INDIVIDUAL-OPTIMUM EQUILIBRIUMS IN GAMES WITH FUSSY PURPOSES OF PLAYERS

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***Abstract**: The notion of fuzzy individual-optimal equilibrium, where every player is considering the interests of fuzzy set of other players, was defined. The notion of a union of fuzzy set of clear relations is put in for this purpose. Structural formulas for the construction of a function of belonging of this relation are developed. The Connection of fuzzy equilibrium with the set of individual-optimal equilibriums was researched and the existence of maximizing fuzzy equilibrium was defined.*

***Keywords:** fuzzy set, fuzzy goal, fuzzy game, membership function, fuzzy set of type 2, decision making.*

***ACM Classification Keywords:** I. Computing Methodologies – I.6. Simulation and modeling (*Time series analysis*) – I.6.8 Types of Simulation – Gaming.*

Introduction

The notion of Nesh’s equilibrium found wide application in the decision of many applied problems in conflict conditions. Its “absolute absence of compromise” is the peculiarity of Nesh’s equilibrium. If there is the sole situation of a game which allows players to adhere to “optimum strategies”, it can indisputably be the basis of stable agreement between players. But, firstly, numerous examples show that there can be situations which are “better” than Nesh’s equilibrium, and in order for these situations to become stable, players must agree to the compromise. Secondly, when Nesh’s equilibriums doesn’t exist, or, opposite, them – much, on the basis of compromise between players it is possible to build a stable agreement between them. Thirdly, often enough in the real conflicts players a priori are in compromise relations and the question lies in how to make it more stable.

As the classic theory of compromises is created only for the collective conduct of players, the problem of its expansion in case of their non-cooperative conduct is topical. In the work, non-cooperative games in which players can choose the strategies individually are explored, but, unlike the classic theory, fussy interests of the partners are taken into account.

Individual-optimal equilibriums

Let’s consider the general game G in the normal form $(X_i, R_i; i \in N)$, where $N = \{1,2,...,n\}$ - is a set from n players; X_i - a set of player $i \in N$ ’s strategies; R_i - a complete binary preference relation of player $i \in N$, which is definite on the set $X = \prod_{i \in N} X_i$ of situations. Let’s consider that the game takes place in the conditions of being completely informed. We will also consider that players operate not cooperatively, that is, everybody chooses the strategies independently.

One of optimum principles in non-cooperative games is the concept of individual optimum [Mashchenko, 2009].

According to this concept every player chooses the strategies individually (non-cooperatively), but takes into account the interests of all other players (compromise for the sake of resolution of conflict).

The players' binary preference relations $R_i, i \in N$, are shown by the aggregated relation $R = \bigcup_{i \in N} R_i$ for the formalization of the individual-optimum equilibrium notion. Obviously, the relation R will also be full. Let S - be the prevailing relation, induced by the aggregated preference relation R. Then $S = \overline{R^{-1}} = \bigcap_{i \in N} S_i$, where $S_i = \overline{(R_i)^{-1}}$ - is the player prevailing relation is induced by the preference relation R_i.

We will say that situations $x, y \in X$ are in relation of a strong *NE*-prevailing of player $i \in N$, that is generated by the aggregated prevailing relation *S*, and to mark it $xS^{NE(i)}y$, if $xSy \wedge (x_{N \setminus i} = y_{N \setminus i})$.

We will name a situation x^* the weak individual-optimum equilibrium of game *G* [Mashchenko, 2009] (we will mark their set through *WIOE*), if $y\overline{S^{NE(i)}}x^*, \forall y \in X, \forall i \in N$. Stability of individual-optimum equilibriums is grounded in a so called one-purpose game. In this game at all players have one purpose, but it is characterized for every player by the preference relation. Ideally, this purpose consists in the choice player strategies so that there is a most preferable situation for all players. Because such situations do not exist often enough, players agree to go on a compromise for the sake of the general purpose. Because the players operate non-cooperatively, each of them can see this compromise in his or her own way, which leads to conflict.

In that and only in that case, when a weak individual-optimum equilibrium x^* will be the basis of an agreement between players, the change by any player $i \in N$, agreed with other players, strategies x_i^* to another, will always result in a situation which will not prevail x^* at least for one player (him in particular). That is player $i \in N$'s purpose, which consists of his personal interests and interests of other players which he takes into account, can be not satisfied and an attained during previous negotiations compromise can be blasted.

In this work the notion of weak individual-optimum equilibrium is summarized for the case, when players can fuzzily take into account their partners' interests. In other words, every player cannot confidently say that he will search for a compromise with the other players, but he can set the fuzzy set of players, the interests of whom he is going to take into account. Let $\eta_i : N \to [0,1]$ - be a belonging function of players $\tilde{N}_i$ fuzzy set, the interests of which player $i \in N$ is going to take into account. For the formalization of the individual-optimum equilibrium notion in the general game *G* for every player $i \in N$ we aggregate players' $j \in \tilde{N}_i$ preference relation R_j in the preference relation of their association $\tilde{R}_{N_i} = \bigcup_{j \in \tilde{N}_i} R_j$. The acquired relation is a fuzzy set $\tilde{N}_i$ union of clear relations R_j, $j \in N$, is a new notion, requires determination and research.

The union of fuzzy set of clear relations

Let's formalize the operation of union $\tilde{R} = \bigcup_{i \in \tilde{N}} R_i$ of the fuzzy set $\tilde{N}$ of clear relations R_j, $j \in N$, where $\tilde{N}$ - is a fuzzy set with the belonging function $\eta : N \to [0,1]$. Let $r_j : X \times X \to \{0,1\}$ - be characteristic function of relation R_j, $j \in N$ (that is $xR_jy \Leftrightarrow r_j(x,y) = 1$). For each $x, y \in X$ we will mark:

$$N^{PO}(x,y)=\{i\in N|\ \nexists j\in N:(r_j(x,y),\eta(j))\succ(r_i(x,y),\eta(i))\} \quad (1)$$

- set players indexes which do not get better in a increasing characteristic functions $r_i(x,y)$ of relations R_i, $i\in N$, and belonging functions $\eta(i)$ of fuzzy set $\tilde{N}$;

$$\tilde{\eta}(x,y,i)=\begin{cases}\eta(i),\ i\in N^{PO}(x,y),\\ 0,\quad i\notin N^{PO}(x,y),\end{cases} \quad (2)$$

-fuzzy subset of set N belonging function with a transmitter $N^{PO}(x,y)$.

A next definition will be related to the known [Mashchenko, 2010] fuzzy set of type 2 notion, the value of belonging function of which is the fuzzy set in classic sense (type 1).

By the union of fuzzy set $\tilde{N}$ of fuzzy relations, according to [Mashchenko, 2010] we will name - fuzzy relation of type 2, which is defined on the set X and fixed by the three $(x,y,r(x,y,z))$, where

- $r:X\times Y\times Z\rightarrow[0,1]$ - is the unclear reflection $\Re$ belonging function, which executes the role of fuzzy belonging function of which is definite thus:

$$r(x,y,z)=\begin{cases}\max\limits_{i\in N}\{\tilde{\eta}(x,y,i))|\ r_i(x,y)=z\},\quad \exists i\in N:r_i(x,y)=z,\\ \qquad 0,\qquad\qquad r_i(x,y)\neq z,\forall i\in N.\end{cases} \quad (3)$$

- $x,y\in X$ - pair of game situations;
- z - element of the universal set $Z=\{0,1\}$ of belonging reflection $\Re$ values of type 2 fuzzy relation $\tilde{R}$.

Values of fuzzy belonging reflection $\Re$ for the fixed situations $x^0,y^0\in X$ pair form a fuzzy subset $\Re_Z(x^0,y^0)$ of the $Z=\{0,1\}$ set with the belonging function $r(x^0,y^0,z)$. The value $r(x^0,y^0,1)$ can be understood as a degree of that the x^0,y^0 pair is in the relation $\tilde{R}$. Accordingly the value $r(x^0,y^0,0)$ has the sense of not belonging degree of the pair x^0,y^0 to the relation $\tilde{R}$.

On the other hand, if in the functions $r(x,y,z)$ are fixed $z=1$, then we will get the belonging function $r(x,y,1)$ of alternatives x,y pairs fuzzy set which are found in the relation $\tilde{R}$. Let's indicate this set $\Re_X(1)$. Analogical, for the fixed value $z=0$ we will get the alternatives x,y pairs fuzzy set which are not found in the relation $\tilde{R}$, with the belonging function $r(x,y,0)$. We will indicate it $\Re_X(0)$. Interestingly, that in the general case $\Re_X(0)\neq\overline{\Re_X(1)}$, and, accordingly $r(x,y,0)\neq 1-r(x,y,1)$.

A next theorem allows structurally to build the belonging function $r(x,y,z)$.

Theorem 1. Let R_i, $i\in N$, - be clear relations which are set on the set X by the appropriate characteristic functions $r_i(x,y)$, $x,y\in X$, $i\in N$; $\eta(i)$, $i\in N$, - fuzzy set $\tilde{N}$ belonging function. For the fuzzy set $\tilde{R}$ of type 2, which is set by the fuzzy reflection $\Re$ with the belonging function $r(x,y,z)$; $x,y\in X$; $z\in[0,1]$, to be the union of fuzzy set $\tilde{N}$ relations R_i, $i\in N$, that is $\tilde{R}=\bigcup\limits_{i\in\tilde{N}}R_i$, it is necessary and it is enough, for $x,y\in X$:

$$r(x,y,1)=\begin{cases}\max\limits_{r_i(x,y)=1}\eta(i), & \exists i\in N: r_i(x,y)=1,\\ 0, & r_i(x,y)=0,\forall i\in N,\end{cases}$$
$$r(x,y,0)=\begin{cases}\max\limits_{i\in N}\eta(i), & r_i(x,y)=0,\forall i\in \operatorname{Argmax}\limits_{i\in N}\eta(i),\\ 0, & \exists i\in \operatorname{Argmax}\limits_{i\in N}\eta(i): r_i(x,y)=1.\end{cases} \quad (4)$$

Proof. We will show at first, that formula (3) is equivalent to such

$$r(x,y,z)=\begin{cases}\max\limits_{i\in N^{PO}(x,y,z)}\eta(i), & N(x,y,z)\neq\varnothing,\\ 0, & N(x,y,z)=\varnothing,\end{cases} \quad (5)$$

where for $\forall x,y\in X\ \ \forall z\in[0,1]$

$$N(x,y,z)=\{i\in N \mid\ z=r_i(x,y)=\max_{\eta(j)\geq\eta(i)} r_j(x,y),\ \eta(i)=\max_{r_j(x,y)\geq r_i(x,y)}\eta(j)\} \quad (6)$$

We will note that from (2), (3) follows, that

$$\tilde{R}=\bigcup_{i\in\tilde{N}}R_i \Leftrightarrow r(x,y,z)=\begin{cases}\max\limits_{i\in N^{PO}(x,y)}\{\eta(i)\mid\ r_i(x,y)=z\}, & \exists i\in N^{PO}(x,y): r_i(x,y)=z,\\ 0, & r_i(x,y)\neq z,\forall i\in N^{PO}(x,y),\end{cases} \quad (7)$$

$x,y\in X\ \ z\in[0,1]$. Therefore for the proving of equivalence (3) and (5) it is enough to show that (7) is equivalent to (5), (6). Let’s show that

$$N^{PO}(x,y)=\{i\in N \mid r_i(x,y)=\max_{\eta(j)\geq\eta(i)} r_j(x,y),\ \eta(i)=\max_{r_j(x,y)\geq r_i(x,y)}\eta(j)\}\ ,\ x,y\in X \quad (8)$$

Assume that for some $x,y\in X$, $i\in N$, the correlation is executed:

$$r_i(x,y)=\max_{\eta(j)\geq\eta(i)} r_j(x,y),\ \eta(i)=\max_{r_j(x,y)\geq r_i(x,y)}\eta(j) \quad (9)$$

We will assume the opposite, that $i\notin N^{PO}(x,y)$. Then according to (1) $\exists l\in N$, for which $r_l(x,y)>r_i(x,y)$, $\eta(l)\geq\eta(i)$, or $r_l(x,y)\geq r_i(x,y)$, $\eta(l)>\eta(i)$. In the first case, from here follows, that $r_l(x,y)>\max\limits_{\eta(j)\geq\eta(i)} r_j(x,y)$. In the second case, we get $\eta(l)>\max\limits_{r_j(x,y)\geq r_i(x,y)}\eta(j)$, which contradicts (9).

Assume that $i\in N^{PO}(x,y)$. We will assume the opposite, that $r_i(x,y)<\max\limits_{\eta(j)\geq\eta(i)} r_j(x,y)$ or $\eta(i)<\max\limits_{r_j(x,y)>r_i(x,y)}\eta(j)$. In the first case from here follows, that $\exists l\in N$, for which $\eta(l)\geq\eta(i)$, $r_l(x,y)>r_i(x,y)$. Then $(r_l(x,y),\eta(l))\succ(r_i(x,y),\eta(i))$ and $i\notin N^{PO}(x,y)$ from (1). Analogical in the second case, $\exists k\in N$, for which $r_k(x,y)>r_i(x,y),\ \eta(k)\geq\eta(i)$. Then $(r_k(x,y),\eta(k))\succ(r_i(x,y),\eta(i))$ and $i\notin N^{PO}(x,y)$ from (1). Thus, we have obtained the contradiction and there is (8). As from (8) follows $N(x,y,z)=N^{PO}(x,y)\bigcap\{i\in N \mid\ r_i(x,y)=z\}$, then (7) is equivalent to (5), (6), and therefore (3) and (5) are equivalent.

Now for proof of theorem it is sufficient to show the equivalence between formulas (4) and (5).

At first we will write down (6) for $z=1$ in two possible cases. Let’s assume at first $r_i(x,y)=0,\forall i\in N$. Then from (4) $r(x,y,1)=0$. On other hand, from (6) directly follows, that $N(x,y,1)=\varnothing$ and then from (5) $r(x,y,1)=0$.

In the second case, assume that $\exists i \in N : r_i(x,y) = 1$. We will indicate $\eta_1^*(x,y) = \max\limits_{r_j(x,y)=1} \eta(j)$, $I_1^*(x,y) = \{ j \in N \mid \eta(j) = \eta_1^*(x,y) \}$. We will define the value $r(x,y,1)$ from (5). For this purpose we will build from (6) $N(x,y,1) = \{ i \in N \mid 1 = r_i(x,y) = \max\limits_{\eta(j) \geq \eta(i)} r_j(x,y),\ \eta(i) = \eta_1^*(x,y) \}$. We will show that $N(x,y,1) = I_1^*(x,y)$. Assume that $i \in I_1^*(x,y)$. Then $\eta(i) = \eta_1^*(x,y)$ and $\max\limits_{\eta(j) \geq \eta(i)} r_j(x,y) = \max\{ \max\limits_{\eta(j) = \eta_1^*(x,y)} r_j(x,y), \max\limits_{\eta(j) > \eta_1^*(x,y)} r_j(x,y) \} =$ $= \max\{1, \max\limits_{\eta(j) > \eta_1^*(x,y)} r_j(x,y)\} = 1 = r_i(x,y)$. From here it is obvious that $i \in N(x,y,1)$.

On the contrary, assume that $i \in N(x,y,1)$. Then $1 = r_i(x,y) = \max\limits_{\eta(j) \geq \eta(i)} r_j(x,y)$ and $\eta(i) = \eta_1^*(x,y)$. From here follows $i \in I_1^*(x,y)$. Then from (5) $r(x,y,1) = \eta_1^*$. Therefore formulas (4), (5) are equivalent for $z = 1$.

Now we will write down (6) for $z = 0$ in two possible cases. We will indicate $\eta_0^* = \max\limits_{j \in N} \eta(j)$, $I_0^* = \{ i \in N \mid \eta(i) = \max\limits_{j \in N} \eta(j) \}$. Assume at first that $r_i(x,y) = 0, \forall i \in I_0^*$. Then from (4) $r(x,y,0) = \eta_0^*$. We will define the value $r(x,y,0)$ from the formula (5). For this purpose we will build the set $N(x,y,0) = \{ i \in N \mid 0 = r_i(x,y) = \max\limits_{\eta(j) \geq \eta(i)} r_j(x,y),\ \eta(i) = \max\limits_{j \in N} \eta(j) \} = \{ i \in I_0^* \mid 0 = r_i(x,y) = \max\limits_{j \in I_0^*} r_j(x,y) \} = I_0^*$ from the formula (6) . From here according to (5) also $r(x,y,0) = \eta_0^*$.

In the second case, assume that $\exists i \in I_0^* : r_i(x,y) = 1$. Then according to (4) $r(x,y,0) = 0$. We will define the value $r(x,y,0)$ from (5). For this purpose we will build $N(x,y,0) = \{ i \in N \mid 0 = r_i(x,y) = \max\limits_{\eta(j) \geq \eta(i)} r_j(x,y),$ $\eta(i) = \max\limits_{j \in N} \eta(j) = \eta_0^* \} = \{ i \in I_0^* \mid 0 \neq r_i(x,y) = \max\limits_{j \in I_0^*} r_j(x,y) = 1 \} = \varnothing$ from the formula (6). From here according to (5) also $r(x,y,0) = 0$. Therefore formulas (4), (5) are equivalent for $z = 0$. The theorem has been proved.

For the illustration of a fuzzy set of clear sets union notion we will consider such an example.

Example. Assume that $N = \{1,2\}$ - is a set of players. We will set on N a fuzzy set $\tilde{N}$ by the belonging function with the values: $\eta(1) = 0{,}3$, $\eta(2) = 0{,}7$. We will find the union of a fuzzy set $\tilde{N}$ of clear relations R_1, R_2, which are defined on the set $X = \{A,B\}$ and set by characteristic functions, according to $r_1(x,y)$, $r_2(x,y)$ (tabl. 1).

Table 1. Union of fuzzy set of clear relations.

Functions and sets	(A,A)	(A,B)	(B,A)	(B,B)
$r_1(x,y)$	1	0	1	0
$r_2(x,y)$	1	1	0	0
$N^{PO}(x,y)$	{2}	{2}	{1,2}	{2}
$\tilde{\eta}(x,y,1)$	0	0	0.3	0
$\tilde{\eta}(x,y,2)$	0.7	0.7	0.7	0.7

In table 1 the set $N^{\cup}(x,y)$ and belonging function $\tilde{\eta}(x,y,i)$ are also indicated. The values of the fuzzy relation of type 2 belonging function are indicated in table 2.

Table 2. Value of belonging function.

Belonging function	(A,A)	(A,B)	(B,A)	(B,B)
$r(x,y,0)$	0	0	0.7	0.7
$r(x,y,1)$	0.7	0.7	0.3	0

The obtained result, also corresponds with theorem 1.

Fuzzy individual-optimum equilibriums and their choice

Assume that $S=\overline{R^{-1}}$ - prevailing relation which is induced by the aggregated preference relation $R=\bigcup_{i\in N} R_i$; $S^{NE(i)}$ - relation of player $i\in N$ *NE*-prevailing, which is generated by S. We will present the set of weak individual-optimum equilibriums *WIOE* of the general game G in a as $WIOE=\bigcap_{i\in N} BR_i$, where $BR_i=\{x\in X \mid y\overline{S^{NE(i)}}x,\ \forall y\in X\}=\{x\in X \mid (y_i,x_{N\setminus i})\overline{S}x,\ \forall y_i\in X_i\}$, $i\in N$. Thus, the individually optimal player's $i\in N$ conduct with a fixed other players' strategies set $x_{N\setminus i}=(x_j)_{j\in N\setminus\{i\}}$ consists in the choice of strategies which form situations, that are not prevailed according to $S=\overline{R^{-1}}$.

We will pass on to the fuzzy individual-optimum equilibrium notion formalization. Assume that $\tilde{P}_i=\bigcup_{j\in\tilde{N}_i} R_j$ - is a fuzzy preference relation of all the player's association (as shown higher than type 2), the interests of which player $i\in N$ is going to take into account. Let it be set by the fuzzy reflection with the belonging function $p_i(x,y,z)$, $x,y\in X$, $z\in[0,1]$. We will build for a player $i\in N$ the fuzzy set (we will indicate it ND_i) of situations x which are not prevailed after the relation $\tilde{S}_i=\tilde{P}_i\setminus\tilde{P}_i^{-1}$ (which is asymmetric part of $\tilde{P}_i$) by other situations $(y_i,x_{N\setminus i})$, that are obtained from situation x by the change of the strategy x_i by this player on other $y_i\in X_i$.

The fuzzy relation of type 2 $\tilde{S}_i=\tilde{P}_i\setminus\tilde{P}_i^{-1}=\tilde{P}_i\cap\overline{\tilde{P}_i^{-1}}$ belonging reflection (asymmetric part of preference relation $\tilde{P}_i$) is set by function $s_i(x,y,z)=\max_{\substack{z_1,z_2\in\{0,1\},\\ z=\min\{z_1,z_2\}}}\min\{p_i(x,y,z_1),p_i(y,x,1-z_2)\}$ according to operations on the fuzzy sets of type 2 according to [Zadeh, 1973].

We will express $s_i(x,y,0)=\max\{\min\{p_i(x,y,0),p_i(y,x,1),\ \min\{p_i(x,y,0),p_i(y,x,0),\ \min\{p_i(x,y,1),p_i(y,x,1)\}$ in terms of player's $i\in N$ preference relation by means of (4). For this purpose we will consider the following cases.

We will assume that $r_j(x,y)=0$, $\forall j\in N$. Then because of the completeness of player's $j\in N$ preference relation R_j, we will get $r_j(y,x)=1$ for $\forall j\in N$. From here according to (4) $p_i(x,y,0)=\max_{j\in N}\eta_i(j)$, $p_i(x,y,1)=0$, $p_i(y,x,0)=0$, $p_i(y,x,1)=\max_{j\in N}\eta_i(j)$. Therefore $s_i(x,y,0)=\max_{j\in N}\eta_i(j)$.

We will assume that $r_j(y,x)=0$, $\forall j\in N$. Then because of completeness of player's $j\in N$ preference relation R_j, $j\in N$, obsessed $r_j(x,y)=1$, $\forall j\in N$. From here after (4) $p_i(x,y,1)=\max_{j\in N}\eta_i(j)$, $p_i(x,y,0)=0$, $p_i(y,x,1)=0$, $p_{ii}(y,x,0)=\max_{j\in N}\eta_i(j)$. Therefore $s_i(x,y,0)=0$.

We will assume that $\exists j\in N:\ r_j(x,y)=1$ and $\exists k\in N:\ r_k(y,x)=1$. Then it is quite clear, that $p_i(x,y,1)=\max\{\eta_i(j)\,|\,j\in N, r_j(x,y)=1\}$ and also $p_i(y,x,1)=\max\{\eta_i(k)\,|\,k\in N, r_k(y,x)=1\}$. We will indicate $K_i=\{k\in N\,|\ \eta_i(k)=\max_{j\in N}\eta_i(j)\}$ and we will consider following cases.

In the first case, assume that $r_j(y,x)=0$, $\forall j\in K_i$. Then because of the completeness of player's $j\in N$ preference relation R_j, $j\in N$, we get $r_j(x,y)=1$, $\forall j\in K_i$. Therefore $p_i(x,y,0)=\max_{j\in N}\eta_i(j)$, $p_i(y,x,0)=0$. Hence we have $s_i(x,y,0)=\max\{\max_{r_k(y,x)=1}\eta_i(k),\ 0,\ \min\{\max_{r_j(x,y)=1}\eta_i(j),\ \max_{r_k(y,x)=1}\eta_i(k)\}\}$. Because $\max\{a,\min\{a,b\}\}=a$, then finally we get $s_i(x,y,0)=\max_{r_k(y,x)=1}\eta_i(k)$.

In the second case, assume that $r_j(x,y)=0$, $\forall j\in K_i$. Then because of the completeness of players' $j\in N$ preference relation R_j, $j\in N$, we have $r_j(y,x)=1$, $\forall j\in K_i$. Therefore $p_i(y,x,0)=\max_{j\in N_i}\eta_i(j)$, $p_i(x,y,0)=0$. Then we get $s_i(x,y,0)=\max\{0,0,\ \min\{\max_{r_k(y,x)=1}\eta_i(k),\ \max_{r_j(x,y)=1}\eta_i(j)\}\}$. Because there is $r_j(y,x)=1$, $\forall j\in K_i$, then $\max_{r_k(y,x)=1}\eta_i(k)\geq\ \geq\max_{r_j(x,y)=1}\eta_i(j)$. Therefore $s_i(x,y,0)=\max_{r_k(y,x)=1}\eta_i(k)$.

In the third case, assume that $\exists j\in K_i:\ r_j(x,y)=1$ and $\exists k\in K_i:\ r_k(y,x)=1$. Then $p_i(x,y,0)=0$, $p_i(x,y,0)=0$. Therefore $s_i(x,y,0)=\max\{0,0,\min\{\max_{r_j(x,y)=1}\eta_i(j),\ \max_{r_k(y,x)=1}\eta_i(k)\}\}$. Because according to condition $\exists j\in K_i$, for which $r_j(x,y)=1$, then $\max_{r_k(x,y)=1}\eta_i(k)=\max\{\eta_i(j)\,|\,j\in N\}$. Therefore $s_i(x,y,0)=\max_{r_k(y,x)=1}\eta_i(k)$.

From the above considered cases it is obvious, that

$$s_i(x,y,0)=\begin{cases}\max\limits_{r_j(y,x)=1}\eta_i(j), & \exists j\in N: r_j(y,x)=1,\\ 0, & r_j(y,x)=0, \forall j\in N,\end{cases} \tag{10}$$

It would be logical to define the set ND_i based on the following reasoning.

Because the value $s_i(x,y,0)$ is a degree, with which situation y is not prevailed by x, then with the fixed variable $y\in X$ the function $s_i(y,x,0)$ can be considered a fuzzy set belonging function of all situations x which are not prevailed by situation y. From here follows, that the subset of situations, each of which is not prevailed by any of the situations of set X, can be the set by the belonging function $\min_{y\in X}s_i(y,x,0)$, $x\in X$. Thus, we see that the fuzzy set ND_i will be set by the belonging function $\mu_i: X\to[0,1]$ of such a kind

$$\mu_i(x)=\min_{y_i\in X_i}s_i((y_i,x_{N\setminus i}),x,0),\ x\in X \tag{11}$$

The value $\mu_i(x)$ can be understood as a “degree of not being prevailed” of the situation $x\in X$ for the player $i\in N$ by another situation $(y_i,x_{N\setminus i})$ which is obtained by his change of strategy x_i to other $y_i\in X_i$. The idea of the set $FIOE=\bigcap_{i\in N}ND_i$ which consists of not prevailed situations by fuzzy relations S_i, $i\in N$, results in the following definition.

The fuzzy set with the belonging function $\mu : X \to [0,1]$ of kind $\mu(x) = \min_{i \in N} \mu_i(x)$, $x \in X$, we will name the set of fuzzy individual-optimum equilibriums of the game $\tilde{G}$ and mark $FIOE$. We will name the set $\operatorname{supp}(FIOE) = \{x \in X \mid \mu(x) > 0\}$ a transmitter of $FIOE$.

We will set the connection of sets: $FIOE$ - fuzzy individual-optimum equilibriums of the game $\tilde{G}$ and $WIOE$ - weak individual-optimum equilibriums of the game G.

Theorem 2. The transmitter $\operatorname{supp}(FIOE)$ of fuzzy individual-optimum equilibriums set of the game $\tilde{G}$ coincides with the weak individual-optimum equilibriums set $WIOE$ of the game G.

Proof. Assume that $x \in WIOE$. We will assume the opposite, i.e. that $x \notin \operatorname{supp}(FIOE)$. Then according to the definition $\exists i \in N$: $\mu_i(x) = 0$. From here according to the formula (11) $\exists y_i \in X_i$, that $\min_{y_i \in X_i} s_i((y_i, x_{N\setminus i}), x, 0) = 0$. Therefore according to (10) $r_j\,(x,(y_i, x_{N\setminus i})) = 0,\ \forall j \in N$. Then relations $x\bar{R}_j\,(y_i, x_{N\setminus i})$, $\forall j \in N$, are executed. From here $x\bar{R}(y_i, x_{N\setminus i})$. Therefore, because of completeness of relation R, we will get $(y_i, x_{N\setminus i})Sx$. Then for the situation $y = (y_i, x_{N\setminus i})$ the relation $yS^{NE(i)}x$ takes place and according to the definition $x \notin WIOE$. We have a contradiction and therefore $WIOE \subseteq \operatorname{supp}(FIOE)$.

Assume that $x \in \operatorname{supp}(FIOE)$. We will assume opposite, that $x \notin WIOE$. Then according to the definition $\exists i \in N\ \exists y = (y_i, x_{N\setminus i})$, for which $yS^{NE(i)}x$. It means that $\exists y_i \in X_i$, for which $(y_i, x_{N\setminus i})Sx$. Therefore $x\bar{R}(y_i, x_{N\setminus i})$. From here follows $x\bar{R}_j\,(y_i, x_{N\setminus i})$, $\forall j \in N$, that means $r_j\,(x,(y_i, x_{N\setminus i})) = 0,\ \forall j \in N$. From here according to formula (10) $s_i((y_i, x_{N\setminus i}), x) = 0$. Therefore according to (11) $\mu_i(x) = 0$. Then according to the definition $x \notin \operatorname{supp}(FIOE)$. A contradiction was obtained. Thus $\operatorname{supp}(FIOE) \subseteq WIOE$ and therefore $\operatorname{supp}(FIOE) = WIOE$. The theorem has been proved.

Because players, as a rule, are interested in the choice of some sole situation of a game which would become the basis of stable agreement between them, they need to choose the fuzzy individual-optimum equilibrium x^* with the maximal degree $\mu(x^*)$ of not prevailing. This reasoning leads to the following notion.

We will name $x^* \in X$ the maximizing fuzzy individual-optimum equilibrium of game $\tilde{G}$, if $\mu(x^*) = \max_{x \in X} \mu(x)$.

It is easy to check that for the general game $\tilde{G}$ there is always a maximizing fuzzy individual-optimum equilibrium. Indeed, according to definitions $\mu(x^*) = \max_{x \in X} \min_{i \in N} \mu_i(x) = \ = \max_{x \in X} \min_{i \in N} \min_{y_i \in X_i} s_i((y_i, x_{N\setminus i}), x, 0)$. From the formula (10) it is easy to see that for $\forall x \in X\ \ \forall i \in N\ \ \forall y_i \in X_i$ the function $s_i((y_i, x_{N\setminus i}), x, 0)$ adopts a finite set of volumes. From here follows, that there is always $\max_{x \in X} \min_{i \in N} \min_{y_i \in X_i} s_i((y_i, x_{N\setminus i}), x, 0)$.

It should be noted that although a maximizing fuzzy individual-optimum equilibrium x^* exists always, it can be that $\mu(x^*) = 0$. Therefore we will consider the following theorem.

Theorem 3 . Assume that $\eta_i(j) \neq 0$, $\forall i \in N$. If a situation x^* meets condition

$$\mu(x^*) = \max_{x \in X} \min_{i \in N} \min_{y_i \in X_i} \max_{r_j\,((y_i, x_{N\setminus i}), x) = 1} \eta_i(j) \tag{12}$$

then it is the maximizing fuzzy individual-optimum equilibrium of game $\tilde{G}$, thus $\mu(x^*) > 0$.

If a situation x^* is the maximizing fuzzy individual-optimum equilibrium of game $\tilde{G}$, then it satisfies (12).

Proof. We will consider the problem

$$\mu(x) = \min_{k \in N} \min_{y_k \in X_k} \max_{r_i^j((y_i, x_{N \setminus i}), x) = 1} \eta_i(j). \tag{13}$$

We will mark Q the set of its decisions. For proof of the theorem it is sufficient to show that $Q = \mathrm{supp}(FIOE)$.

Assume that $x \in Q$, and values i, y_i^*, j^* satisfy (13). We will show that $x \in WIOE$. From (13) follows, that $\eta_i(j^*) \ge \eta_i(j)$, $\forall j \in M_i(x, (y_i^*, x_{N \setminus i}))$, where $M_i(x, (y_i^*, x_{N \setminus i})) = \{ j \in N \mid r_j(x, (y_i^*, x_{N \setminus i})) = 1 \}$. Thus

$$M_k(x, (y_k, x_{N \setminus k})) \supseteq M_i(x, (y_i^*, x_{N \setminus i})), \ \forall x \in X, \ \forall k \in N, \ \forall y_k \in X_k \tag{14}$$

We will consider the following cases.

1. Let's assume that $M_i(x, (y_i^*, x_{N \setminus i})) = N$. Then $r_j(x, (y_i^*, x_{N \setminus i})) = 1$ for $\forall j \in N$. Because $M_k(x, (y_k, x_{N \setminus k})) \supseteq M_i(x, (y_i^*, x_{N \setminus i}))$ for $\forall k \in N$, $\forall y_k \in X_k$, then $r_j(x, (y_k, x_{N \setminus k})) = 1$ for $\forall k \in N$, $\forall j \in N$, $\forall y_k \in X_k$. Therefore $xR_j(y_k, x_{N \setminus k})$ for $\forall k \in N$, $\forall j \in N$, $\forall y_k \in X_k$. From here $xR(y_k, x_{N \setminus k})$ for $\forall k \in N$, $\forall y_k \in X_k$. Thus, $y\overline{S^{NE(k)}}x^*$, $\forall y \in X$, $\forall k \in N$ and according to the definition $x \in WIOE$.

2. Assume that $M_i(x, (y_i^*, x_{N \setminus i})) \subset N$. We will assume opposite, that $x \notin WIOE$. Then according to the definition $\exists i \in N \ \exists y = (y_i, x_{N \setminus i})$, for which $yS^{NE(i)}x$. It means that $\exists y_i \in X_i$, for which $(y_i, x_{N \setminus i})Sx$. Therefore $x\overline{R}(y_i, x_{N \setminus i})$. From here follows, that $x\overline{R}_j(y_i, x_{N \setminus i})$, $\forall j \in N$, that means $r_j(x, (y_i, x_{N \setminus i})) = 0, \ \forall j \in N$. From here we will get $M_i(x, (y_i, x_{N \setminus i})) = \varnothing$ that contradicts (14). Therefore $x \in WIOE$. Thus $Q \subseteq WIOE$.

Assume that $x \in WIOE$. We will show that $x \in Q$. If $x \in WIOE$, then according to the definition $y\overline{S^{NE(k)}}x^*$, $\forall y \in X$, $\forall k \in N$. Therefore for $\forall k \in N$, $\forall y_k \in X_k$ $(y_k, x_{N \setminus k})\overline{S}x$ takes place. From here, because of asymmetric of the relation S, the preference relation $R = \overline{S}^{-1}$, therefore $xR(y_k, x_{N \setminus k})$. Because $R = \bigcup_{j \in N} R_i$, then $\exists j \in N$: $xR_j(y_k, x_{N \setminus k})$. That is $\forall k \in N$ $\forall y_k \in X_k$ $\exists j \in N$: $r_j(x, (y_k, x_{N \setminus k})) = 1$. Hence according to (9) $s_k(x, y, 0) = \max\{ \eta_k(j) \mid r_j((y_k, x_{N \setminus k}), x) = \ = 1 \}$ for $\forall k \in N$, $\forall y_k \in X_k$. Thus, according to the definition of fuzzy individual-optimum equilibrium and formula (13) we will see, that x satisfies (13). Hence $x \in Q$. Thus $Q \supseteq WIOE$, therefore $Q = WIOE$. Then after the theorem 2 $Q = \mathrm{supp}(FIOE)$. The theorem has been proved.

Conclusion

The fuzzy individual-optimum equilibriums considered in the given work allow players to make the stable agreement in which they can fuzzily take into account the interests of one other. It allows substantially simplifying the problem of choice of concrete individual-optimum equilibrium due to the use by the player of subjective estimations of importance of interests of partners which are expressed by a fuzzy set belonging function of players' interests which he is going to take into account. It should also be mentioned that the notion of fuzzy individual-optimum equilibriums will be correct and also has definite interest in games with the purposes of players, which are set by fuzzy preference relations.

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