CROSS INTERSECTION SEQUEL OF DISCRETE ISOPERIMETRY[1]

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Abstract: This work inspired by a specifically constrained communication model. Given collections of communicating objects, and communication is by means of several relay centres. The complete cross connectivity of elements of different collections is the target, supposing that communicating objects differ by their connections to the relay centres. Such models exist only for proper object groups – when they have specific sizes and there is a corresponding number of relay points. We consider optimization problems studying the validity boundaries. Terms are combinatorial – geometry of binary cube, lexicographical orders, shadowing and isoperimetry. The main interest is methodological and aims at extending the consequences that can be delivered from the solution of the well known discrete isoperimetry problem.

Keywords: communication, optimization, isoperimetry.

ACM Classification Keywords: G.2.1 Discrete mathematics: Combinatorics

Introduction

Mass communication models with resource limitations require a proper design and analysis stage. Practical examples are populations with sophisticated communication means, wireless sensor models with requirements of connectivity, coverage and energy efficiency, and other ad hoc networks with different additional requirements. Resource limitations which appear everywhere need to be checked against the existence of a valid network, and when it is, - optimization that brings the resource minimization and the quality enhancement.

Communication model which we consider consists of several independent (none intersecting) societies $\Xi_1, \Xi_2, ..., \Xi_m$ whose elements are to be cross connected totally. This means that each pair $s_{i_1 p} \in \Xi_{i_1}$ and $s_{i_2 q} \in \Xi_{i_2}$, $i_1 \neq i_2$ is connected. Connection is through the set of n relay points $X_1, X_2, ..., X_n$. If object s is connected to the relay points $X_{j_1}, X_{j_2}, ..., X_{j_k}$ then we code it as the binary n-vector $\tilde{\alpha}(s) = (\alpha_1, \alpha_2, ..., \alpha_n)$ with coordinates $j_1, j_2, ..., j_k$ equal to 1, and all other coordinates - to 0. Now for objects $\tilde{\alpha}$ and $\tilde{\beta}$ connectivity means that they (their code vectors) intersect by their sets of 1 coordinates. Connectivity that we described requires all connections between the societies $\Xi_1, \Xi_2, ..., \Xi_m$. Inside the society we require that the binary codes defined for objects are all different. One more condition may require that objects are connected to a fixed number of relay points and this condition is also applied to our model.

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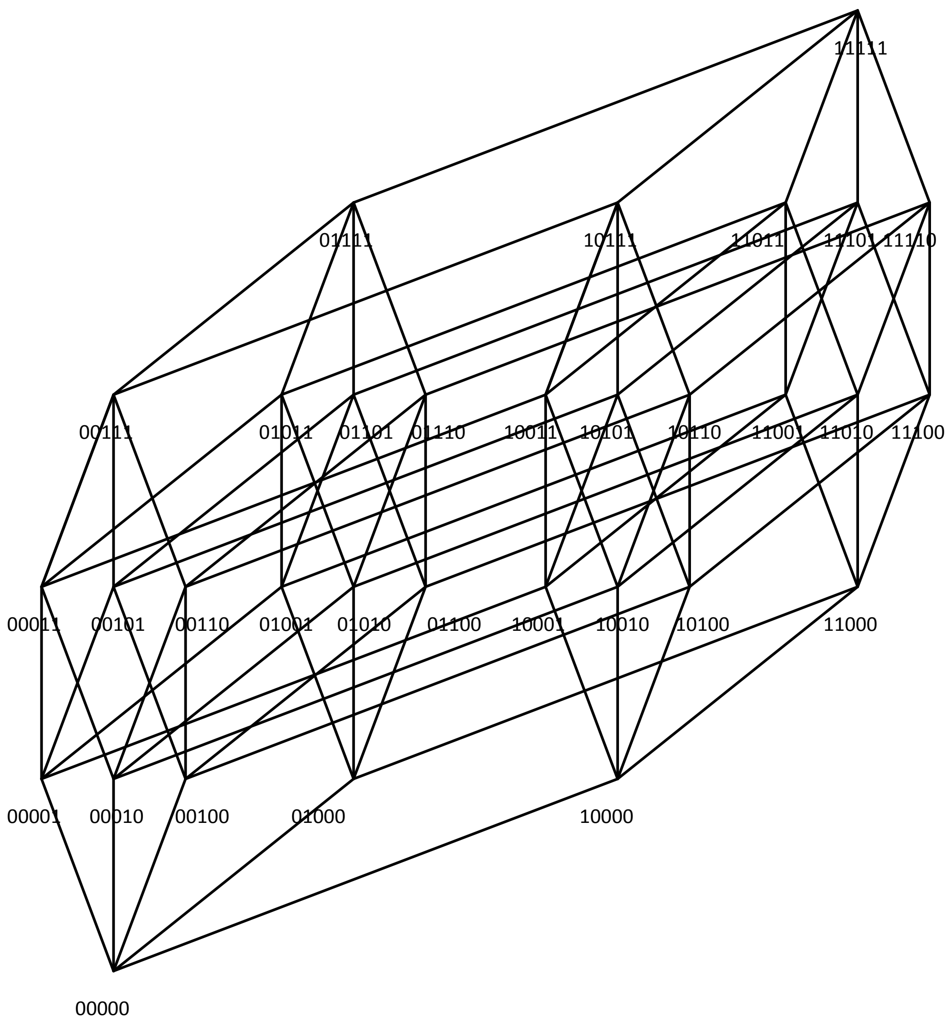
Given above - is a particular communication model for societies. There can be a broad diversity of models varying in requirements. For instance, one can require a balanced use of relay points. Diversity of object binary codes can be applied on the total integrated society, etc. But our goal is to see the optimization framework of models of this type. It is shown that combinatorial optimization appears as the instrument of design of such networks. The particular technique that appears is the binary n-cube geometry being linked to the fundamental results of combinatorial optimization in that area.

The Formal Model

Boolean domain: First formal model that we consider to interpret the cross intersection property is in terms of n-dimensional binary cube, Boolean functions and systems of Boolean functions. Let $E = \{0,1\}$. Cartesian degree E^n represents the set of all n- dimensional binary vectors $\tilde{e} = (e_1, e_2, ..., e_n)$. As usual we define the <u>weight</u> of vector and the Hamming <u>distance</u> of a pair of vectors in E^n. Weight of $\tilde{e}$ is the number of its 1 coordinates. The Hamming distance $h(\tilde{e}_1, \tilde{e}_2)$ between $\tilde{e}_1$ and $\tilde{e}_2$ is the number of coordinates where these vectors are different. Consider the Hasse diagram of E^n. An example for E^5 is given below.



The diagram consists of $n+1$ layers $0,1,...,n$ placed vertically. Each layer is composed of the same weight vertices placed on that layer horizontally. k-th layer consists of C_n^k vertices denoted by E_k^n. Edges connect 2 neighbour vertices – those that have distance 1 (differing exactly in 1 coordinate). Vertices $\tilde{e}_1$ and $\tilde{e}_2$ are comparable, in particular $\tilde{e}_1 \prec \tilde{e}_2$, if all corresponding coordinate vice similar inequalities hold. In a special case when $\tilde{e}_1$ and $\tilde{e}_2$ are comparable and their distance equals 1, then they are connected by an edge in the diagram.

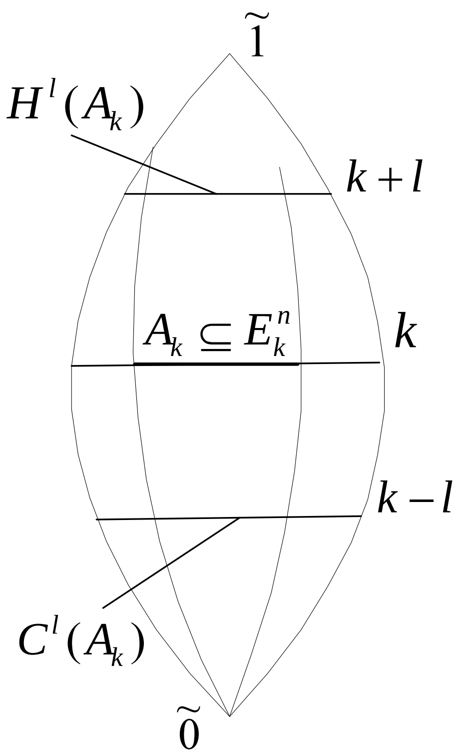
Consider the list of vertices of layer 2 in the order we see them on the diagram:

$$00011,00101,00110,01001,01010,01100,10001,10010,10100,11000.$$

It is easy to check that this is lexicographic sequence of all vertices of layer 2. In a similar way we may compose the lexicographic sequence of all n-dimensional words of weight k over the alphabet $\{0,1\}$. Here we suppose the usual precedence $0 \prec 1$. We denote this sequence as L_k^n, and let $L_k^n(\delta)$ is the initial δ-segment of L_k^n. And let T_k^n and $T_k^n(\delta)$ denote the reverse sequence to L_k^n and its initial δ-segment. In area of discrete isoperimetry it is common for T_k^n the term <u>standard placement</u>.

It is well known the specific (and unique in this form) decomposition of set $T_k^n(\delta)$ and the number δ itself in the following way:

$$\delta = C_{n-m_1}^{k-m_1} + C_{n-m_2}^{k-m_2+1} + ... + C_{n-m_r}^{k-m_r+r-1}, \text{ where } 1 \le m_1 < m_2 < ... < m_r < n. \quad (1)$$

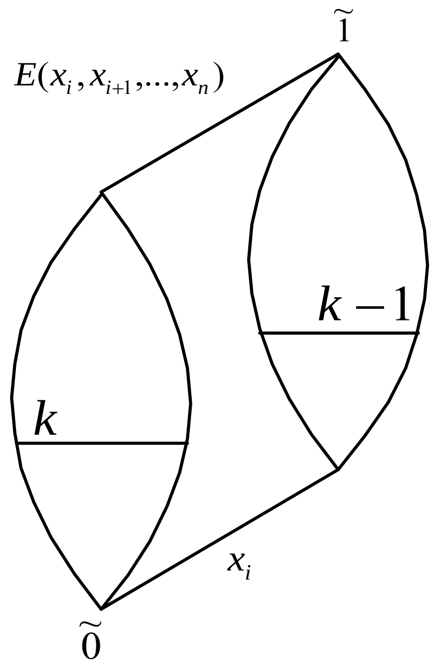


Consider arbitrary vertex subsets $A \subseteq E^n$. A_k denote the intersection $A \cap E_k^n$. Two type of concepts are introduced – internal (blocked) area $C^l(A_k)$, and bordering (shadow) area $H^l(A_k)$. <u>$C^l(A_k)$</u> is the set of all those vertices of layer <u>$k-l$</u> that are internal by the set of vertices of A_k. In other terms $\tilde{\alpha} \in C^l(A_k)$ iff <u>all</u> vertices of layer k comparable with $\tilde{\alpha}$ belong to A_k. <u>$H^l(A_k)$</u> belongs to layer <u>$k+l$</u> and consists of vertices $\tilde{\alpha}$ that <u>at least one</u> of elements of layer k comparable with $\tilde{\alpha}$ belongs to A_k. Below we suppose that $k \le n/2$. When after some transformations we receive subsets above the layer $n/2$, then $C^l(A_k)$ and $H^l(A_k)$ are in some sense bottom up constructions. In this case $C^l(A_k)$ belongs to layer $k+l$ and $H^l(A_k)$ is from layer $k-l$ (we refer to this by notion).

Consider the set $A_k = T_k^n(\delta)$. How this is related to the formula (1)? Consider series of splits in E^n. Coordinates $x_1, x_2, ..., x_n$ are applied consequently. Initially we are given the cube size n and δ vertices to be on layer k. Split of E^n by $x_1 = 0$ and $x_1 = 1$ brings two $n-1-$ subcubes. A_k (similarly any other subset of vertices of E^n) is split to one part on the layer k of subcube with $x_1 = 0$, a the reminder vertices of that are on layer k of (right) subcube with $x_1 = 1$. Depending on whether one of these parts completes the layer we choose the left or the right subcube for continuation.

Let after several splits that use coordinates $x_1, x_2, ..., x_i$ we face the situation that the $k-i$-th layer of right subcube is filled by elements of A_k. Here appears the first term C_{n-i}^{k-i} in formula (1). After that we continue splitting in left subcube, where we have the cube size $n-i$ and $\delta - C_{n-i}^{k-i}$ vertices to be on layer $k-i+1$.

Continuation of splitting that finally concludes the formula (1) is a series of splits similar to the case we considered.



Decomposition of δ given by formula (1) have two major properties. Explain them in terms of a particular step of split process. In a situation, when the right subcube become completed by δ reminder vertices, this part creates several new internal vertices on layer $k-i-l$, and several new bordering vertices on layer $k-i+l$ (in global cube E^n, not the splitted ones, these are layers $k-l$ and $k+l$). Internal a bordering vertices created during this step are non intersecting with the ones created during the previous splitting steps.

So the following formulas are consequences of (1).

First we suppose the case $l = 1$. Introduce the formulas:

$$c_{k-1}(\delta) = c_{k-1}^1(\delta) = C_{n-m_1}^{k-m_1-1} + C_{n-m_2}^{k-m_2} + ... + C_{n-m_r}^{k-m_r+r-2}$$
$$h_{k+1}(\delta) = h_{k+1}^1(\delta) = C_{n-m_1}^{k-m_1+1} + C_{n-m_2}^{k-m_2+2} + ... + C_{n-m_r}^{k-m_r+r} \quad (2)$$

For l general we have:

$$c_{k-l}^l(\delta) = c_{k-l}(c_{k-l+1}(...(c_{k-1}(\delta)))) = C_{n-m_1}^{k-m_1-l} + C_{n-m_2}^{k-m_2-l+1} + ... + C_{n-m_r}^{k-m_r-l+r-1}$$
$$h_{k+l}^l(\delta) = h_{k+l}(h_{k+l-1}(...(h_{k+1}(\delta)))) = C_{n-m_1}^{k-m_1+l} + C_{n-m_2}^{k-m_2+l-1} + ... + C_{n-m_r}^{k-m_r+l-r+1} \quad (3)$$

Here we define the base isoperimetry problem, give its one, very easy solution, and mention some consequences that we will use. Vertex $\widetilde{\alpha} \in A \subseteq E^n$ is interior, if all 1 distance vertices from $\widetilde{\alpha}$ belong to A. In general, $S_r^n(\widetilde{\alpha})$ denotes the sphere of radius r with center $\widetilde{\alpha}$. So $\widetilde{\alpha}$ is interior in A, if $S_1^n(\widetilde{\alpha}) \subseteq A$. $\mathrm{I}nt(A)$ will denote the set of all points interior in A. The reminder vertices $A \setminus \mathrm{I}nt(A)$ of A we call boundary vertices. The base discrete isoperimetry problem (DIP) by a given size $a,\ 0 \le a \le 2^n$ is seeking for subsets A so that

$$\left|\mathrm{I}nt(A)\right| \ge \left|\mathrm{I}nt(B)\right|_{B \subseteq E^n, |B|=a}.$$

Theorem 1. $S_{k-1}^n(\widetilde{0}) \cup T_k^n(\delta)$ is a DIP solution for $a = \sum_{i=0}^{k-1} C_n^i + \delta,\ \delta < C_n^k$.

Consequence 1. If $A \subseteq E_k^n, |A| = \delta, \delta < C_n^k$ then

$$\left|C^l(A)\right| \le \left|C^l(T_k^n(\delta))\right| = c_{k-l}^l(\delta).$$

Consequence 2 (Kruscal-Katona theorem). If $A \subseteq E_k^n, |A| = \delta, \delta < C_n^k$ then

$$\left|H^l(A)\right| \ge \left|H^l(T_k^n(\delta))\right| = h_{k+l}^l(\delta).$$

Given is the basic knowledge that we need from the discrete isoperimetry area. We may use several extensions of results but we prefer to stay on basic postulations to be more or less transparent and understandable.

We may also use an equivalent terminology given in terms of Boolean functions. n-dimensional Boolean function f accepts value 1 (true) in some subset $\Xi_f \subseteq E^n$. Denote $E^n \setminus \Xi_f$ by $\overline{\Xi}_f$ which is now the set of all 0 values of function f (and 1 vertices of the inversion of function f). Introduce spectral characteristics for function f as the sequence $t_0, t_1, \ldots, t_n$ of sizes of sets $\Xi_f[k] = E_k^n \cap \Xi_f$, $k = 0,1,\ldots,n$.

In terms of cross connected network design we associate one Boolean function to one society. Ξ_f consists of all codes of object in one society f. These codes are all different. An object code belongs to the layer k means that it is connected to the k relay points. In a simplest case we suppose that only the sets $\Xi_f[k_0]$ for one fixed values of k are nontrivial (non zero). In cross societal connectivity problem we posted above a system $f_1, f_2, \ldots, f_m$ of Boolean functions is considered.

Set-theoretical domain: The set-theoretical framework related to the applied communication model defined above was introduced and studied firstly in [HIL,1977]. Here interpretation is as follows. Let $k, m \geq 1$. k is the number of links from objects to the set of n relays. m is the number of societies. Let

$$\Xi_1 = \{S_{11}, S_{12}, ..., S_{1t_1}\}$$

...

$$\Xi_m = \{S_{m1}, S_{m2}, ..., S_{mt_m}\}$$

be a list of collections of subsets (societies) with subsets of set $\{1,2,...,n\}$. Suppose that

Condition 1.

$|S_{ir}| = k \leq n/2$ for $1 \leq i \leq m$ and $1 \leq r \leq t_i$ /objects are linked to the same number k of relays/

$S_{ir_1} \neq S_{ir_2}$, $1 \leq i \leq m, 1 \leq r_1 < r_2 \leq t_i$ /objects in one society are different by their connections to relays/ (4)

$S_{i_1 r'} \cap S_{i_2 r''} \neq 0$ $1 \leq i_1 < i_2 \leq m, 1 \leq r' \leq t_{i_1}, 1 \leq r'' \leq t_{i_2}$ /objects from different societies are intersecting/.

Theorem 2. [HIL,1977] proves that under the Condition 1.

$$t_1 + t_2 + ... + t_m \leq \begin{cases} C_n^k \ if \ n/k \geq m, \\ mC_{n-1}^{k-1} \ if \ n/k \leq m. \end{cases} \tag{5}$$

Here $C_n^k = n/k \cdot C_{n-1}^{k-1}$ so that the maximum between the C_n^k and mC_{n-1}^{k-1} is correlated with maximum between the n/k and m.

The proof of this postulation is hard in [HIL,1977], with intensive formula manipulations. After the result was achieved, a series of publications appeared bringing more simple and transparent results. We aim to demonstrate that the most suitable technique for this research area is the discrete isoperimetry technique [ASL,1979]. This not only gives the numerical estimates but also explains the structural properties of cross connectivity collections.

Intersection – Isoperimetry Relations

Theorem 3. If a collection of sets $\{\Xi_1, \Xi_2, ..., \Xi_m\}$ of characteristics $\{t_1, t_2, ..., t_m\}$ exists under the Condition 1. then the same Condition 1. properties obeyed by the set $\{T_k^n(t_1), T_k^n(t_2), ..., T_k^n(t_m)\}$.

To prove this consider an induction on m. If $m=1$, it is simply evident that as the set Ξ_1 we can take initial fragment $T_k^n(t_1)$. This choice is valid due to $t_1 \le C_n^k$ by Condition 1.

Suppose that the theorem postulation is correct for arbitrary collections $\{\Xi_1, \Xi_2, ..., \Xi_{m'}\}$ of characteristics $\{t_1, t_2, ..., t_{m'}\}$ under the Condition 1., when $m' < m$. Consider the proof for values $m,\ m \ge 2$.

Let $\{\Xi_1, \Xi_2, ..., \Xi_m\}$ be an arbitrary collection of subsets of connections of objects that satisfies Condition 1. Consider the sub-collection $\{\Xi_1, \Xi_2, ..., \Xi_{m-1}\}$ and construct in accord to this collection the collection of compliments/negations $\{\overline{\Xi}_1, \overline{\Xi}_2, ..., \overline{\Xi}_{m-1}\}$ of initial sub-collections, where $\overline{\Xi}_i = \{\overline{S}_{i1}, \overline{S}_{i2}, ..., \overline{S}_{it_i}\}$, $1 \le i \le m-1$. It is clear *that the relation* $\left|\overline{S}_{ij}\right| = n-k \ge n/2 \ge k$ holds so that during this negations all points are transferring from layer k to the layer $n-k$.

Consider the set $\breve{S} = \bigcup_{i=1}^{m-1} \Xi_i$ and let $\widehat{S} = \bigcup_{i=1}^{m-1} \overline{\Xi}_i$. Compose by this the set $H^l\left(\bigcup_{i=1}^{m-1} \overline{\Xi}_i\right)$ for the value $l = n-2k$. Recall that if $\widehat{S} \subseteq E_{n-k}^n$, $k \le n/2$ (notion), then $H^l(\widehat{S})$ consists of some elements of E_k^n, those that are covered by elements of set $\widehat{S}$. Hence we received that $H^l\left(\bigcup_{s \in \breve{S}} \overline{s}\right) \subseteq E_k^n$, with the general requirement that

$$C_n^k - \left|H^l\left(\bigcup_{s \in \breve{S}} \overline{s}\right)\right| \ge t_m . \tag{6}$$

The last requirement takes into account that sets $S_{mj_1}, 1 \le j_1 \le t_m$ doesn't belong (are not covered) to any $H^l\left(\overline{S}_{ij_2}\right), 1 \le i \le m-1, 1 \le j_2 \le t_i$ and so, they doesn't belong to the union of these sets. If an inclusion $S_{mj_1} \in H^l\left(\overline{S}_{ij_2}\right)$ holds for some $1 \le i \le m-1$, then we receive that the initial vectors are not intersecting, $S_{mj_1} \cap S_{ij_2} = 0$, which contradicts to the theorem conditions. This means that the set Ξ_m is to be out of $H^l\left(\bigcup_{s \in \breve{S}} \overline{s}\right)$ and the corresponding relation of sizes of these sets is introduced in formula (6).

By the induction suppositions, there exists a collection $\{\Xi'_1, \Xi'_2, ..., \Xi'_{m-1}\}$ of characteristics $\{t_1, t_2, ..., t_{m-1}\}$ that satisfy Condition 1. and that $\Xi'_i = T_k^n(t_i)$. It is easy to check that $\bigcup_{i=1}^{m-1} \Xi'_i = T_k^n(t)$ for some value $t = \max(t_1, t_2, ..., t_{m-1})$.

Now the set $\widehat{S} = \bigcup_{i=1}^{m-1} \overline{\Xi}_i$ is represented as a finite sequence $L_{(n-k)}^n(t)$ of the $n-k$ -th layer of E^n.

According to Consequence 2.

$$C_n^k - \left|H^l\left(T_{n-k}^n(t)\right)\right| \ge C_n^k - \left|H^l\left(\bigcup_{s\in\bar{S}} \bar{s}\right)\right| \ge t_m . \quad (7)$$

Moreover, each subset $S \in E_k^n$, not belonging to the set $H^l\left(\bigcup_{s\in\bar{S}} \bar{s}\right)$ intersects with some subset S_{ij}, $1 \le i \le m-1$, $1 \le j \le t_i$.

In addition, $H^l\left(\bigcup_{s\in\bar{S}} \bar{s}\right) = H^l\left(\bigcup_{i=1}^{m-1} L_{n-k}^n(t_i)\right) = L_{n-k}^n(t)$, - and the compliment of $L_k^n(t)$ in the layer E_k^n is some $T_k^n(t')$. Now, constructing the proper $\{\Xi'_1, \Xi'_2, ..., \Xi'_m\}$ it is enough to take the Ξ'_m as the set $T_k^n(t_m)$ taking into account the proven relation $t' \ge t_m$. This proves the Theorem 3.

Denote by $R(m)$ the number of all those vectors $t_1 t_2, ..., t_m$ which correspond to some sets $\{\Xi_1, \Xi_2, ..., \Xi_m\}$ as the characteristics and obey the Condition 1.

Theorem 4. A necessary and sufficient condition for existence of a collection $\{\Xi_1, \Xi_2, ..., \Xi_m\}$ of characteristics $t_1 \ge t_2 \ge ... \ge t_m$ with Condition 1. is the existence of a collection $\{\Omega_1, \Omega_2\}$ of characteristics (t_1, t_2) that accords Condition 1.

The necessity point of theorem postulation is evident. To prove the sufficiency suppose that we are give collections $\{\Omega_1, \Omega_2\}$ of k-subsets, and collections have sizes t_1 and t_2 with Condition 1, satisfied. Theorem 3. implies, that without loss of generality we may suppose that $\Omega_i = T_k^n(t_i), i = 1,2.$ Take $\Xi_i = T_k^n(t_i), i = 3,...,m$ and prove that the resulting system $\{\Omega_1, \Omega_2, \Xi_3, ..., \Xi_m\}$ obeys Condition 1. According to construction of collections $\{\Omega_1, \Omega_2, \Xi_3, ..., \Xi_m\}$ we have that first 2 points of Condition 1. are satisfied. Then, each pair of elements from $\{\Omega_1, \Omega_2\}$ are proven intersecting. Similarly, elements of Ω_1 and Ω_2 are intersecting with elements of other sets because of sets Ω_i, $j \ge 3$ are subsets of Ω_2. Last to prove is that subsets from Ξ_{i_1} intersect with subsets from Ξ_{i_2}, when $3 \le i_1 < i_2 \le m$. This happens because of $\Xi_{i_1} \subseteq \Omega_1$ and $\Xi_{i_2} \subseteq \Omega_2$.

Now we combine (1), (3) and (7) to achieve a quantitative condition for cross intersections. Consider again values $t_1 \ge t_2 \ge ... \ge t_m$ and the formula (1) for value t_1:

$t_1 = C_{n-m_1}^{k-m_1} + C_{n-m_2}^{k-m_2+1} + ... + C_{n-m_r}^{k-m_r+r-1}$, where $1 \le m_1 < m_2 < ... < m_r < n$.

Apply a simple transformation of parameters. Replace k by $n-k$ taking into account that $0 \le n-k \le n$ and that $C_n^{n-k} - C_n^k$. In a similar way replace r values m_i by $\mu_i = n - m_i$. Based on (1) and (3) the modified formulas appear as:

$$t_1 = C_{\mu_1}^{k} + C_{\mu_2}^{k-1} + ... + C_{\mu_r}^{k-r+1} \text{ and } h_{k+l}^{l}(t_1) = C_{\mu_1}^{k+l} + C_{\mu_2}^{k+l-1} + ... + C_{\mu_r}^{k+l-(r-1)}.$$

When t_1 points T_1 belong to the layer $n-k$ and we consider $H^l(T_1)$ downward by the layers (note), then without major changes of parameters we receive that $h_{n-k-l}^{l}(t_1) = C_{\mu_1}^{k+l} + C_{\mu_2}^{k+l-1} + ... + C_{\mu_r}^{k+l-(r-1)}$ for the value $l = n-2k$ applied.

This representation of t_1 above and the one in (3) is used to formulate a necessary and sufficient condition for existence of set collections under the Condition 1.

Theorem 5. *A necessary and sufficient condition of existence of collection* $\{\Xi_1, \Xi_2, ..., \Xi_m\}$ of an n-element set, with characteristics $t_1 \ge t_2 \ge ... \ge t_m$ and with Condition 1., is the relation

$$t_2 \le C_n^k - C_{\mu_1}^{k+l} - C_{\mu_2}^{k+l-1} + ... + C_{\mu_p}^{k+l-(p-1)}.$$

The theorem, initially, can be given in terms of a two set collection, $m = 2$, by the Theorem 4. To prove the postulation it is essential to know the real volume of points projected from the t_1 set $T_{n-k}^{n}(t_1)$ onto the layer k.

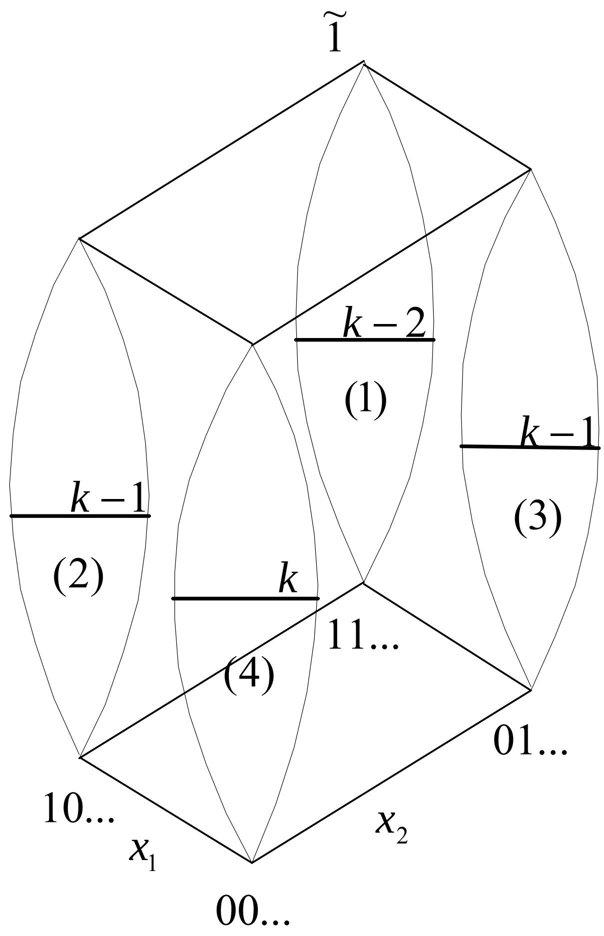
Suppose, that the desired pair $\{\Omega_1, \Omega_2\}$ with (t_1, t_2) exists. Then, by Theorem 1. $C_n^k - \left|H^l(\overline{\Omega}_1)\right| \ge t_2$.

Write the Consequence 2. inequality for this case:

$$\left|H^l(\overline{\Omega}_1)\right| \ge C_n^k - C_{\mu_1}^{k+l} - C_{\mu_2}^{k+l-1} + ... + C_{\mu_p}^{k+l-(p-1)}.$$

Further we come with this to the necessity point of theorem. Concluding the inequalities we receive the requirements

$$t_2 \le C_n^k - C_{\mu_1}^{k+l} - C_{\mu_2}^{k+l-1} + ... + C_{\mu_p}^{k+l-(p-1)}.$$



To consider the sufficiency part, and suppose that the last inequality is valid. Take $\Omega_1 = T^n_{n-k}(t_1)$. Now the possibility to choose a set Ω_2 that consists of t_2 elements and obeys the Condition 1. follows from the given inequality, taking into accounts considerations with complementary subsets, which was used regularly in the given descriptions.

Theorem 5 gives a technique to check the cross intersection for given (t_1, t_2). We can consider the problem of maximising the $t_1 + t_2$. Increasing t_1 points that we have on the layer $n-k$, and composing the shadow of set $T^n_{n-k}(t_1)$ to the layer k we may take all the reminder part as the set $L^n_k(t_2)$.

Maximum is when these quantities are approximately equal. Practically there is a very simple construction explaining this convergence. Consider the two dimension split of the unite cube. Increase t_1 and consider the corresponding set $L^n_k(t_1)$. Find for this the corresponding maximal value of t_2. $L^n_k(t_1)$ starts by the point from (1) continued then by (2), (3), and (4). When it is in area of (1), then t_2 maximum equals t_1 but still this is not the total maximum. The same postulation is also true for area (1)+(2). Here intersection is provided by the value $x_1 = 1$.

Consider the next to the (1)+(2) vertex $\tilde{\alpha}$. $\tilde{\alpha}$ starts with 01 followed by $k-1$ entities of 1, and then 0's. $\tilde{\alpha} = 01 \quad \underbrace{11...1}_{k-1} \quad 00...0$. Due to condition $k < n/2$ number of right 0's are not less than $k-1$ so that $k-1$ 1's can be shifted right without an intersection by the initial set of 1 coordinates. Do this shift, and replace first 2 coordinates by 10. We receive a vertex which belongs to (2) $\tilde{\beta} = 10 \quad 00...0 \quad \underbrace{11...1}_{k-1}$. It is easy to check that $\tilde{\alpha}$ and $\tilde{\beta}$ are non-intersecting vertices which proves that $t_1 \le C^{k-1}_{n-1}$ when $m \ge 2$. Further increase of Ω_1 leads to a similar decrease of Ω_2. This construction appears in Lemma 2.2 of [5] proving the inequality indicated in (5).

Conclusion

The set theoretical issue of complete cross intersecting set systems is considered. This is one of cases of applied societies' connectivity model but variations of models are possible and their analysis come to similar set theoretical optimizations. The paper can be characterised as methodological and it continues the line of possible application of Discrete Isoperimetry Property started at [5]. [8] represents another important application in area of search engines. The cross intersection topic itself is not yet expired. Extensions to consider include nodes with different number of connections to relays. Instead of artificial requirement that nodes are different by their connections to relays, more realistic models are required. Optimization of use of relays and their balancing can be studied which can bring natural conditions of nodes to be different inside the societies. The proof technique will move from the flat layer k consideration to the space constructions in entire cube E^n.

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