possible PDs and the second object is dependent on the first and is characterized by RV X_2 which can have one of $L_1\times L_2$ conditional PDs $G_{l_2/l_1}=\{G_{l_2/l_1}(x^2\mid x^1), x^1,x^2\in X\}$, $l_1=\overline{1,L_1}$, $l_2=\overline{1,L_2}$. Joint PDs are $G_{l_1}oG_{l_2/l_1}=\{G_{l_1,l_2}(x^1,x^2), x^1,x^2\in X\}$, $l_1=\overline{1,L_1}, l_2=\overline{1,L_2}$, where $G_{l_1,l_2}(x^1,x^2)=G_{l_1}(x^1)G_{l_2/l_1}(x^2\mid x^1)$. Let $(x_1,x_2)=((x_1^1,x_1^2),(x_2^1,x_2^2),...(x_N^1,x_N^2))$ be a sequence of results of N independent observations of the pair of objects. The probability $G_{l_1,l_2}^N(x_1,x_2)$ of vector (x_1,x_2) is the following product:

$$G_{l_1,l_2}^N(x_1,x_2)=G_{l_1}^N(x_1)G_{l_2/l_1}^N(x_2\mid x_1)=\prod_{n=1}^N G_{l_1}(x_n^1)G_{l_2/l_1}(x_n^2\mid x_n^1),$$

with $G_{l_1}^N(x_1)=\prod_{n=1}^N G_{l_1}(x_n^1)$ and $G_{l_2/l_1}^N(x_2\mid x_1)=\prod_{n=1}^N G_{l_2/l_1}(x_n^2\mid x_n^1)$.

For the object characterized by X_1 the non-randomized test $\varphi_1^N(x_1)$ can be determined by partition of the sample space X^N on L_1 disjoint subsets $A_{l_1}^N=\{x_1:\varphi_1^N(x_1)=l_1\}$, $l_1=\overline{1,L_1}$, i.e. the set $A_{l_1}^N$ consists of vectors x_1 for which the PD G_{l_1} is adopted. The probability $\alpha_{l_1|m_1}^N(\varphi_1^N)$ of the erroneous acceptance of PD G_{l_1} provided that G_{m_1} is true, $l_1,m_1=\overline{1,L_1}, m_1\neq l_1$, is defined by the probability $G_{m_1}^N$ of the set $A_{l_1}^N$:

$$\alpha_{l_1|m_1}^N(\varphi_1^N)\stackrel{\Delta}{=}G_{m_1}^N(A_{l_1}^N). \tag{1}$$

We define the probability to reject G_{m_1}, when it is true, as follows

$$\alpha^N_{m_1|m_1}(\varphi^N_1) \stackrel{\Delta}{=} \sum_{l_1:l_1\neq m_1} \alpha^N_{l_1|m_1}(\varphi^N_1) = G^N_{m_1}(\overline{A^N_{m_1}}). \tag{2}$$

Denote by φ_1 the infinite sequences of tests for the first object. Corresponding error probability exponents, which we call reliabilities $E_{l_1|m_1}(\varphi_1)$ for test φ_1 are defined as

$$E_{l_1|m_1}(\varphi_1) \stackrel{\Delta}{=} \overline{\lim_{N\to\infty}}\{-\frac{1}{N}\log \alpha^N_{l_1|m_1}(\varphi^N_1)\},\ m_1, l_1 = \overline{1, L_1}. \tag{3}$$

It follows from (2) and (3) that

$$E_{m_1|m_1}(\varphi_1) = \min_{l_1:l_1\neq m_1} E_{l_1|m_1}(\varphi_1),\ l_1, m_1 = \overline{1, L_1}, \quad l_1 \neq m_1. \tag{4}$$

We shall reformulate now the Theorem from [3] for the case of one object with L_1 hypotheses. This requires some additional notions and notations. For some PD $Q = \{Q(x^1), x^1 \in X\}$ the entropy $H_Q(X_1)$ and the informational divergence $D(Q \,||\, G_{l_1}),\ l_1 = \overline{1, L_1}$, are defined as follows:

$$H_Q(X_1) \stackrel{\Delta}{=} -\sum_{x^1\in X} Q(x^1)\log Q(x^1),$$

$$D(Q \,||\, G_{l_1}) \stackrel{\Delta}{=} \sum_{x^1\in X} Q(x^1)\log\frac{Q(x^1)}{G_{l_1}(x^1)}.$$

For given positive numbers $E_{1|1},\ldots,E_{L-1|L-1}$, let us consider the following sets of PDs $Q = \{Q(x^1), x^1 \in X\}$:

$$R_{l_1} \stackrel{\Delta}{=} \{Q : D(Q \,||\, G_{l_1}) \leq E_{l_1|l_1}\},\ l_1 = \overline{1, L_1 - 1}, \tag{5a}$$

$$R_{L_1} \stackrel{\Delta}{=} \{Q : D(Q \,||\, G_{l_1}) > E_{l_1|l_1},\ l_1 = \overline{1, L_1 - 1}\}, \tag{5b}$$

and the elements of the reliability matrix $E(\varphi_1^*)$ of the LAO test φ_1^*:

$$E^*_{l_1|l_1} = E^*_{l_1|l_1}(E_{l_1|l_1}) \stackrel{\Delta}{=} E_{l_1|l_1},\ l_1 = \overline{1, L_1 - 1}, \tag{6a}$$

$$E^*_{l_1|m_1} = E^*_{l_1|m_1}(E_{l_1|l_1}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} D(Q \,||\, G_{m_1}),\ m_1 = \overline{1, L_1},\ m_1 \neq l_1,\ l_1 = \overline{1, L_1 - 1}, \tag{6b}$$

$$E^*_{L_1|m_1} = E^*_{L_1|m_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \stackrel{\Delta}{=} \inf_{Q \in R_{L_1}} D(Q \,||\, G_{m_1}),\ m_1 = \overline{1, L_1 - 1}, \tag{6c}$$

$$E^*_{L_1|L_1} = E^*_{L_1|L_1}(E_{1|1}, E_{2|2}, ..., E_{L_1-1|L_1-1}) \stackrel{\Delta}{=} \min_{l_1 = \overline{1, L_1 - 1}} E^*_{l_1|L_1}. \tag{6d}$$

Theorem 1 [3]: *If all distributions G_{l_1}, $l_1 = \overline{1, L_1}$, are different in the sense that $D(G_{l_1} \,||\, G_{m_1}) > 0$, $l_1 \neq m_1$, and the positive numbers $E_{1|1}, E_{2|2}, ..., E_{L-1|L-1}$ are such that the following inequalities hold*

$$E_{1|1} < \min_{l_1 = \overline{2, L_1}} D(G_{l_1} \,||\, G_1),$$

$$\text{---} \tag{7}$$

$$E_{m_1|m_1} < \min\left(\min_{l_1 = \overline{m_1+1, L_1}} D(G_{l_1} \,||\, G_{m_1}),\ \min_{l_1 = \overline{1, m_1 - 1}} E^*_{l_1|m_1}(E_{l_1|l_1})\right),\ \ m_1 = \overline{2, L_1 - 1},$$

then there exists a LAO sequence of tests φ_1^*, the reliability matrix of which $E(\varphi_1^*) = \{E_{l_1|m_1}(\varphi_1^*)\}$ is defined in (6) and all elements of it are positive.Inequalities (7) are necessary for existence of tests sequence with reliability matrix having in diagonal given elements $E_{l_1|l_1}$, $l_1 = \overline{1, L_1 - 1}$, and all other elements positive .

Corollary 1 [3]: *If, in contradiction to condition of strict positivity, one, or several diagonal elements $E_{m_1|m_1}$, $m_1 = \overline{1, L_1 - 1}$, of the reliability matrix are equal to zero, then the elements of the matrix determined in functions of this $E_{m_1|m_1}$ will be given as in the case of Stein's lemma [11], [12]*

$$E_{l_1|m_1}(E_{m_1|m_1}) = D(G_{l_1} \,||\, G_{m_1}),\ m_1 \neq l_1,$$

and the remaining elements of the matrix $E(\varphi_1^*)$ will be defined by $E_{l_1|l_1} > 0$, $l_1 \neq m_1$, $l_1 = \overline{1, L_1 - 1}$, as follows from Theorem 1.

Now we formulate the concept of LAO approach to the identification problem for one object, which was introduced in [4]. We have one object, and there are known $L_1 \geq 2$ possible PDs. Identification is the answer to the question whether r_1-th distribution is correct, or not. As in the testing problem, the answer must be given on the base of a sample x with the help of an appropriate test.

There are two error probabilities for each $r_1 \in [1, L_1]$: the probability $\alpha_{l_1 \neq r_1 | m_1 = r_1}(\varphi_N)$ to accept l-th PD different from r_1, when PD r_1 is correct, and the probability $\alpha_{l_1 = r_1 | m_1 \neq r_1}(\varphi_N)$ that r_1 is accepted, when it is not correct.

The probability $\alpha_{l_1 \neq r_1 | m_1 = r_1}(\varphi_N)$ coincides with the probability $\alpha_{r_1 | r_1}(\varphi_N)$ which is equal to $\sum\limits_{l_1 : l_1 \neq r_1} \alpha_{l_1 | r_1}(\varphi_N)$.

The corresponding reliability $E_{l_1 \neq r_1 | m_1 = r_1}(\varphi)$ is equal to $E_{r_1 | r_1}(\varphi)$ which satisfies equality (4).

The reliability approach to identification assumes determining the optimal dependence of $E^*_{l_1 = r_1 | m_1 \neq r_1}$ upon given $E^*_{l_1 \neq r_1 | m_1 = r_1} = E^*_{r_1 | r_1}$, which can be an assigned value satisfying conditions (7). The solution of this problem assumes knowledge of some a priori PDs of the hypotheses.

The result from paper [4] is valid for the first object.

Theorem 2 [4]: *In case of distinct hypothetical PDs $G_1, G_2, \ldots, G_{L_1}$, under condition that the probabilities of all L_1 hypotheses are strictly positive for given $E^*_{l_1 \neq r_1 | m_1 = r_1} = E^*_{r_1 | r_1}$ the reliability $E^*_{l_1 = r_1 | m_1 \neq r_1}$ is the following:*

$$E^*_{l_1 = r_1 | m_1 \neq r_1}(E^*_{r_1 | r_1}) = \min_{m_1 : m_1 \neq r_1} \inf_{Q : D(Q \| G_{r_1}) \leq E_{r_1 | r_1}} D(Q \| G_{m_1}), \; r_1 = \overline{1, L_1}.$$

In Section 2 we consider two related objects as one complex object and we obtain corresponding reliabilities for LAO testing and identification. In Section 3 we will obtain the lower estimates of the reliabilities for LAO testing and in Section 4 for identification for the dependent object. These estimates serve for deducing of lower estimates of the reliabilities for LAO testing (in Section 5) and identification (in Section 6) of distributions of two related objects. Results of certain calculations for an example will be graphically presented in Section 7.

2. LAO Testing and Identification of the Probability Distributions for Two Stochastically Coupled Objects

We expose the direct approach for LAO testing and identification of PDs for two related objects. It consists in considering the pair of objects as one composite object [10]. The test, which we denote by Φ^N , is a procedure of making decision about unknown indices of PDs on the base of results of N observations (x_1,x_2) . For the objects characterized by X_1,X_2 the non-randomized test $\Phi^N(x_1,x_2)$ can be determined by partition of the sample space $(X \times X)^N$ on $L_1 \times L_2$ disjoint subsets $A^N_{l_1,l_2} = \{(x_1,x_2) : \Phi^N(x_1,x_2) = (l_1,l_2)\}$, $l_1 = \overline{1,L_1}$, $l_2 = \overline{1,L_2}$, i.e. the set $A^N_{l_1,l_2}$ consists of vectors (x_1,x_2) for which the PD G_{l_1,l_2} must be adopted. The probability $\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N)$ of the erroneous acceptance of PD G_{l_1,l_2} provided that G_{m_1,m_2} is true, $l_1,m_1 = \overline{1,L_1},\ l_2,m_2 = \overline{1,L_2}$, $(m_1,m_2) \neq (l_1,l_2)$ is defined by the set $A^N_{l_1,l_2}$

$$\alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N) \stackrel{\Delta}{=} G^N_{m_1,m_2}(A^N_{l_1,l_2}). \tag{8}$$

We define the probability to reject G_{m_1,m_2} , when it is true, as follows

$$\alpha^N_{m_1,m_2|m_1,m_2}(\Phi^N) \stackrel{\Delta}{=} \sum_{(l_1,l_2)\neq(m_1,m_2)} \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N) = G^N_{m_1,m_2}(\overline{A^N_{m_1,m_2}}). \tag{9}$$

Our intention is to study the reliabilities of the infinite sequence of tests Φ

$$E_{l_1,l_2|m_1,m_2}(\Phi) \stackrel{\Delta}{=} \overline{\lim_{N\to\infty}} \{-\frac{1}{N}\log \alpha^N_{l_1,l_2|m_1,m_2}(\Phi^N)\},\ l_1,m_1 = \overline{1,L_1},\ l_2,\ m_2 = \overline{1,L_2}. \tag{10}$$

From (9) and (10) we deduce that

$$E_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2)\neq(m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1,m_1 = \overline{1,L_1},\ l_2,m_2 = \overline{1,L_2}. \tag{11}$$

The matrix $E(\Phi)=\{E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1,m_1=\overline{1,L_1},\ l_2,m_2=\overline{1,L_2}\}$ is called the reliability matrix of the sequence of tests Φ. Our aim is to investigate the reliability matrix of optimal tests, and the conditions ensuring positivity of all its elements.

For given positive numbers $E_{1,1|1,1},\dots,E_{L_1,L_2-1|L_1,L_2-1}$, let us consider the following sets of PDs

$$QoV \stackrel{\Delta}{=} \{Q(x^1)V(x^2 \mid x^1), x^1, x^2 \in X\}:$$
$$R_{l_1,l_2} \stackrel{\Delta}{=} \{QoV : D(QoV \,\|\, G_{l_1,l_2}) \le E_{l_1,l_2|l_1,l_2}\},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{12a}$$

$$R_{L_1,L_2} \stackrel{\Delta}{=} \{QoV : D(QoV \,\|\, G_{l_1,l_2}) > E_{l_1,l_2|l_1,l_2},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}\}, \tag{12b}$$

and the elements of the reliability matrix E^* of the LAO test:

$$E^*_{l_1,l_2|l_1,l_2} = E^*_{l_1,l_2|l_1,l_2}(E_{l_1,l_2|l_1,l_2}) \stackrel{\Delta}{=} E_{l_1,l_2|l_1,l_2},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{13a}$$

$$E^*_{l_1,l_2|m_1,m_2} = E^*_{l_1,l_2|m_1,m_2}(E_{l_1,l_2|l_1,l_2}) \stackrel{\Delta}{=} \inf_{QoV\in R_{l_1,l_2}} D(QoV \,\|\, G_{m_1,m_2}),\ m_1=\overline{1,L_1},$$
$$m_2=\overline{1,L_2},\ (l_1,l_2)\ne(m_1,m_2),\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2-1}, \tag{13b}$$

$$E^*_{L_1,L_2|m_1,m_2} = E^*_{L_1,L_2|m_1,m_2}(E_{1,1|1,1},E_{1,2|1,2},E_{1,3|1,3}\dots,E_{L_1,L_2-1|L_1,L_2-1})$$
$$\stackrel{\Delta}{=} \inf_{QoV\in R_{L_1,L_2}} D(QoV \,\|\, G_{m_1,m_2}),\ \ m_1=\overline{1,L_1},\ m_2=\overline{1,L_2-1}, \tag{13c}$$

$$E^*_{L_1,L_2|L_1,L_2} = E^*_{L_1,L_2|L_1,L_2}(E_{1,1|1,1},E_{1,2|1,2},E_{1,3|1,3}\dots,E_{L_1,L_2-1|L_1,L_2-1}) \stackrel{\Delta}{=} \min_{l_1=\overline{1,L_1}}\ \min_{l_2=\overline{1,L_2-1}} E^*_{l_1,l_2|L_1,L_2}. \tag{13d}$$

For simplicity we can take $(X_1, X_2) = \mathrm{Y}$, $X \times X = Y$ and $y = (y_1, y_2, ..., y_N) \in Y^N$, where $y_n = (x_n^1, x_n^2), n = \overline{1, N}$, then we will have $L_1 \times L_2 = L$ new hypotheses for one object

$$G_{l_1,l_2} = \{G_{l_1}(x^1)G_{l_2/l_1}(x^2 \mid x^1), x^1, x^2 \in X\},\ l_1 = \overline{1, L_1}, l_2 = \overline{1, L_2}, \text{ where } G_{1,1} = K_1,$$
$$G_{1,2} = K_2,\ G_{1,3} = K_3,\ ...,\ G_{1,L_2} = K_{L_2},\ G_{2,1} = K_{L_2+1},\ ...,\ G_{l_1,l_2} = K_{(l_1-1)L_2+l_2},$$
$$l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2},\ \alpha_{l_1,l_2|m_1,m_2} = \alpha_{(l_1-1)L_2+l_2|(m_1-1)L_2+m_2},\ l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2}$$
$$E_{l_1,l_2|m_1,m_2} = E_{(l_1-1)L_2+l_2|(m_1-1)L_2+m_2},\ l_1 = \overline{1, L_1},\ l_2 = \overline{1, L_2}$$

and thus we have brought the original problem to the case of one object with $L_1 \times L_2$ hypotheses.

So applying Theorem 1 we can deduce that there exists a LAO sequence of tests Φ^*, the reliability matrix of which $E^* = \{E_{l|m}(\Phi^*)\}$ is defined in (13) and all elements of it are positive.

Using Theorem 2 for this composite object we can deduce that identification reliabilities are connected with the following formula

$$E_{l=r|m\neq r}(E_{r|r}) = \min_{m: m\neq r}\ \inf_{QoV: D(QoV||K_r) \leq E_{r|r}} D(QoV \,||\, K_m),\ r \in [1, L]. \tag{14}$$

Now let us consider the more general particular model, when X_1 and X_2 are related statistically, in the following way $G_{l_1,l_2}(x^1, x^2) = G_{l_1}(x^1)G_{l_2/l_1}(x^2)$. The probability of vector (x_1, x_2) is defined by the following PD G_{l_1,l_2}

$$G_{l_1,l_2}^N(x_1, x_2) = G_{l_1}^N(x_1)G_{l_2/l_1}^N(x_2) = \prod_{n=1}^{N} G_{l_1}(x_n^1)G_{l_2/l_1}(x_n^2),$$

where $G_{l_1}^N(x_1) = \prod_{n=1}^{N} G_{l_1}(x_n^1)$ and $G_{l_2/l_1}^N(x_2) = \prod_{n=1}^{N} G_{l_2/l_1}(x_n^2)$.

In this case we can analogously bring the problem to the problem on one object with $L_1 \times L_2 = L$ hypotheses, where $G_{l_1,l_2} = \{G_{l_1}(x^1)G_{l_2/l_1}(x^2), x^1, x^2 \in X\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$, and for the sets R_{l_1,l_2}, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$ of PDs $QoV \overset{\Delta}{=} \{Q(x^1)V(x^2), x^1, x^2 \in X\}$:

When the objects X_1 and X_2 can have only different distributions from same L given probability distributions (PD) G_1, G_2, G_3, ..., G_L from $P(X)$, [4], [7] we can reduce the problem to the problem of one object and $L \times (L-1)$ hypotheses, where $G_{l_1,l_2} = \{G_{l_1}(x^1)G_{l_2}(x^2), x^1, x^2 \in X\}$, $l_1, l_2 = \overline{1,L}$, $l_1 \neq l_2$ (see [4], [7]).

3. An Approach to Multiple Hypotheses Testing for the Second (Dependent) Object

Let us remark that test Φ^N can be composed of a pair of tests φ_1^N and φ_2^N for the separate objects: $\Phi^N = (\varphi_1^N, \varphi_2^N)$. For the second object characterized by RV X_2 depending on X_1 the non-randomized test $\varphi_2^N(x_2, x_1, l_1)$ based on vectors x_1, x_2 and on the index of the hypothesis l_1 adopted for X_1, can be given for each l_1 and x_1 by division of the sample space X^N on L_2 disjoint subsets $A_{l_2/l_1}^N(x_1) = \{x_2 : \varphi_2^N(x_2, x_1, l_1) = l_2\}$, $l_1 = \overline{1, L_1}$, $l_2 = \overline{1, L_2}$. We upper estimate the error probabilities for second object proceeding from definition (8).

$$G_{m_1,m_2}^N(A_{l_1,l_2}^N) = \sum_{x_1 \in A_{l_1}^N} G_{m_1}^N(x_1) G_{m_2/m_1}^N(A_{l_2/l_1}^N(x_1) \mid x_1) \leq \max_{x_1 \in A_{l_1}^N} G_{m_2/m_1}^N(A_{l_2/l_1}^N(x_1) \mid x_1) \sum_{x_1 \in A_{l_1}^N} G_{m_1}^N(x_1)$$

$$= G_{m_1}(A_{l_1}^N) \max_{x_1 \in A_{l_1}^N} G_{m_2/m_1}^N(A_{l_2/l_1}^N(x_1) \mid x_1) \stackrel{\Delta}{=} \beta_{l_1,l_2|m_1,m_2}^N(\Phi^N), (l_2, l_1) \neq (m_1, m_2). \tag{15}$$

Consequently we can deduce that "reliabilities"

$$F_{l_1,l_2|m_1,m_2}(\Phi) \stackrel{\Delta}{=} \overline{\lim_{N \to \infty}} \{-\frac{1}{N} \log \beta_{l_1,l_2|m_1,m_2}^N(\Phi^N)\}, (l_2, l_1) \neq (m_1, m_2),$$

$$l_1, m_1 = \overline{1, L_1}, \ l_2, m_2 = \overline{1, L_2},$$

and

$$F_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2) \neq (m_1,m_2)} F_{l_1,l_2|m_1,m_2}(\Phi) \tag{16}$$

are lower estimates for $E_{l_1,l_2|m_1,m_2}(\Phi)$.

We can also introduce

$$\beta^N_{l_2|l_1,m_1,m_2}(\varphi^N_2) \overset{\Delta}{=} \max_{x_1 \in A^N_{l_1}} G^N_{m_2/m_1}(A^N_{l_2/l_1}(x_1) \mid x_1),\ l_2 \neq m_2,\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2},\ m_2 \neq l_2,$$

We define also

$$\beta^N_{m_2|l_1,m_1,m_2}(\varphi^N_2) \overset{\Delta}{=} \max_{x_1 \in A^N_{l_1}} G^N_{m_2/m_1}(\overline{A^N_{m_2/l_1}(x_1)} \mid x_1) = \sum_{l_2 \neq m_2} \beta^N_{l_2|l_1,m_1,m_2}(\varphi^N_2). \tag{17}$$

The corresponding estimates of the reliabilities of test φ^N_2 , are the following

$$F_{l_2|l_1,m_1,m_2}(\varphi_2) \overset{\Delta}{=} \overline{\lim_{N \to \infty}}\{-\frac{1}{N} \log \beta^N_{l_2|l_1,m_1,m_2}(\varphi^N_2)\},\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2},\ m_2 \neq l_2. \tag{18}$$

It is clear from (17) that

$$F_{m_2|l_1,m_1,m_2}(\varphi_2) = \min_{l_2: l_2 \neq m_2} F_{l_2|l_1,m_1,m_2}(\varphi_2),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}. \tag{19}$$

We need some notions and estimates from the method of types [11], [12]. The type of a vector x_1 is a PD

$$Q_{x_1} = \{Q_{x_1}(x^1) = \frac{1}{N} N(x^1 \mid x_1), x^1 \in X\},$$

where $N(x^1 \mid x^1)$ is the number of repetitions of the symbol x^1 in vector x_1. The subset of $P(X)$ consisting of the possible types of sequences $x_1 \in X^N$ is denoted by $P_N(X)$. The set of all vectors x_1 of the type Q_{x_1} is denoted by $T^N_{Q_{x_1}}(X_1)$, remark that $T^N_Q(X_1) = \varnothing$ for $Q \notin P_N(X)$. The following estimates for the set $T^N_{Q_{x_1}}(X_1)$ hold

$$(N+1)^{-|X|}\exp\{NH_{Q_{x_1}}(X_1)\}\le|T^N_{Q_{x_1}}(X_1)|\le\exp\{NH_{Q_{x_1}}(X_1)\}.$$

For a pair of sequences $(x_1,x_2)\in X^N\times X^N$ let $N(x^1,x^2\,|\,x_1,x_2)$ be the number of occurrences of pair $(x^1,x^2)\in X\times X$ in the similar places in the pair of vectors (x_1,x_2). The joint type of the pair (x_1,x_2) is PD $Q_{x_1,x_2}=\{Q_{x_1,x_2}(x^1,x^2),x^1,x^2\in X\}$ defined by

$$Q_{x_1,x_2}(x^1,x^2)\stackrel{\Delta}{=}\frac{1}{N}N(x^1,x^2\,|\,x_1,x_2),\quad x^1,x^2\in X.$$

The conditional type of x_2 for given x_1 is the conditional distribution

$V_{x_1,x_2}\stackrel{\Delta}{=}\{V_{x_2,x_1}(x^2\,|\,x^1),x^1,x^2\in X\}$ defined as follows:

$$V_{x_1,x_2}(x^2\,|\,x^1)\stackrel{\Delta}{=}\frac{Q_{x_1,x_2}(x^1,x^2)}{Q_{x_1}(x^1)}=\frac{N(x^1,x^2\,|\,x_1,x_2)}{N(x^1\,|\,x_1)},\quad x^1,x^2\in X.$$

The conditional entropy of RV X_2 for given X_1 is:

$$H_{Q_{x_1},V_{x_1,x_2}}(X_2\,|\,X_1)=-\sum_{x^1,x^2}Q_{x_1}(x^1)V_{x_1,x_2}(x^2\,|\,x^1)\log V_{x_1,x_2}(x^2\,|\,x^1).$$

For some conditional PD $V=\{V(x^2\,|\,x^1),x^1,x^2\in X\}$ the conditional divergences of PD $\{Q(x^1)V(x^2\,|\,x^1),x^1,x^2\in X\}$ with respect to PD $\{Q(x^1)G_{l_2/l_1}(x^2\,|\,x^1),x^1,x^2\in X\}$ for all l_1,l_2 are defined as follows

$$D(V\,\|\,G_{l_2/l_1}\,|\,Q)\stackrel{\Delta}{=}\sum_{x^1,x^2}Q(x^1)V(x^2\,|\,x^1)\log\frac{V(x^2\,|\,x^1)}{G_{l_2/l_1}(x^2\,|\,x^1)},$$

also

$$D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q) \stackrel{\Delta}{=} \sum_{x^1,x^2} Q(x^1) G_{l_2/l_1}(x^2 \mid x^1) \log \frac{G_{l_2/l_1}(x^2 \mid x^1)}{G_{m_2/m_1}(x^2 \mid x^1)}.$$

The family of vectors x_2 of the conditional type V for given x_1 of the type Q_{x_1} is denoted by $T^N_{Q_{x_1},V}(X_2 \mid x_1)$ and called V-shell of x_1. The set of all possible V-shells for x_1 of type Q_{x_1} is denoted by $V_N(X, Q_{x_1})$. For any conditional type V and $x_1 \in T^N_{Q_{x_1}}(X_1)$ it is known that

$$(N+1)^{-|X|^2} \exp\{NH_{Q_{x_1},V}(X_2 \mid X_1)\} \le | T^N_{Q_{x_1},V}(X_2 \mid x_1) | \le \exp\{NH_{Q_{x_1},V}(X_2 \mid X_1)\}. \quad (20)$$

For given positive numbers $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$, for $Q \in R_{l_1}$ (5.a),(5.b) and for each pair $l_1, m_1 = \overline{1, L_1}$ let us define the following regions and values:

$$R_{l_2/l_1}(Q) \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1} \mid Q) \le F_{l_2|l_1,m_1,l_2}\},\ l_2 = \overline{1, L_2 - 1}, \quad (21a)$$

$$R_{L_2/l_1}(Q) \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1} \mid Q) > F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}\}, \quad (21b)$$

$$R^N_{l_2/l_1}(Q_{x_1}) = R_{l_2/l_1}(Q) \cap V_N(X, Q_{x_1})$$

$$F^*_{l_2|l_1,m_1,l_2} = F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,l_2}) \stackrel{\Delta}{=} F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}, \quad (22a)$$

$$F^*_{l_2|l_1,m_1,m_2} = F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V \in R_{l_2/l_1}(Q)} D(V \| G_{m_2/m_1} \mid Q),\ m_2 = \overline{1, L_2},\ m_2 \ne l_2,$$
$$l_2 = \overline{1, L_2 - 1}, \quad (22b)$$

$$F^*_{L_2|l_1,m_1,m_2} = F^*_{L_2|l_1,m_1,m_2}(F_{1|l_1,m_1,1}, E_{2|l_1,m_1,2}, \ldots, F_{L_2-1|l_1,m_1,L_2-1})$$
$$\stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V \in R_{L_2/l_1}(Q)} D(V \| G_{m_2/m_1} \mid Q),\ m_2 = \overline{1, L_2 - 1}, \quad (22c)$$

$$F^*_{L_2|l_1,m_1,L_2} = F^*_{L_2|l_1,m_1,L_2}(F_{1|l_1,m_1,1}, F_{2|l_1,m_1,2}, \ldots, F_{L_2-1|l_1,m_1,L_2-1}) \stackrel{\Delta}{=} \min_{l_2 = \overline{1, L_2 - 1}} F^*_{l_2|l_1,m_1,L_2}. \quad (22d)$$

We denote by $F(\varphi_2)$ the matrix of lower estimates for $E(\varphi_2)$.

Theorem 3: *If for fixed* $m_1, l_1 = \overline{1, L_1}$ *all conditional PDs* G_{l_2/l_1}, $l_2 = \overline{1, L_2}$, *are different in the sense that* $D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q) > 0$, *for all* $Q \in R_{l_1}$, $l_2 \neq m_2$, $m_2 = \overline{1, L_2}$, *when the numbers* $F_{1|l_1,m_1,1}, F_{2|l_1,m_1,2}, \ldots, F_{L_2-1|l_1,m_1,L_2-1}$ *are such that the following inequalities hold*

$$0 < F_{1|l_1,m_1,1} < \min_{l_2=\overline{2,L_2}} \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{1/m_1} \mid Q), \tag{23a}$$

$$0 < F_{m_2|l_1,m_1,m_2} < \min\Big(\min_{l_2=\overline{m_2+1,L_2}} \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} \mid Q), \min_{l_2=\overline{1,m_2-1}} F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2})\Big),$$
$$for\ m_2 = \overline{2, L_2-1}, \tag{23b}$$

then there exists a LAO sequence of tests φ_2^*, the matrix of lower estimate of which $F(\varphi_2^*)$ is defined in (22) with all elements of it strictly positive.

Inequalities (23) are necessary for existence of test sequence with matrix of lower estimates $F(\varphi_2^*)$ having in diagonal given elements $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2-1}$, and other elements positive.

Proof: For $x_1 \in X^N$, $x_2 \in T^N_{Q_{x_1},V}(X_2 \mid x_1)$ the conditional probability $G^N_{m_2/m_1}(x_2 \mid x_1)$ can be presented as follows

$$G^N_{m_2/m_1}(x_2 \mid x_1) = \prod_{n=1}^{N} G_{m_2/m_1}(x_n^2 \mid x_n^1)$$
$$= \prod_{x^1,x^2} G_{m_2/m_1}(x^2 \mid x^1)^{N(x^1,x^2|x_1,x_2)} = \prod_{x^1,x^2} G_{m_2/m_1}(x^2 \mid x^1)^{NQ_{x_1}(x^1)V(x^2|x^1)}$$
$$= \exp\{N \sum_{x^1,x^2} [-Q_{x_1}(x^1)V(x^2 \mid x^1) \log \frac{V(x^2 \mid x^1)}{G_{m_2/m_1}(x^2 \mid x^1)} + Q_{x_1}(x^1)V(x^2 \mid x^1) \log V(x^2 \mid x^1)]\}$$
$$= \exp\{-N[D(V \| G_{m_2/m_1} \mid Q_{x_1}) + H_{Q_{x_1},V}(X_2 \mid X_1)]\}. \tag{24}$$

We shall prove that the sequence of tests φ_2^*, defined for each $x_1 \in B^N_{l_1} = \bigcup_{Q \in R_{l_1}} T^N_Q(X_1)$ by the following collection of sets constructed of conditional types

$$B^{(N)}_{l_2/l_1}(x_1) = \bigcup_{V \in R^N_{l_2/l_1}(Q_{x_1})} T^N_{Q_{x_1},V}(X_2 \mid x_1), l_2 = \overline{1, L_2}, \tag{25}$$

is optimal with respect to lower estimates of corresponding reliabilities and the lower estimate matrix $F(\varphi_2^*)$ is defined in (22). First we show that each N -vector x_2 is in one and only one of $B^{(N)}_{l_2/l_1}(x_1)$, that is

$$B^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{m_2/l_1}(x_1) = \varnothing, l_2 = \overline{1, L_2 - 1},\ m_2 = \overline{l_2 + 1, L_2},\ and\ \bigcup_{l_2=1}^{L_2} B^{(N)}_{l_2/l_1}(x_1) = X^N.$$

Really, (21.b) and (25) show that

$$B^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{L_2/l_1}(x_1) = \varnothing,\ l_2 = \overline{1, L_2 - 1}.$$

For $l_2 = \overline{1, L_2 - 2}$, $m_2 = \overline{l_2 + 1, L_2 - 1}$, for each $x_1 \in B^N_{l_1}$ let us consider arbitrary $x_2 \in B^{(N)}_{l_2/l_1}(x_1)$. It follows from (17.a) and (21) that if $Q_{x_1} \in P_N(X)$ there are $V \in V_N(X, Q_{x_1})$ such that $D(V \| G_{l_2/l_1} \mid Q_{x_1}) \le F_{l_2|l_1,m_1,l_2}$ and $x_2 \in T^N_{Q_{x_1},V}(X_2 \mid x_1)$. From (21) -- (23) we have $F_{m_2|l_1,m_1,m_2} < F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}) < D(V \| G_{m_2/l_1} \mid Q_{x_1})$. From definition (25) for each m_1 we see that $x_2 \notin B^{(N)}_{m_2/m_1}(x_1)$, that is $x_2 \notin B^{(N)}_{m_2/l_1}(x_1)$.

Now for $m_2 = \overline{1, L_2 - 1}$, $l_1 \ne m_1$ using (17), (20), (21), (23) -- (25) we can upper estimate $\beta^{*N}_{m_2|l_1,m_1,m_2}$ as follows:

$$\beta^{*N}_{m_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}(\overline{B^{(N)}_{m_2/l_1}(x_1)} \mid x_1) \le \max_{x_1 \in B^N_{l_1}} G^N_{m_2/m_1}\left(\bigcup_{V: D(V \| G_{m_2/m_1} | Q_{x_1}) > E_{m_2|l_1,m_1,m_2}} T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1 \right)$$

$$\le (N+1)^{|X|^2} \max_{x_1 \in B_{l_1}^N} \sup_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} G_{m_2/m_1}(T_{Q_{x_1},V}^N(X_2\,|\,x_1)\,|\,x_1)$$

$$\le (N+1)^{|X|^2} \sup_{Q_{x_1}\in R_{l_1}^N} \sup_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} \exp\{-ND(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1})\}$$

$$\le \exp\{-N[\inf_{Q_{x_1}\in R_{l_1}^N} \inf_{V:D(V\|G_{m_2/m_1}|Q_{x_1})>E_{m_2|l_1,m_1,m_2}} D(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1}) - o_N(1)]\} \le \exp\{-N[F_{m_2|l_1,m_1,m_2} - o_N(1)]\}$$

For $l_2 \ne m_2$ we estimate by analogy

$$\beta_{l_2|l_1,m_1,m_2}^{*N} = \max_{x_1\in B_{l_1}^N} G_{m_2/m_1}^N(B_{l_2/l_1}^{(N)}(x_1)\,|\,x_1) = \max_{x_1\in B_{l_1}^N} G_{m_2/m_1}^N(\bigcup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} T_{Q_{x_1},V}^N(X_2\,|\,x_1)\,|\,x_1)$$

$$\le (N+1)^{|X|^2} \max_{x_1\in B_{l_1}^N} \sup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} G_{m_2/m_1}^N(T_{Q_{x_1},V}^N(X_2\,|\,x_1)\,|\,x_1)$$

$$\le (N+1)^{|X|^2} \sup_{Q_{x_1}\in R_{l_1}^N} \sup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} \exp\{-ND(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1})\} \tag{26}$$

$$\le \exp\{-N[\inf_{Q_{x_1}\in R_{l_1}^N} \inf_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} D(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1}) - o_N(1))]\}.$$

Now we want to deduce the lower estimate

$$\beta_{l_2|l_1,m_1,m_2}^{*N} = \max_{x_1\in B_{l_1}^N} G_{m_2/m_1}^N(B_{l_2/l_1}^{(N)}(x_1)\,|\,x_1) = \max_{x_1\in B_{l_1}^N} G_{m_2/m_1}^N(\bigcup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} T_{Q_{x_1},V}^N(X_2\,|\,x_1)\,|\,x_1)$$

$$\ge \max_{x_1\in B_{l_1}^N} \sup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} G_{m_2/l_1}^N(T_{Q_{x_1},V}(X_2\,|\,x_1)\,|\,x_1) \ge (N+1)^{-|X|^2} \sup_{Q_{x_1}\in R_{l_1}^N} \sup_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} \exp\{-ND(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1})\}. \tag{27}$$

$$\ge \exp\{-N[\inf_{Q_{x_1}\in R_{l_1}^N} \inf_{V:V\in R_{l_2/l_1}^N(Q_{x_1})} D(V\,\|\,G_{m_2/m_1}\,|\,Q_{x_1}) + o_N(1))]\}.$$

Taking into account (26), (27) and the continuity of the functional $D(V\,\|\,G_{m_2/l_1}\,|\,Q)$ we obtain that $\lim_{N\to\infty}\{-N^{-1}\log\beta_{l_2|l_1,m_1,m_2}^{*N}\}$ exists and in correspondence with (22.b) equals to $F_{l_2|l_1,m_1,m_2}^*$. Thus $F_{l_2|l_1,m_1,m_2}(\varphi_2^*) = F_{l_2|l_1,m_1,m_2}^*$, $m_2 = \overline{1,L_2}$, $l_2 = \overline{1,L_2}$.

The proof of the first part of the theorem will be accomplished if we show that the sequence of the tests φ_2^* for given $F_{1|l_1,m_1,1},\ldots,F_{L_2-1|l_1,m_1,L_2-1}$ and for any sequence of tests φ_2^{**} is such that for all $m_2, l_2 = \overline{1, L_2}$, $F^{**}_{l_2|l_1,m_1,m_2} \leq F^{*}_{l_2|l_1,m_1,m_2}$.

Consider sequence φ_2^{**} of tests, which is defined by the sets $D^{(N)}_{1/l_1}(x_1), D^{(N)}_{2/l_1}(x_1), \ldots, D^{(N)}_{L_2/l_1}(x_1)$ such that $F^{**}_{l_2|l_1,m_1,m_2} \geq F^{*}_{l_2|l_1,m_1,m_2}$ for some l_2, m_2. For a large enough N we can replace this condition by the following inequality

$$\beta^{**N}_{l_2|l_1,m_1,m_2} \leq \beta^{*N}_{l_2|l_1,m_1,m_2}. \tag{28}$$

Examine the sets $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{l_2/l_1}(x_1)$, $l_2 = \overline{1, L_2 - 1}$. This intersection cannot be empty, because in that case

$$\beta^{**N}_{l_2|l_1,m_1,l_2} = \max_{x_1 \in B^N_{l_1}} G^N_{l_2/l_1}(\overline{D}^{(N)}_{l_2/l_1}(x_1) \mid x_1) \geq \max_{x_1 \in B^N_{l_1}} G^N_{l_2/l_1}(B^{(N)}_{l_2/l_1}(x_1) \mid x_1)$$

$$\geq \max_{x_1 \in B^N_{l_1}} \sup_{V: D(V \| G_{l/l_1} | Q_{x_1}) \leq F_{l_2|l_1,m_1,l_2}} G^N_{l_2/l_1}(T^N_{Q_{x_1},V}(X_2 \mid x_1) \mid x_1) \geq \exp\{-N(F_{l_2|l_1,m_1,l_2} + o_N(1))\},$$

and we have a contradiction with (28). Let us show that $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{m_2/l_1}(x_1) = \varnothing, m_2, l_2 = \overline{1, L_2 - 1},$ $l_2 \neq m_2$. If there exists V such that $D(V \| G_{m_2/l_1} \mid Q) \leq F_{m_2|l_1,m_1,m_2}$ and $T^N_{V,Q_{x_1}}(X_2 \mid x_1) \in D^{(N)}_{l_2/l_1}(x_1)$, then

$$\beta^{**N}_{l_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(D^{(N)}_{l_2/l_1}(x_1) \mid x_1) > \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(T^N_{V,Q_{x_1}}(X_2 \mid x_1) \mid x_1) \geq \exp\{-N[F_{m_2|l_1,m_1,m_2} + o_N(1)]\}.$$

When $\varnothing \neq D^{(N)}_{l_2/l_1} \bigcap T^N_{V,Q_{x_1}}(X_2 \mid x_1) \neq T^N_{V,Q_{x_1}}(X_2 \mid x_1)$, we also obtain that

$$\beta^{**N}_{l_2|l_1,m_1,m_2} = \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(D^{(N)}_{l_2/l_1} | x_1) > \max_{x_1 \in B^N_{l_1}} G^N_{m_2/l_1}(D^{(N)}_{l_2/l_1} \bigcap T^N_{V,Q_{x_1}}(X_2 | x_1) | x_1) \geq \exp\{-N(F_{m_2|l_1,m_2,m_2} + O_N(1))\}.$$

Thus we conclude that $F^{**}_{l_2|l_1,m_1,m_2} < F_{m_2|l_1,m_1,m_2}$, which contradicts to (19). Hence we obtain that $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{l_2/l_1}(x_1) = B^{(N)}_{l_2/l_1}(x_1)$ for $l_2 = \overline{1, L_2 - 1}$.

The following intersection $D^{(N)}_{l_2/l_1}(x_1) \bigcap B^{(N)}_{L_2/l_1}(x_1)$ is empty too, because otherwise we arrive to

$$\beta^{**N}_{L_2|l_1,m_1,m_2} \geq \beta^{*N}_{L_2|l_1,m_1,m_2},$$

which contradicts to (28), it means that $D^{(N)}_{l_2/l_1}(x_1) = B^{(N)}_{l_2/l_1}(x_1)$, for all $l_2 = \overline{1, L_2}$.

The proof of the second part of the Theorem is simple. If one of the conditions (23) is violated, then from (21), (22) and (23) -- (26) it follows that at least one of the elements $F_{l_2|l_1,m_1,m_2}$ is equal to 0. For example, let $F_{m_2|l_1,m_1,m_2} \geq \min\limits_{\underline{l_2 = m_2+1, L_2}} \min\limits_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} | Q)$, then there is $l_2^{'} \in \overline{m_2 + 1, L_2}$ such that $F_{m_2|l_1,m_1,m_2} \geq \min\limits_{Q \in R_{l_1}} D(G_{l_2^{'}/l_1} \| G_{m_2/m_1} | Q)$. After using (22b) we obtain that $F^{*}_{m_2|m_1,l_1,l_2^{'}} = 0$. From (19) we see that $F_{m_2|l_1,m_1,m_2} \leq \min\limits_{\underline{l_2 = 1, m_2 - 1}} F^{*}_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2})$. Theorem is proved.

Corollary 2: *If in contradiction to conditions (23) one, or several diagonal elements* $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$, *of the reliability matrix are equal to zero, then the elements of the matrix determined in functions of this* $F_{l_2|l_1,m_1,l_2}$ *are given as in the case of Stein's lemma [11], [12]*

$$F_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,l_2}) = \inf_{Q \in R_{l_1}} D(G_{l_2/l_1} \| G_{m_2/m_1} | Q), \ m_1 = \overline{1, L_1}, \ m_1 \neq l_1,$$

and the remaining elements of the matrix $F(\varphi_2^*)$ are defined in function of positive $F_{l_2|l_1,m_1,l_2} > 0$, $l_1 \neq m_1$, $l_1 = \overline{1, L_1 - 1}$, as follows from Theorem 3.

Proof: Really, if $F_{l_2|l_1,m_1,l_2}=0$, then $\beta^N_{l_2|l_1,m_1,l_2}$ is not exponentially decreasing . Thus using Stein's lemma we have

$$\lim_{N\to\infty}\log\frac{1}{N}\beta^{(N)}_{l_2|l_1,m_1,m_2}(\beta^{(N)}_{l_2|l_1,m_1,l_2}(\varphi_2)=c)=-\inf_{Q\in R_{l_1}} D(G_{l_2/l_1}\,\|\,G_{m_2/m_1}\,|\,Q),\ l_2\neq m_2.$$

So the corollary is proved.

4. On Identification of the Probability Distribution of the Dependent Object

In this section we will obtain the lower estimates of the reliabilities of LAO identification for dependent object. Then we deduce the lower estimates of the reliabilities for LAO identification of two related objects.

There exist two error probabilities for each $r_2=\overline{1,L_2}$: the probability $\alpha_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi_N)$ to accept l_2 different from r_2, when r_2 is in reality, and the probability $\alpha_{l_2=r_2|l_1,m_1,m_2\neq r_2}(\varphi_N)$ to accept r_2, when it is not correct.

The upper estimate $\beta_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi_2^N)$ of $\alpha_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi_2^N)$ is already known, it coincides with the $\beta_{r_2|l_1,m_1,r_2}(\varphi_2^N)$ which is equal to $\sum\limits_{l_2:l_2\neq r_2}\beta_{l_2|l_1,m_1,r_2}(\varphi_2^N)$. The corresponding reliability $F_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi_2)$ is equal to $F_{r_2|l_1,m_1,r_2}(\varphi_2)$ which satisfies the equality (19).

The reliability approach to identification of lower estimates assumes determining the optimal dependence of $F^*_{l_2=r_2|l_1,m_1,m_2\neq r_2}$ upon given $F^*_{l_2\neq r_2|l_1,m_1,m_2=r_2}=F^*_{r_2|l_1,m_1,r_2}$, which can be an assigned values satisfying conditions (23).

Theorem 4: *In case of distinct PDs $G_{1|l_1},G_{2|l_1},\ldots,G_{L_2|l_1}$, for every l_1 under condition that the upper estimates of probabilities of all L_2 hypotheses are strictly positive the ``reliability" $F_{l_2=r_2|l_1,m_1,m_2\neq r_2}$ for given $F_{l_2\neq r_2|l_1,m_1,m_2=r_2}=F_{r_2|l_1,m_1,r_2}$ is the following*:

$$F_{l_2=r_2|l_1,m_1,m_2\neq r_2}(F_{r_2|l_1,m_1,r_2})=\min_{m_2:m_2\neq r_2}\inf_{Q\in R_{l_1}}\inf_{V:D(V\|G_{r_2/l_1}|Q)\leq F_{r_2|l_1,m_1,r_2}} D(V\,\|\,G_{m_2|m_1}\,|\,Q),\ r_2=\overline{1,L_2}.$$

Proof: We have

$$\beta^N_{l_2=r_2|l_1,m_1,m_2\neq r_2}=\frac{Pr^N(m_2\neq r_2, l_2=r_2/l_1,m_1)}{Pr(m_2\neq r_2/m_1)}=\frac{\sum\limits_{m_2:m_2\neq r_2}\beta_{r_2|l_1,m_1,m_2}Pr(m_2/m_1)}{\sum\limits_{m_2\neq r_2}Pr(m_2/m_1)}.$$

Consequently, we obtain that

$$F_{l_2=r_2|l_1,m_1,m_2\neq r_2}(F_{r_2|l_1,m_1,r_2})\overset{\Delta}{=}\overline{\lim_{N\to\infty}}-\frac{1}{N}\log\beta^N_{l_2=r_2|l_1,m_1,m_2\neq r_2}$$
$$=\overline{\lim_{N\to\infty}}-\frac{1}{N}(\log\sum_{m_2:m_2\neq r_2}\beta_{r_2|l_1,m_1,m_2}Pr(m_2/m_1)-\log\sum_{m_2\neq r_2}Pr(m_2/m_1))$$

$$=\overline{\lim_{N\to\infty}}-\frac{1}{N}(\log\max\beta_{r_2|l_1,m_1,m_2}+\log\sum_{m_2:m_2\neq r_2}\frac{\beta_{r_2|l_1,m_1,m_2}Pr(m_2/m_1)}{\max\beta_{r_2|l_1,m_1,m_2}}-\log\sum_{m_2\neq r_2}Pr(m_2/m_1))=\min_{m_2:m_2\neq r_2}F_{r_2|l_1,m_1,m_2}.$$

And using $(22.b)$ we prove the theorem.

5. LAO Hypotheses Testing for Two Stochastically Dependent Objects

In this section we find the ``reliabilities" $F_{l_1,l_2|m_1,m_2}$ for LAO testing which will be lower bounds for corresponding $E_{l_1,l_2|m_1,m_2}$. Using (15) we can prove the following lemma

Lemma: If the elements $E_{l_1|m_1}(\varphi_1)$ *and* $F_{l_2|l_1,m_1,m_2}(\varphi_2)$ are positive, then

$$F_{l_1,l_2|m_1,m_2}(\Phi) = E_{l_1|m_1}(\varphi_1) + F_{l_2|l_1,m_1,m_2}(\varphi_2),\ m_1 \neq l_1,\ m_2 \neq l_2, \quad (29a)$$

$$F_{l_1,l_2|m_1,m_2}(\Phi) = E_{l_1|m_1}(\varphi_1),\ m_1 \neq l_1,\ m_2 = l_2, \quad (29b)$$

$$F_{l_1,l_2|m_1,m_2}(\Phi) = F_{l_2|l_1,m_1,m_2}(\varphi_2),\ \ m_1 = l_1,\ \ m_2 \neq l_2. \quad (29c)$$

Proof: The following relations hold for upper estimates of error probabilities

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N) = \alpha^N_{l_1|m_1}(\varphi_1^N)\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N),\ m_1 \neq l_1,\ \ m_2 \neq l_2, \quad (30a)$$

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N) = \alpha^N_{l_1|m_1}(\varphi_1^N)(1-\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N)),\ m_1 \neq l_1,\ m_2 = l_2, \quad (30b)$$

$$\beta^N_{l_1,l_2|m_1,m_2}(\Phi^N) = (1-\alpha^N_{l_1|m_1}(\varphi_1^N))\beta^N_{l_2|l_1,m_1,m_2}(\varphi_2^N),\ \ m_1 = l_1,\ m_2 \neq l_2. \quad (30c)$$

Thus, in light of (3) and (18), we obtain (29). The lemma is proved.

Let us define the following subsets of $P(X)$ for given strictly positive elements

$E_{L_1,l_2|l_1,l_2}$, $F_{l_1,L_2|l_1,l_2}$, $l_1 = \overline{1, L_1-1},\ l_2 = \overline{1, L_2-1}$:

$$R_{l_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) \leq E_{L_1,l_2|l_1,l_2}\},\ \ l_1 = \overline{1, L_1-1},\ l_2 = \overline{1, L_2-1},$$

$$R_{l_2/l_1}(Q) \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1} | Q) \leq F_{l_1,L_2|l_1,l_2}\},\ \ l_1 = \overline{1, L_1-1},\ l_2 = \overline{1, L_2-1},$$

$$R_{L_1} \overset{\Delta}{=} \{Q : D(Q \| G_{l_1}) > E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1-1},\ l_2 = \overline{1, L_2-1}\},$$

$$R_{L_2/l_1}(Q) \overset{\Delta}{=} \{V : D(V \| G_{l_2/l_1} | Q) > F_{l_1,L_2|l_1,l_2},\ l_1 = \overline{1, L_1-1},\ l_2 = \overline{1, L_2-1}\}.$$

Assume also

$$F^*_{l_1,L_2|l_1,l_2} \stackrel{\Delta}{=} F_{l_1,L_2|l_1,l_2}, \; E^*_{L_1,l_2|l_1,l_2} \stackrel{\Delta}{=} E_{L_1,l_2|l_1,l_2}, \; l_1 = \overline{1,L_1-1}, \;\; l_2 = \overline{1,L_2-1}, \tag{31a}$$

$$E^*_{l_1,l_2|m_1,l_2} \stackrel{\Delta}{=} \inf_{Q:Q\in R_{l_1}} D(Q\,\|\,G_{m_1}), \;\; m_1 \neq l_1 \tag{31b}$$

$$F^*_{l_1,l_2|l_1,m_2} \stackrel{\Delta}{=} \inf_{Q\in R_{l_1}} \inf_{V:V\in R_{l_2/l_1}(Q)} D(V\,\|\,G_{m_2/m_1}\,|\,Q), \; m_2 \neq l_2 \tag{31c}$$

$$F^*_{l_1,l_2|m_1,m_2} \stackrel{\Delta}{=} F^*_{m_1,l_2|m_1,m_2} + E^*_{l_1,m_2|m_1,m_2}, \;\; m_i \neq l_i, \; i=1,2, \tag{31d}$$

$$F^*_{m_1,m_2|m_1,m_2} \stackrel{\Delta}{=} \min_{(l_1,l_2)\neq(m_1,m_2)} F^*_{l_1,l_2|m_1,m_2}. \tag{31e}$$

Theorem 5: *If all distributions* G_{m_1}, $m_1 = \overline{1,L_1}$, *are different, that is* $D(G_{l_1}\,\|\,G_{m_1}) > 0$, $l_1 \neq m_1$, $l_1, m_1 = \overline{1,L_1}$, *and all conditional distributions* G_{l_2/l_1}, $l_2 = \overline{1,L_2}$, *are also different for all* $l_1 = \overline{1,L_1}$, *in the sense that* $D(G_{l_2/l_1}\,\|\,G_{m_2/l_1}\,|\,Q) > 0$, $l_2 \neq m_2$, *then the following statements are valid.*

When given elements $E_{L_1,l_2|l_1,l_2}$ and $F_{l_1,L_2|l_1,l_2}$, $l_1 = \overline{1,L_1-1}, \; l_2 = \overline{1,L_2-1}$, meet the following conditions

$$0 < E_{L_1,l_2|1,l_2} < \min_{l_1=2,L_1} D(G_{l_1}\,\|\,G_1), \tag{32a}$$

$$0 < F_{l_1,L_2|l_1,1} < \min_{l_2=2,L_2} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1}\,\|\,G_{1/m_1}\,|\,Q), \tag{32b}$$

$$0 < E_{L_1,l_2|l_1,l_2} < \min[\min_{l_1=1,m_1-1} E^*_{l_1,l_2|m_1,l_2}, \min_{l_1=m_1+1,L_1} D(G_{l_1}\,\|\,G_{m_1})], \; l_1 = \overline{2,L_1-1}, \tag{32c}$$

$$0 < F_{l_1,L_2|l_1,l_2} < \min[\min_{l_2=1,m_2-1} F^*_{l_1,l_2|l_1,m_2}, \min_{l_2=m_2+1,L_2} \inf_{Q\in R_{l_1}} D(G_{l_2/l_1}\,\|\,G_{m_2/m_1}\,|\,Q)], \; l_2 = \overline{2,L_2-1}, \tag{32d}$$

then there exists a LAO test sequence Φ^*, the lower estimate matrix of which $F(\Phi^*) = \{F_{l_1,l_2|m_1,m_2}(\Phi^*)\}$ is defined in (31) and all elements of it are positive.

When even one of the inequalities (32) is violated, then at least one element of the lower estimate matrix $F(\Phi^*)$ is equal to 0.

Proof: It is proved in [7] that $E_{l_1|l_1} = E_{L_1|l_1}$, $l_1 = \overline{1, L_1 - 1}$. By analogy we can deduce that

$$F_{l_2|l_1,m_1,l_2} = F_{L_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}. \tag{33}$$

Applying the theorem of Kuhn-Tucker in (22.b) we can show that the elements $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, L_2 - 1}$ can be determined by elements $F_{l_2|l_1,m_1,m_2}$, $m_2 \neq l_2$, $l_2 = \overline{1, L_2}$,

$$F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,m_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1} | Q) \le F_{l_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q).$$

From (23) it is clear that $F_{m_2|l_1,m_1,m_2}$ can be equal only to one of $F_{l_2|l_1,m_1,m_2}$, $l_2 = \overline{m_2 + 1, L_2}$. Assume that (33) is not correct, that is $F_{m_2|l_1,m_1,m_2} = F_{l_2|l_1,m_1,m_2}$, $l_2 = \overline{m_2 + 1, L_2 - 1}$.

From (22.b) it follows that

$$F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,m_2}) \stackrel{\Delta}{=} \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1} | Q) \le F_{l_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q)$$

$$= \inf_{Q \in R_{l_1}} \inf_{V: D(V \| G_{m_2/l_1} | Q) \le F_{m_2|l_1,m_1,m_2}} D(V \| G_{l_2/l_1} | Q) = F_{m_2|l_1,m_1,l_2},\ m_2, l_2 = \overline{1, L_2 - 1},\ m_2 < l_2,$$

but from conditions (23) it follows that $F_{l_2|l_1,m_1,l_2} < F_{m_2|l_1,m_1,l_2}$ for $m_2 = \overline{1, l_2 - 1}$. Our assumption is not true, thus (33) is valid.

Hence we can rewrite the inequalities (7) and (23) as follows:

$$0 < E_{L_1|1} < \min_{l_1=\overline{2,L_1}} D(G_{m_1} \| G_1), \tag{34a}$$

$$0 < F_{L_2|l_1,m_1,1} < \inf_{Q\in R_{l_1}} \min_{l_2=\overline{2,L_2}} D(G_{m_2/l_1} \| G_{1/l_1} \mid Q), \tag{34b}$$

$$0 < E_{L_1|l_1} < \min[\min_{l_1=\overline{1,l_1-1}} E^*_{l_1|m_1}, \min_{l_1=\overline{l_1+1,L_1}} D(G_{m_1} \| G_{l_1})], \quad l_1 = \overline{2,L_1-1}, \tag{34c}$$

$$0 < F_{L_2|l_1,m_1,l_2} < \min[\min_{l_2=\overline{1,l_2-1}} F^*_{l_2|l_1,m_1,m_2}, \inf_{Q\in R_{l_1}} \min_{l_2=\overline{l_2+1,L_2}} D(G_{l_2/l_1} \| G_{m_2/l_1} \mid Q)], \; l_2 = \overline{2,L_2-1}. \tag{34d}$$

According to Theorem 1 and Theorem 2 there exist LAO sequences of tests φ_1^* and φ_2^*, for the first and second objects, such that the elements of the matrices $E(\varphi_1^*)$ are determined in (6) and the lower estimate matrix $F(\varphi_2^*)$ is determined in (22). The inequalities (34.a), (34.c) are equivalent to the inequalities (7) and (34.b), (34.d) are equivalent to the inequalities (23). Then using Lemma we deduce that the lower estimate matrix $F(\Phi^*)$ is determined in (31). The proof of the second assertion of the Theorem is obvious.

6 . On Identification of the Probability Distributions of Two Stochastically Dependent Objects

In this section we study an approach to deducing optimal interdependencies of lower estimates of corresponding reliabilities for LAO identification. The LAO test Φ^* is the compound test consisting of the pair of LAO tests φ_1^* and φ_2^* for respective separate objects, and for it the equalities (29) take place. The statistician has to answer to the question whether the pair of distributions (r_1, r_2) occurred or not. Let us consider two types of error probabilities for each pair (r_1, r_2), $r_1 = \overline{1, L_1}, r_2 = \overline{1, L_2}$. We denote by $\alpha^N_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)}$ the probability, that pair (r_1, r_2) is true, but it is rejected. Note that this probability is equal to $\alpha_{r_1,r_2|r_1,r_2}(\Phi^N)$. Let $\alpha^N_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ be the probability that (r_1, r_2) is accepted, when it is not correct. The corresponding reliabilities are $E_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)} = E_{r_1,r_2|r_1,r_2}$ and $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$. Our aim is to determine the dependence of $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ on given $E_{r_1,r_2|r_1,r_2}(\Phi^N)$.

Now let us suppose that hypotheses $G_1, G_2, ..., G_{L_1}$ have a priori positive probabilities Pr (r_1), $r_1 = \overline{1, L_1}$ and

$G_{1|l_1}, G_{2|l_1}, ..., G_{L_2|l_1}$ have a priori positive conditional probabilities Pr $(r_2 \mid l_1)$, $r_2 = \overline{1, L_2}$, and consider the probability, which we are interested

$$\beta^N_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \frac{Pr^N((m_1,m_2)\neq(r_1,r_2),(l_1,l_2)=(r_1,r_2))}{Pr((m_1,m_2)\neq(r_1,r_2))} = \frac{\sum\limits_{(m_1,m_2):(m_1,m_2)\neq(r_1,r_2)} \beta_{(r_1,r_2)|(m_1,m_2)} Pr((m_1,m_2))}{\sum\limits_{(m_1,m_2)\neq(r_1,r_2)} Pr(m_1,m_2)}$$

Consequently, we obtain that

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \min_{(m_1,m_2):(m_1,m_2)\neq(r_1,r_2)} F_{r_1,r_2|m_1,m_2}. \tag{35}$$

For every LAO test Φ^* from (11), (29) and (35) we obtain that

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)} = \min_{m_1\neq r_1, m_2\neq r_2} \left(E_{r_1|m_1}(E_{r_1|r_1}), F_{r_2|l_1,m_1,m_2}(F_{r_2|l_1,m_1,r_2})\right), \tag{36}$$

where $E_{r_1|m_1}(E_{r_1|r_1}), F_{r_2|l_1,m_1,m_2}(F_{r_2|l_1,m_1,r_2})$ are determined by (6) and (22) for, correspondingly, the first and the second objects. For every LAO test Φ^* from (16) and (29) we deduce that

$$F_{r_1,r_2|r_1,r_2} = \min_{m_1\neq r_1, m_2\neq r_2} \left(E_{r_1|m_1}, F_{r_2|l_1,m_1,m_2}\right) = \min\left(E_{r_1|r_1}, F_{r_2|l_1,m_1,r_2}\right), \tag{37}$$

and each of $E_{r_1|r_1}, E_{r_2|l_1,m_1,r_2}$ satisfies the following conditions:

$$0 < E_{r_1|r_1} < \min\left[\min_{l_1=\overline{1,r_1-1}} E^*_{l_1|m_1}(E_{l_1|l_1}), \min_{l_1=\overline{r_1+1,L_1}} D(G_{l_1} \| G_{r_1})\right], \tag{38a}$$

$$0 < F_{r_2|l_1,m_1,r_2} < \min\left[\min_{l_2=\overline{1,r_2-1}} F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}), \inf_{Q\in R_{l_1}} \min_{l_2=\overline{r_2+1,L_2}} D(G_{l_2|l_1} \| G_{r_2|m_1} \mid Q)\right]. \tag{38b}$$

From (6.b) and (22.b) we see that the elements $E^*_{l_1|m_1}(E_{l_1|l_1}),\ l_1=\overline{1,r_1-1}$ and $E^*_{l_2|l_1,m_1,m_2}(E_{l_2|l_1,m_1,l_2}),$ $l_2=\overline{1,r_2-1}$ are determined only by $E_{l_1|l_1}$ and $F_{l_2|l_1,m_1,l_2}$. But we are considering only elements $E_{r_1|r_1}$ and $F_{r_2|l_1,m_1,r_2}$. We can use Corollary 1, Corollary 2 and upper estimates (38.a), (38.b) as follows:

$$0 < E_{r_1|r_1} < \min\left[\min_{l=\overline{1,r_1-1}} D(G_{r_1} \| G_{l_1}),\ \min_{l_1=\overline{r_1+1,L_1}} D(G_{l_1} \| G_{r_1})\right], \tag{39a}$$

$$0 < F_{r_2|l_1,m_1,r_2} < \min\left[\inf_{Q\in R_{l_1}} \min_{l_2=\overline{1,r_2-1}} D(G_{r_2|m_1} \| G_{l_2|l_1} \mid Q),\ \inf_{Q\in R_{l_1}} \min_{l_2=\overline{r_2+1,L_2}} D(G_{l_2|l_1} \| G_{r_2|m_1} \mid Q)\right]. \tag{39b}$$

From (37) we have that $F_{r_1,r_2|r_1,r_2} = E_{r_1|r_1}$, when $E_{r_1|r_1} \le F_{r_2|l_1,m_1,r_2}$, and when

$F_{r_1,r_2|r_1,r_2} = F_{r_2|l_1,m_1,r_2}$, then $F_{r_2|l_1,m_1,r_2} \le E_{r_1|r_1}$. Hence, it can be implied that given strictly positive element $F_{r_1,r_2|r_1,r_2}$ must meet both inequalities (39.a) and (39.b).

Using (37) we can determine reliability $F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ in function of $F_{r_1,r_2|r_1,r_2}$ as follows:

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}\left(F_{r_1,r_2|r_1,r_2}\right) = \min_{m_1\neq r_1,m_2\neq r_2}\left[E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2}), F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})\right] \tag{40}$$

where $E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2})$ and $F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})$ are determined respectively by (6.b) and by (22.b). Finally we obtained

Theorem 6: *If the distributions G_{m_1}, and $G_{m_2|m_1}$, $m_1=\overline{1,L_1}$, $m_2=\overline{1,L_2}$ are different and the given strictly positive number $F_{r_1,r_2|r_1,r_2}$ satisfies condition (39.a) or (39.b), then the lower estimate $F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ of $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ can be calculated by (40).*

In the particular case, when X_1 and X_2 are related statistically [8], [9] that is the second object depending on PD of the first is characterized by RV X_2 which can have one of $L_1 \times L_2$ conditional PDs $G_{l_2/l_1} = \{G_{l_2/l_1}(x^2), x^2 \in X\},\ l_1=\overline{1,L_1},\ l_2=\overline{1,L_2}$, we will have $A^N_{l_2/l_1} = \{x_2 : \varphi^N_2(x_2,l_1)=l_2\},$ $l_1=\overline{1,L_1},\ l_2=\overline{1,L_2}$, in place of the set $A^N_{l_2/l_1}(x_1)$ and in that case from [8] we have

$$G^N_{m_1,m_2}(A^N_{l_1,l_2}) = \sum_{(x_1,x_2)\in A^N_{l_1,l_2}} G^N_{m_1}(x_1)G^N_{m_2/m_1}(x_2) = \sum_{x_1\in A^N_{l_1}} G^N_{m_1}(x_1) \sum_{x_2\in A^N_{l_2/l_1}} G^N_{m_2/m_1}(x_2)$$

$$= G^N_{m_2/m_1}(A^N_{l_2/l_1})G_{m_1}(A^N_{l_1}), \quad (l_1,l_2)\neq(m_1,m_2).$$

The probabilities of the erroneous acceptance of PD $G_{l_/l_1}$ provided that G_{m_2/m_1} is true, $l_1, m_1 = \overline{1, L_1}$, are denoted by

$$\alpha^N_{l_2|l_1,m_1,m_2}(\varphi^N_2) = G^N_{m_2/m_1}(A^N_{l_2/l_1}),\ l_2 \neq m_2.$$

The probability to reject G_{m_2/m_1} , when it is true is denoted as follows

$$\alpha^N_{m_2|l_1,m_1,m_2}(\varphi^N_2) = G^N_{m_2/m_1}(\overline{A_{m_2/l_1}}) \stackrel{\Delta}{=} \sum_{l_2\neq m_2} \alpha^N_{l_2|l_1,m_1,m_2}(\varphi^N_2).$$

Thus in the conditions and in the results of Theorems 3-6, instead of conditional divergences

$\inf\limits_{Q\in R_{l_1}} D(G_{l_2/l_1} \,||\, G_{m_2/m_1} \,|\, Q)$, $\inf\limits_{Q\in R_{l_1}} D(V \,||\, G_{m_2/m_1} \,|\, Q)$ we will have just divergences

$D(G_{l_2/l_1} \,||\, G_{m_2/m_1}), D(V \,||\, G_{m_2/m_1})$ and in place of $F_{l_2|l_1,m_1,m_2}(\Phi), F_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},$

$l_2, m_2 = \overline{1, L_2}$, will be $E_{l_2|l_1,m_1,m_2}(\Phi), E_{l_1,l_2|m_1,m_2}(\Phi),\ l_1, m_1 = \overline{1, L_1},\ l_2, m_2 = \overline{1, L_2}$.

And in that case regions defined in (21) will be changed as follows:

$$R_{l_2/l_1} \stackrel{\Delta}{=} \{V : D(V \,||\, G_{l_2/l_1}) \leq E_{l_2|l_1,m_1,l_2}\},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{L_2/l_1} \stackrel{\Delta}{=} \{V : D(V \,||\, G_{l_2/l_1} \,|\, Q) > E_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, L_2 - 1}\},$$

$$R^N_{l_2/l_1} = R_{l_2/l_1} \cap P_N(X).$$

In case of two statistically dependent objects the corresponding regions will be

$$R_{l_1} \stackrel{\Delta}{=} \{Q : D(Q \| G_{l_1}) \le E_{L_1,l_2|l_1,l_2}\}, l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1},$$

$$R_{l_2/l_1} \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1}) \le E_{l_1,L_2|l_1,l_2}\}, l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1},$$

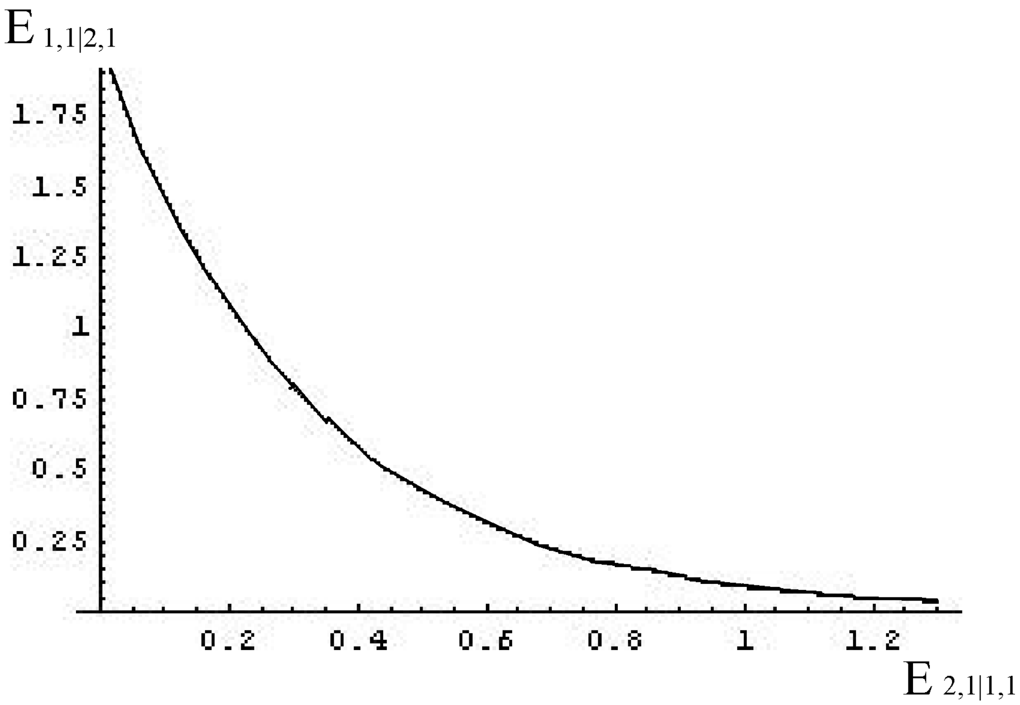
$$R_{L_1} \stackrel{\Delta}{=} \{Q : D(Q \| G_{l_1}) > E_{L_1,l_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}\},$$

$$R_{L_2/l_1} \stackrel{\Delta}{=} \{V : D(V \| G_{l_2/l_1}) > E_{l_1,L_2|l_1,l_2},\ l_1 = \overline{1, L_1 - 1},\ l_2 = \overline{1, L_2 - 1}\}.$$

So in this case we obtain the optimal interdependencies of reliabilities. The results were shown in [8] and in [9]. For this model in next section will present some results of calculations.

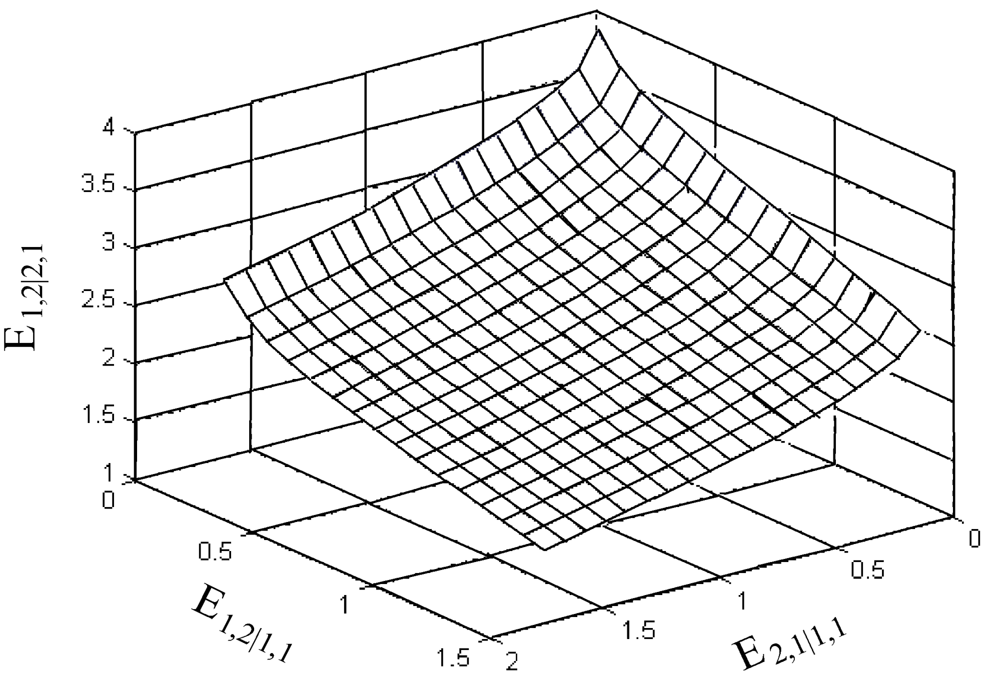
7. Example

. Let us consider the set of two elements $X = \{0,1\}$ and the following probability distributions given on X: $G_1 = \{0.84;0.16\}$, $G_2 = \{0.23;0.77\}$, $G_{1/1} = \{0.78;0.22\}$, $G_{2/1} = \{0.21;0.79\}$, $G_{1/2} = \{0.59;0.41\}$ $G_{2/2} = \{0.32;0.68\}$. In Fig.1 and Fig.2 the results of calculations of functions $E_{1,1|2,1}(E_{2,1|1,1})$ and $E_{1,2|2,1}(E_{2,1|1,1}, E_{1,2|1,1})$ are presented. For these distributions we have $D(G_2 \| G_1) \approx 1.3$ and $D(G_{2/1} \| G_{1/1}) \approx 1.06.$ We see in Fig.1 that when an analog of the inequality (32.a) of Theorem 5 (for statistically dependent objects) is violated then $E_{1,1|2,1} = 0$ and in Fig.2 we see that when analogs of (32.a) and (32.b) equalities are violated then $E_{1,2|2,1} = 0$.



E 1,1|2,1
1.75
1.5
1.25
1
0.75
0.5
0.25
0.2
0.4
0.6
0.8
1
1.2
E 2,1|1,1

Fig. 1



4
3.5
3
2.5
2
1.5
1
0
E1,2|2,1
0.5
1
1.5
E1,2|1,1
0
0.5
1
1.5
2
E2,1|1,1

Fig. 2

8. Conclusion

We studied the more general model of stochastically dependence of two discrete random variables. For this model reliability requirements to multiple hypotheses testing and identification are investigated. By the first approach optimal interdependencies of elements of reliability matrix of test Φ can be found when its L_1L_2-1 diagonal elements are given. But by this approach we do not have information about the reliabilities of the first and the second objects. By the second approach at first we find optimal interdependencies of reliabilities of the first object and then interdependencies of lower estimates of reliabilities of the second object. Similarly we also solve the identification problem for two objects. Results of the second approach are applied to finding the optimal interdependencies of lower estimates of reliabilities of two objects when L_1L_2-1 non diagonal elements of lower estimate matrix are given. If random variables X_1 and X_2 take values in different sets $\mathcal{X}_1$ and $\mathcal{X}_2$ only the notations become more complicated, so we omit this "generalization". The correspondence with other, less general, cases of objects relation is discussed in [5] -- [10].

Bibliography

[1] W. Hoeffding, "Asymptotically optimal tests for multinomial distributions", *The Annals of Mathematical Statistics*, vol. 36, pp. 369--401, 1965.

[2] I. Csiszár and G. Longo, "On the error exponent for source coding and for testing simple statistical hypotheses", *Studia Scientiarum Math. Hungarica*, vol. 6, pp. 181--191, 1971.

[3] E. A. Haroutunian, "Logarithmically asymptotically optimal testing of multiple statistical hypotheses" *Problems of Control and Information Theory*, vol. 19(5-6), pp. 413--421, 1990.

[4] R. F. Ahlswede and E. A. Haroutunian, "On logarithmically asymptotically optimal testing of hypotheses and identification". *Lecture Notes in Computer Science, vol. 4123, "General Theory of Information Transfer and Combinatorics"*, pp. 462-- 478, Springer, 2006.

[5] E. A. Haroutunian and P. M. Hakobyan, "Multiple hypotheses LAO testing for many independent object", *International journal ``Scholarly Research Exchange"*, vol. 2009, pp. 1-6, 2009.

[6] E. A. Haroutunian and P. M. Hakobyan, "On identification of distributions of two independent objects", *Mathematical Problems of Computer Science*, vol. XXVIII, pp. 114--119, 2007.

[7] E. A. Haroutunian and A. O. Yessayan, "On hypotheses testing for two differently distributed objects". *Mathematical Problems of Computer Science*, vol. XXVI, pp. 91--96, 2006.

[8] E. A. Haroutunian and A. O. Yessayan, "On logarithmically asymptotically optimal hypothesis testing for pair

of statistically dependent objects”, *Mathematical Problems of Computer Science*, vol. XXIX, pp. 97--103, 2007.

[9] A. O. Yessayan, “On reliability approach to identification of probability distributions of two statistically dependent objects”, *Mathematical Problems of Computer Science*, vol. XXXII, pp. 65--69, 2009.

[10] E. A. Haroutunian and A. O. Yessayan, “On optimal hypotheses testing for pair of stochastically coupled objects”, *Mathematical Problems of Computer Science*, vol. XXXI, pp. 49--59, 2008.

[11] E. A. Haroutunian, M. E. Haroutunian, and A. N. Harutyunyan, “Reliability criteria in information theory and in statistical hypotheses testing”, *Foundations and Trends in Communications and Information Theory*, vol. 4, no. 2-3, 2008.

[12] I. Csiszár and J. Körner, *Information theory: Coding Theorems for Discrete Memoryless Systems.* New York: Academic Press, 1981.

Authors' Information

Evgueni Haroutunian – *Head of Laboratory for Information Theory and Applied Statistics, Institute for Informatics and Automation Problems, Armenian National Academy of Sciences;*
e-mail: evhar@ipia.sci.am

Major Fields of Scientific Research: Information theory, Statistics

Aram Yessayan – *Scientific Collaborator of Laboratory for Information Theory and Applied Statistics, Institute for Informatics and Automation Problems, Armenian National Academy of Sciences;*
e-mail: armfrance@yahoo.fr

Major Fields of Scientific Research: Information theory, Statistics

Parandzem Hakobyan – *Scientific Collaborator of Laboratory for Information Theory and Applied Statistics, Institute for Informatics and Automation Problems, Armenian National Academy of Sciences;*
e-mail: par_h@ipia.sci.am

Major Fields of Scientific Research: Information theory, Statistics

PROOF COMPLEXITIES OF SOME PROPOSITIONAL FORMULAE CLASSES IN DIFFERENT REFUTATION SYSTEMS[1]

Ashot Abajyan, Anahit Chubaryan

Abstract: *In this paper the proof complexities of some classes of quasi-hard determinable ($Tsgf_n$) and hard determinable (ψ_n) formulas are investigated in some refutation propositional systems. It is proved that 1) the number of proof steps of $Tsgf_n$ in $R(lin)$ (Resolution over linear equations) and $GCNF'+$ permutation (cut-free Gentzen type with permutation) systems are bounded by p($\log_2|Tsgf_n|$) for some polynomial p(), 2) the formulas ψ_n require exponential size proofs in $GCNF'+$ permutation.*

It is also shown that Frege systems polynomially simulate $GCNF'+$ permutation and any Frege system has exponential speed-up over the $GCNF'+$ permutation.

Keywords: *determinative conjunct, hard determinable formula, quasi-hard determinable formula, proof complexity, refutation system, polynomial simulation.*

ACM Classification Keywords: *F.4.1 Mathematical Logic and Formal Languages, Mathematical Logic, Proof theory*

Introduction

The interest in the complexity of propositional proofs has arisen, in particular, from two fields connected with computers: automated theorem proving and computational complexity theory, the most famous open problems of which is the $P = NP$ problem.

In 1979 Cook and Reckhow studied the relationship between the lengths of propositional proofs and computational complexity, and observed that $NP = co - NP$ iff there exists a propositional system in which proofs are all polynomially bounded [Cook, Reckhow, 1979].

Cut-free sequent and resolution systems are the most frequently used proof systems for automated theorem proving, but they are "weak" systems. There are some formulas which require exponential proof complexities in these systems.

[1] Supported by grant 11-1b023 of Government of The Republic of Armenia

Due to the popularity of these systems it is natural to consider some of their extensions. Resolution over linear equations ($R(lin)$) [Raz, Tzameret, 2008] and cut-free Gentzen type calculus with permutation ($GCNF'+$ permutation) [Arai, 1996] can be considered as such extensions. These systems are stronger than the original systems.

In this paper we investigate the proof complexities of some classes of propositional formulas in $R(lin)$ and $GCNF'+$ permutation. In [Abajyan, 2011] and [Aleksanyan, Chubaryan, 2009] the notions of quasi-hard determinable and hard determinable formulas are introduced and proof complexities of such formulas are investigated in some propositional systems. In particular, it was proved that the complexities of some class of quasi-hard determinable formulas $Tsgf_n$ in Split Tree (Analytic Tableaux) and resolution systems are by order p($|Tsgf_n|$) for some polynomial p() [Abajyan, 2011] and in [Aleksanyan, Chubaryan, 2009] it was proved that complexities of some class of hard determinable formulas ψ_n are polynomially bounded in Frege systems.

Now we show that the minimal steps of $Tsgf_n$ proofs in $R(lin)$ and in $GCNF'+$ permutation are bounded by p($\log_2|Tsgf_n|$) for some polynomial p() and the formulas ψ_n require exponential size proofs in $GCNF'+$ permutation. We also show that any Frege system $p-$simulates $GCNF'+$ permutation and has exponential speed-up over the last one.

Note that $R(lin)$ and $GCNF'+$ permutation are refutation systems, that is, these systems intend to prove the unsatisfiability of formulas (negations of tautologies), therefore sometimes we shall speak about refutations and proofs interchangeably.

2. Main notions and notations

2.1 Hard determinable and quasi-hard determinable formulas

To prove our main results, we recall some notions and notations. We will use the current concept of the unit Boolean cube (E^n), a propositional formula, a tautology, a proof system for Classical Propositional Logic (CPL) and proof complexity.

By $|\varphi|$ we denote the size of a formula φ , defined as the number of all variable entries. It is obvious that the full length of a formula, which is understood to be the number of all symbols and the number of all entries of logical signs, is bounded by some linear function in $|\varphi|$.

A tautology φ is called minimal if φ is not an instance of a shorter tautology.

Following the usual terminology we call the variables and negated variables *literals*. The conjunct K can be simply represented as a set of literals (no conjunct contains a variable and its negation at the same time).

In [Aleksanyan, Chubaryan, 2009] the following notions were introduced.

We call a replacement-rule each of the following trivial identities for a propositional formula φ.

$$0 \,\&\, \psi = 0,\ \psi \,\&\, 0 = 0,\ 1 \,\&\, \psi = \psi,\ \psi \,\&\, 1 = \psi,\ \psi \,\&\, \psi = \psi,\ \psi \,\&\, \overline{\psi} = 0,\ \overline{\psi} \,\&\, \psi = 0,$$

$$0 \vee \psi = \psi,\ \psi \vee 0 = \psi,\ 1 \vee \psi = 1,\ \psi \vee 1 = 1,\ \psi \vee \psi = \psi,\ \psi \vee \overline{\psi} = 1,\ \overline{\psi} \vee \psi = 1,$$

$$0 \supset \psi = 1,\ \psi \supset 0 = \psi,\ 1 \supset \psi = \psi,\ \psi \supset 1 = 1,\ \psi \supset \psi = 1,\ \psi \supset \overline{\psi} = \overline{\psi},\ \overline{\psi} \supset \psi = \psi,$$

$$\overline{0} = 1,\ \overline{1} = 0\,,\ \overline{\overline{\psi}} = \psi :$$

Application of a replacement-rule to some word consists of replacing some of its subwords, having the form of the left-hand side of one of the above identities by the corresponding right-hand side.

Let φ be a propositional formula, $X = \{x_1, \dots, x_n\}$ be the set of all variables of φ and $X' = \{x_{i_1}, \dots, x_{i_m}\}$ $(1 \le m \le n)$ be some subset of X.

Definition 1. Given $\sigma = \{\sigma_1, \dots, \sigma_m\} \in E^m$, the conjunct $K^\sigma = \{x_{i_1}^{\sigma_1}, x_{i_2}^{\sigma_2}, \dots, x_{i_m}^{\sigma_m}\}$ is called φ-*determinative* if assigning σ_j $(1 \le j \le m)$ to each x_{i_j} and successively using replacement-rules we obtain the value of φ (0 or 1) independently of the values of the remaining variables.

Definition 2. We call the minimal possible number of variables in a φ-determinative conjunct the *determinative size* of φ and denote it by $d(\varphi)$.

Obviously, $d(\varphi) < |\varphi|$ for every formula φ, and the smaller is the difference between these quantities, the "harder" can be considered the formula under study.

Definition 3. Let φ_n $(n \ge 1)$ be a sequence of minimal tautologies. If for some n_0, $\forall n \ge n_0$, $d(\varphi_n) < d(\varphi_{n+1})$ then the formulas $\varphi_{n_0}, \varphi_{n_0+1}, \varphi_{n_0+2}, \dots$ are called *quasi-hard determinable*.

Definition 4. Let φ_n $(n \ge 1)$ be a sequence of minimal tautologies. If for some n_0 there is a constant c such that $\forall n \ge n_0$, $(d(\varphi_n))^c \le |\varphi_n| < (d(\varphi_n))^{c+1}$ then the formulas $\varphi_{n_0}, \varphi_{n_0+1}, \varphi_{n_0+2}, \dots$ are called *hard determinable*.

Example 1. For the well-known tautologies

$$PHP_n = \mathop{\&}_{i=1}^{n+1} \bigvee_{j=1}^{n} x_{ij} \supset \bigvee_{1 \le i < k \le n+1} \bigvee_{1 \le j \le n} \left(x_{ij} \,\&\, x_{kj}\right) \qquad (n \ge 1)$$

presenting the Pigeonhole Principle, the determinative conjunct is, in particular, $\{x_{11}, x_{21}\}$, therefore $d(PHP_n) = 2$ for all $n \ge 1$, hence, PHP_n is neither quasi-hard determinable nor hard determinable.

Example 2. The following tautologies are considered in [Aleksanyan, Chubaryan, 2009].

$$TTM_{n,m} = \bigvee_{(\sigma_1,\ldots,\sigma_n)\in E^n} \mathop{\&}_{j=1}^{m} \bigvee_{i=1}^{n} x_{ij}^{\sigma_i}, \ \left(n \geq 1, 1 \leq m \leq 2^n - 1\right).$$

From the structure of $TTM_{n,m}$ it follows obviously that every $TTM_{n,m}$-determinative conjunct contains at least m literals. Let $\psi_n = TTM_{n,2^n-1}$ for all $n \geq 1$. Then the formulas $\psi_3, \psi_4, \psi_5, \ldots$ are hard determinable [Aleksanyan, Chubaryan, 2009].

The sequence of quasi-hard tautologies can be considered on the base of graphs.

Let us recall the definition of Tseitin graph formulas [Tseitin, 1968]. Let G be a connected and finite graph with no loops and assume distinct literals are attached to its edges.

Definition 5. Graph is called *marked* if each vertex is marked by 0 or 1 and one assigned literal is chosen for each edge.

Let $x_1, \ldots, x_n$ be distinct literals, $\varepsilon \in \{0,1\}$. $[x_1, \ldots, x_n]^\varepsilon$ denotes a set of disjunctions that consists of literals $x_1, \ldots, x_n$ and satisfy the following conditions

1. For each i $\left(1 \leq i \leq n\right)$ either x_i or $\overline{x}_i$ belongs to the disjunction.
2. If ε is odd, then the number of negated literals is even and if ε is even, the number of negated literals is odd.

Let G be a marked graph. Let us construct the set of $[x_1, \ldots, x_n]^\varepsilon$ disjunctions for each vertex where ε is the value assigned to the given vertex and $x_1, \ldots, x_n$ are variables assigned to the incident edges. The set of disjunctions constructed for all vertices of graph G is denoted by $\alpha(G)$ and the sum of values assigned to vertices of the graph by modulo 2 is denoted by $\sigma(G)$. In [Tseitin, 1968] it is proved that $\alpha(G)$ is unsatisfiable iff $\sigma(G) = 1$.

It is obvious that if Tseitin graph formulas are constructed on the base of graphs, minimal degree of which is of the same order as the number of vertices, then such formulas are quasi-hard determinable but not hard determinable.

2.2 Proof complexity, polynomial simulation

In the theory of proof complexity the two main characteristics of the proof are: $t-$*complexity*, defined as the number of proof steps, and $l-$ *complexity*, defined as total number of proof symbols. Let Φ be a proof system and φ be a tautology. We denote by t_φ^Φ (l_φ^Φ) the minimal possible value of $t-$complexity ($l-$complexity) for all the proofs of tautology φ in Φ.

Let Φ_1 and Φ_2 be two different proof systems. Following [Cook, Reckhow, 1979] we recall

Definition 6. Φ_2 $p-t-$simulates ($p-l-$ simulates) Φ_1 if there exists a polynomial $p()$ such that for every formula φ derivable both in Φ_1 and in Φ_2 $t_\varphi^{\Phi_2} \le p(t_\varphi^{\Phi_1})$ ($l_\varphi^{\Phi_2} \le p(l_\varphi^{\Phi_1})$).

Definition 7. The systems Φ_1 and Φ_2 are $p-t-$equivalent ($p-l-$ equivalent) iff Φ_1 $p-t-$simulates ($p-l-$ simulates) Φ_2 and Φ_2 $p-t-$simulates ($p-l-$ simulates) Φ_1.

Definition 8. The system Φ_2 has exponential $t-$speed-up ($l-$speed-up) over the system Φ_1 if there exists a polynomial $p()$ and a sequence of such formulas φ_n , provable both in Φ_1 and in Φ_2, that $t_{\varphi_n}^{\Phi_1} > 2^{p\left(t_{\varphi_n}^{\Phi_2}\right)}$ ($l_{\varphi_n}^{\Phi_1} > 2^{p\left(l_{\varphi_n}^{\Phi_2}\right)}$).

3. Main systems

Let us recall the definitions of some proof systems of CPL which are not well-known.

3.1 Resolution over linear equations

Let us describe $R(lin)$ system following [Raz, Tzameret, 2008]. $R(lin)$ is an extension of well-known resolution which operates with disjunction of linear equations with integer coefficients. A disjunction of linear equations is of the following form

$$\left(a_1^{(1)}x_1 + \ldots + a_n^{(1)}x_n = a_0^{(1)}\right) \vee \ldots \vee \left(a_1^{(t)}x_1 + \ldots + a_n^{(t)}x_n = a_0^{(t)}\right)$$

where $t \ge 0$ and the coefficients $a_i^{(j)}$ are integers (for all $0 \le i \le n$ $1 \le j \le t$). We discard duplicate linear equations from a disjunction of linear equations. Any CNF formula can be translated into a collection of disjunctions of linear equations directly: every clause $\underset{i\in I}{\vee} x_i \vee \underset{j\in J}{\vee} \neg x_j$ (where I and J are sets of indices of variables) involved in the CNF is translated into the disjunction $\underset{i\in I}{\vee}(x_i = 1) \vee \underset{j\in J}{\vee}(x_j = 0)$. For a clause D we denote by $\widetilde{D}$ its translation into a disjunction of linear equations. It is easy to verify that any Boolean assignment of the variables $x_1,\ldots,x_n$ satisfies a clause D iff it satisfies $\widetilde{D}$.

As we wish to deal with Boolean values, we augment the system with axioms, called *Boolean axioms*: $(x_i = 0) \vee (x_i = 1)$ for all $i \in [n]$.

Axioms are not fixed: for any formula φ we obtain $\neg\varphi$, then we obtain $R(lin)$ translation of CNF of $\neg\varphi$. We also add Boolean axioms for each variable.

Definition 9 $(R(lin))$**.** Let $K = \{K_1,\ldots,K_m\}$ be a collection of disjunctions of linear equations. An $R(lin)$-

proof from K *of a disjunction of linear equations* D is a finite sequence $\pi = (D_1,...,D_l)$, of disjunctions of linear equations such that $D_l = D$ and for every $i \in [l]$, either $D_i = K_j$ for some $j \in [m]$, or D_i is a Boolean axiom $(x_h = 0) \vee (x_h = 1)$ for some $h \in [n]$, or D_i was deduced by one of the following $R(lin)$-inference rules, using D_j, D_k for some $j,k < i$.

Resolution. Let A, B be two disjunctions of linear equations (possibly the empty disjunctions) and let L_1, L_2 be two linear equations. From $A \vee L_1$ and $B \vee L_2$ it is derived $A \vee B \vee (L_1 + L_2)$ or $A \vee B \vee (L_1 - L_2)$.

Weakening. From a disjunction of linear equations A derive $A \vee L$, where L is an arbitrary linear equation over X.

Simplification. From $A \vee (0 = k)$ derive A, where A is a disjunction of linear equations and $(k \neq 0)$.

An $R(lin)$ *refutation* of a collection of disjunctions of linear equations K is a proof of the empty disjunction from K. Raz and Tzameret showed that $R(lin)$ is a sound and complete Cook-Reckhow refutation system for unsatisfiable CNF formulas (translated into unsatisfiable collection of disjunctions of linear equations).

3.2 GCNF' system

Let us describe $GCNF'$ system following [Arai, 1996]. $GCNF'$ is a variant of cut-free Gentzen system introduced by Gallier. It is also a refuting system. Here a clause is a set of literals, separated by commas. For example, $\{p_1, \overline{p}_2, p_3\}$ means $p_1 \vee \overline{p}_2 \vee p_3$. A *cedent* is a finite set of clauses, expressed as a sequence of clauses punctuated by commas. The meaning of a cedent is the conjunction of the clauses in the cedent. For example, $C_1, C_2, \ldots, C_n$ means $C_1 \& C_2 \& \ldots \& C_n$. We use capital Greek letters Γ, Δ, Π for cedents. The semantics of cedents implies that a cedent $C_1, \ldots, C_n$ is false iff the formula $C_1 \& \ldots \& C_n \supset \perp$ is valid.

The axioms are of the following form $p, \overline{p}$. And there are two inference rules

Structural: $\dfrac{\Gamma}{\Gamma, \Delta}$.

Logical (*Log*): $\dfrac{\Gamma, C_1, \ldots C_k \Pi, l}{\Gamma \cup \Pi, C_1 l, \ldots, C_k l}(l)$, where l is an arbitrary literal, which is called *auxiliary literal* of this inference rule.

$GCNF'$ is a sound and complete system [Arai, 1996].

3.3 GCNF' + permutation system

$GCNF'$+ permutation system is based on $GCNF'$ with one more inference rule [Arai, 1996].

Permutation (Perm): $\frac{\Gamma(p_1,\ldots,p_m)}{\Gamma(\pi(p_1),\ldots,\pi(p_m))}\pi$, where π is a permutation on $\{p_1,\ldots,p_m\}$ and $\Gamma(\pi(p_1),\ldots,\pi(p_m))$ is the result of replacing every occurrence of p_i , $1\le i\le m$ in $\Gamma(p_1,\ldots,p_m)$ by $\pi(p_i)$.

4. Main results

Let us denote by $Tsgf_n$ $(n\ge 2)$ the Tseitin graph formulas which are constructed on the base of complete n-vertices graph, only one of vertices of which is marked with 1.

Theorem 1:

1. $t_{Tsgf_n}^{R(lin)} \le l_{Tsgf_n}^{R(lin)} \le p(\log_2 |Tsgf_n|)$ for some polynomial $p()$.

2. $t_{Tsgf_n}^{GCNF'+permutation} \le p(\log_2 |Tsgf_n|)$ for some polynomial $p()$ and $l_{Tsgf_n}^{GCNF'+permutation} = \Theta(|Tsgf_n|)$.

Proof: 1. In order to prove the first part, let us recall two additional lemmas following [Raz, Tzameret, 2008].

Lemma 1: Let D_1 be $\underset{i\in[0,n-1]}{\vee}(x_1+x_2+\cdots+x_{n-1}=i)$ and D_2 be $\underset{i\in[0,n-1]}{\vee}(x_1+x_2+\cdots+x_n=i+\alpha)$. Then there exists an $R(lin)$ proof of D_2 from D_1 and $x_n=\alpha$ with n steps.

Lemma 2: Let D_1 be $\underset{i\in[0,n-1]}{\vee}(x_1+x_2+\cdots+x_{n-1}=i)$ and D_2 be $\underset{i\in[0,n]}{\vee}(x_1+x_2+\cdots+x_n=i)$. Then there exists an $R(lin)$ proof of D_2 from D_1 and $(x_n=0)\vee(x_n=1)$ with $2n+2$ steps.

Now we can consider complete marked n-vertices graph. For each vertex we have the following $R(lin)$ formula $x_{i_1}+x_{i_2}+\cdots+x_{i_{n-1}}=\varepsilon_i$, where ε_i is the value assigned to the given vertex and x_{i_j} $\left(1\le j<n, 1\le i\le \frac{n(n-1)}{2}\right)$ are variables assigned to the incident edges.

Using Resolution rule to $R(lin)$ formulas $n-1$ times (or, summarizing those formulas), we obtain

$$2x_1+2x_2+\cdots+2x_{\frac{n(n-1)}{2}}=1 \tag{1}$$

On the other hand, for all the variables, we have the following axioms, $(x_i=0)\vee(x_i=1), i\in\left[1,\frac{n(n-1)}{2}\right]$.

By Lemma 2, there is an $R(lin)$ proof of

$$\bigvee_{i\in\left[0,\frac{n(n-1)}{2}\right]}\left(x_1+x_2+\cdots+x_{\frac{n(n-1)}{2}}=i\right) \tag{2}$$

from the axioms, and the number of proof steps is $\sum_{i=2}^{n(n-1)/2}(2i+2)=\frac{n^4-2n^3+7n^2-6n-16}{4}$. Using Resolution rule $\frac{n(n-1)}{2}+1$ times, every time taking the next linear equation of (2) as $L_1=L_2$, we obtain

$$\bigvee_{i\in\left[0,\frac{n(n-1)}{2}\right]}\left(2x_1+2x_2+\cdots+2x_{\frac{n(n-1)}{2}}=2i\right) \tag{3}$$

Now, let us consider (1) and (3).

Using Resolution rule $\frac{n(n-1)}{2}+1$ times and Simplification rule $\frac{n(n-1)}{2}+1$ times (by using Resolution rule, we take (1) as L_1 and the next linear equation of (3) as L_2), we will cut-off all linear equations in (3) and obtain the empty clause $(0=1)$.

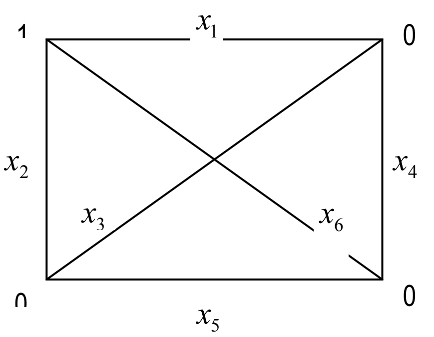
The number of proof steps is

$$n-1+\frac{n^4-2n^3+7n^2-6n-16}{4}+\frac{n(n-1)}{2}+1+\frac{n(n-1)}{2}+1+\frac{n(n-1)}{2}+1=\frac{n^4-2n^3+13n^2-8n}{4}$$

.Taking into consideration that $|Tsgf_n|=n(n-1)2^{n-2}$, we obtain $t_{Tsgf_n}^{R(lin)}\le p(\log_2|Tsgf_n|)$.

The size of the proof of (1) is $O(n^3)$, the size of the proof of (2) is $O(n^8)$. The size of the proof of (3) is $O(n^6)$. And, the size of deducing of the empty clause is $O(n^6)$. So, the size of the proof of the initial formula is $O(n^8)$, hence, $l_{Tsgf_n}^{R(lin)}=O((\log_2|Tsgf_n|)^8)$.□

1. In order to prove the point 2, let us at first demonstrate a proof of $Tsgf_4$ in $GCNF'+$ permutation system. The axioms for this case are indicated as (4).



$$\begin{array}{cccc}\bar{x}_1\bar{x}_2x_6 & \bar{x}_1x_2\bar{x}_6 & x_1\bar{x}_2\bar{x}_6 & x_1x_2x_6\\ \bar{x}_1x_3x_4 & x_1\bar{x}_3x_4 & x_1x_3\bar{x}_4 & \bar{x}_1\bar{x}_3\bar{x}_4\\ \bar{x}_2x_3x_5 & x_2\bar{x}_3x_5 & x_2x_3\bar{x}_5 & \bar{x}_2\bar{x}_3\bar{x}_5\\ \bar{x}_4x_5x_6 & x_4\bar{x}_5x_6 & x_4x_5\bar{x}_6 & \bar{x}_4\bar{x}_5\bar{x}_6\end{array} \tag{4}$$

$$\begin{array}{ll}
 & x_1, \bar{x}_1 \qquad x_2, \bar{x}_3 \\ \cline{2-2}
\text{Log} & \bar{x}_1 \vee x_2, \qquad x_1, \bar{x}_3 \\ \cline{2-2}
\text{Perm} & \bar{x}_1 \vee x_2,\ x_1, \bar{x}_3,\ x_1 \vee \bar{x}_3,\ \bar{x}_1, x_2 \qquad x_2, \bar{x}_2 \\ \cline{2-2}
\text{Log} & \bar{x}_1 \vee x_3,\ x_1 \vee \bar{x}_3,\ \bar{x}_1 \vee \bar{x}_2,\ \bar{x}_2 \vee x_3,\ x_1, \bar{x}_3,\ x_2 \\ \cline{2-2}
\text{Log} & \bar{x}_1 \vee x_3,\ x_1 \vee \bar{x}_3,\ \bar{x}_1 \vee \bar{x}_2,\ \bar{x}_2 \vee x_3,\ x_1 \vee x_2,\ x_2 \vee \bar{x}_3
\end{array} \tag{5}$$

Using $x_1 \to x_2, x_2 \to x_3, x_3 \to x_1$ Permutation rule to (5), we obtain

$$\bar{x}_2 \vee x_1, x_2 \vee \bar{x}_1, \bar{x}_2 \vee \bar{x}_3, \bar{x}_3 \vee x_1, x_2 \vee x_3, x_3 \vee \bar{x}_1 \tag{6}$$

Using $x_1 \to x_3, x_2 \to x_1, x_3 \to x_2$ Permutation rule to (5), we obtain

$$\bar{x}_3 \vee x_2, x_3 \vee \bar{x}_2, \bar{x}_3 \vee \bar{x}_1, \bar{x}_1 \vee x_2, x_3 \vee x_1, x_1 \vee \bar{x}_2 \tag{7}$$

Applying Logical inference rule to (5), (6), (7) and respectively to axioms $x_6, \bar{x}_6$, $x_4, \bar{x}_4$, $x_5, \bar{x}_5$, we obtain first three lines of (4). The last line of (4) we can deduce as follows:

$$\begin{array}{ll}
 & x_4, \bar{x}_4 \qquad x_5, \bar{x}_5 \\ \cline{2-2}
\text{Log} & x_4 \vee x_5,\ \bar{x}_4 \vee x_5,\ \bar{x}_5 \qquad x_4, \bar{x}_4 \\ \cline{2-2}
\text{Log} & x_4 \vee x_5,\ \bar{x}_4 \vee x_5,\ x_4 \vee \bar{x}_5,\ \bar{x}_4 \vee \bar{x}_5 \qquad x_6, \bar{x}_6 \\ \cline{2-2}
\text{Log} & \bar{x}_4 \vee x_5 \vee x_6,\ x_4 \vee \bar{x}_5 \vee x_6,\ x_4 \vee x_5,\ \bar{x}_4 \vee \bar{x}_5 \qquad \bar{x}_6 \\ \cline{2-2}
\text{Log} & \bar{x}_4 \vee x_5 \vee x_6,\ x_4 \vee \bar{x}_5 \vee x_6,\ x_4 \vee x_5 \vee \bar{x}_6,\ \bar{x}_4 \vee \bar{x}_5 \vee \bar{x}_6
\end{array}$$

For $Tsgf_n$ we denote by $t(i)$ the derivation steps of first $i-1$ lines (as above) of the axioms corresponding to the complete graph with i vertices. It is not difficult to see that $t(3)=4$ and $t(n)=t(n-1)+(n-2)+2(n-1)$, hence, $t(n)=\dfrac{n(3n-5)}{2}-2 \le 3n^2$. The last line of the axioms consists of such variables that do not exist in the $n-1$-vertices complete graph, that is, those variables are assigned to the edges which are incident to the newly added vertex. Each clause consists of $n-1$ literals and $2(n-2)$ steps are needed to deduce the last line. So, the number of proof steps is $\dfrac{n(3n-5)}{2}-2+2(n-2)=\dfrac{n(3n-1)}{2}-6 \le 3n^2$, then we obtain $t_{Tsgf_n}^{GCNF'+permutation} \le p(\log_2 |Tsgf_n|)$.

There are at most $(n-1)2^{n-2}$ literals in each step of the proof and the number of proof steps is at most $3n^2$, hence $l_{Tsgf_n}^{GCNF'+permutation} = O(|Tsgf_n|)$. It is obvious that the lower bound is the same by order. □

Theorem 2: $l_{\neg\psi_n}^{GCNF+permutation} = \Omega\left(2^{\sqrt{|\psi_n|}\frac{\log_2 n}{\sqrt{n}}}\right)$.

Proof. It is not difficult to see that CNF of $\neg\psi_n = \underset{(\sigma_1,\ldots,\sigma_n)\in E^n}{\&} \overset{2^n-1}{\underset{j=1}{\vee}} \overset{n}{\underset{i=1}{\&}} x_{ij}^{\sigma_i}$ has at least n^{2^n-1} conjuncts such that neither these conjuncts nor any of their subset can be obtained from each other by Permutation rule (for $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 1$ and for $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$), therefore $l_{\neg\psi_n}^{GCNF'+permutation} > 2(1+2+\cdots+2^n-1)n^{2^n-1} = 2^n(2^n-1)n^{2^n-1} > (2^n-1)^2 2^{(2^n-1)\log_2 n}$. Taking into consideration that $|\neg\psi_n| = 2^n(2^n-1)n$, we obtain the statement of the Theorem. □

Now, let us recall some additional systems.

1. $GCNF'+$ renaming system is based on $GCNF'$ with one more inference rule [Arai,1996].

Renaming: $\dfrac{\Gamma}{\Gamma(p \to q)} p \to q$, where $\Gamma(p \to q)$ is obtained by replacing every occurrence of p by q in Γ.

2. $GCNF'+$ restricted renaming system is based on $GCNF'$ with one more inference rule [Arai, 1996].

Restricted renaming: $\dfrac{\Gamma}{\Gamma(p \Rightarrow q)} p \Rightarrow q$, where $\Gamma(p \Rightarrow q)$ is obtained by replacing every occurrence of p by a variable q which does not appear in Γ.

3. We also use the well-known notions of F – Frege, SF – Substitution Frege and EF – Extended Frege systems (see, for example, [Pudlak, 1998]).

Theorem 3:

1. F has exponential speed-up over the $GCNF'+$ permutation.

2. F $p-$simulates $GCNF'+$ permutation.

Proof of point 1 follows from Theorem 2 and main result of [Aleksanyan, Chubaryan, 2009] where it is proved that F proofs of tautology $TTM_{n,m}$ are $l-$polynomially bounded.

Proof of point 2 follows from some results of [Arai, 1996], [Arai, 2000] and [Cook, Reckhow, 1979], in particular

a) $GCNF'+$ renaming $p-l-$simulates $GCNF'+$ restricted renaming (it is obvious).
b) $GCNF'+$ restricted renaming $p-l-$simulates $GCNF'+$ permutation (see [Arai, 1996]).
c) F $p-l-$simulates $GCNF'+$ renaming iff F polynomially simulates EF (see [Arai, 1996]).
d) SF and EF are $p-l-$equivalent (see [Pudlak 1998]).
e) F and SF are $p-l-$equivalent (see [Chubaryan, Nalbandyan, 2010]). □

Bibliography

[Abajyan, 2011] A. A. Abajyan "Polynomial length proofs for some class of Tseitin formulas", Proceeding of the Yerevan State University, № 3, pp. 3-8, 2011.

[Aleksanyan, Chubaryan, 2009] S. R. Aleksanyan, A. A. Chubaryan "The polynomial bounds of proof complexity in Frege systems", Siberian Mathematical Journal, vol. 50, № 2, pp. 243-249, 2009.

[Arai, 1996] N. H. Arai "Tractability of Cut-free Gentzen-type propositional calculus with permutation inference", Theoretical Computer Science 170, pp. 129-144, 1996.

[Arai, 2000] N. H. Arai "Tractability of Cut-free Gentzen-type propositional calculus with permutation inference II", Theoretical Computer Science 243, pp. 185-197, 2000.

[Chubaryan, Nalbandyan, 2010] A. Chubaryan, H. Nalbandyan "Comparison of proof sizes in Frege systems and substitution Frege system", International Journal "Information Theories and Applications", vol. 17, № 2, pp. 146-153, 2010.

[Cook, Reckhow, 1979] S. A. Cook, A. R. Reckhow, "The relative efficiency of propositional proof systems", Journal of Symbolic Logic, vol. 44, pp. 36-50, 1979.

[Pudlak, 1998] P. Pudlak "Lengths of proofs" Handbook of proof theory, North-Holland, pp. 547-637, 1998.

[Raz, Tzameret, 2008] Ran Raz, Iddo Tzameret, "Resolution over linear equations and multilinear proofs", Ann. Pure Appl. Logic 155(3), pp. 194-224, 2008.

[Tseitin, 1968] G. S. Tseitin "On the complexity of derivation in the propositional calculus", (in Russian), Zap. Nauchn. Semin. LOMI. Leningrad: Nauka, vol. 8, pp. 234-259, 1968.

Authors' Information

Ashot Abajyan *– PhD Student of the Department of Informatics and Applied Mathematics, Yerevan State University, 1. A.Manoogyan, 0025 – Yerevan, Armenia; e-mail:* ashotabajian@rambler.ru.

Major Fields of Scientific Research: Mathematical Logic, Proof Theory, Proof complexity.

Anahit Chubaryan *– Doctor of sciences, full professor of the Department of Informatics and Applied Mathematics, Yerevan State University, 1. A.Manoogyan, 0025 – Yerevan, Armenia; e-mail:* achubaryan@ysu.am.

Major Fields of Scientific Research: Mathematical Logic, Proof Theory, Proof complexity.

Table of contents of IJ ITA Vol. 17, No.: 3

Constraint Convexity Tomography and Lagrangian Approximations

Levon Aslanyan, Artyom Hovsepyan, Hasmik Sahakyan 203

On HYPERSIMPLE wtt -mitotic Sets, WHICH ARE not tt -mitotic

Arsen H. Mokatsian........ 215

A New Algoritm for the Longest Common Subsequence Problem

Vahagn Minasyan 226

Interference Minimization in Physical Model of Wireless Networks

Hakob Aslanyan........ 235

On Measurable Models of Promotion of Negentropic Strategies by Cognition

Pogossian Edward 249

On Reliability Approach to Multiple Hypotheses Testing and to Identification of Probability Distributions of Two Stochastically Related Objects

Evgueni Haroutunian, Aram Yessayan, Parandzem Hakobyan........ 259

Proof Complexities of Some Propositional Formulae Classes in Different Refutation Systems

Ashot Abajyan, Anahit Chubaryan 289