MODELING AND CONTROL OF LYAPUNOV EXPONENTS IN A COUPLED MAP LATTICE

Vitaliy Yatsenko, Olexandr Kochkodan

***Abstract**: We describe a control method based on optimization techniques to control of spatiotemporal chaos in a globally coupled map lattice (CML) system. We have developed a method for updating a CML model emulating complex spatial dynamics in an epileptic brain that exhibits characteristic spatiotemporal changes seen during transitions into a seizure susceptible state. Our updating algorithm uses metaheuristic techniques for obtaining feedback control parameters for controlling spatiotemporal chaos (local and global Lyapunov exponents). This methodology can be used in systems with hidden variables, i.e. where not all variables can be observed, such as the human brain, to reconstruct evolution maps and complex spatial patterns. Results from numerical simulations show that this algorithm is robust and effective in achieving controllability of the lattice model. We discuss the computational aspects of this learning methodology and its potential application in epileptic seizure control.*

***Keywords:** modeling, optimization, Lyapunov exponent, control, chaos, oscillator*

***ACM Classification Keywords:** 1.6. Simulation and modeling, H.1 Models and principles, Optimization*

Introduction

The presence of chaos has been extensively demonstrated in natural and technical systems. Recently, controlling chaos has attracted increased attention over the past few decades due to its broad applications in physical, chemical and biological systems. One of the important and challenging areas for the application of chaos control techniques is the problem of treatment of neurological disorders such as epilepsy. Characterization of the electroencephalogram (EEG) using measures of chaos has been very useful in providing information about the dynamical state of the epileptic brain. Studies in humans [Iasemidis, 1990; Iasemidis,1991; Iasemidis,, 2004; Yatsenko, 2004] and animal models [Nair, 2004] of epilepsy suggest that occurrence of spontaneous seizures correlates with the evolution of the brain to a more temporally ordered state. This has been demonstrated by changes in gross system properties such as Lyapunov exponents calculated from EEG recorded from multiple brain regions. It has been postulated that such spatiotemporal transitions occur due to self organizing transitions in the epileptic brain that drives it from chaos to order. Furthermore, seizures have been considered to be inherent resetting mechanisms of the brain to revert it back from order to the chaotic regime [Iasemidis, 1996]. It is hence obvious as to why the maintenance of chaos in such biological systems is extremely desirable because of its implications from a therapeutic or control point of view. More recently the concept of "anticontrol" schemes, where the goal is the maintenance of chaos, has been the topic of much investigation and several algorithms have been devised to realize this objective [Ramaswamy, 1998; Wang, 2000; Morgu,l 2003]. The design and adaptation of such chaos control algorithms to dynamical systems that exhibit exceedingly complex dynamics, such as the brain, is not a trivial challenge.

Biological systems are inherently adaptive in nature and therefore require adaptive control techniques [Glass,1988]. Recent investigations in human epilepsy have suggested that effective modulation of brain dynamics needs new control techniques that rely on robust prediction and adaptive optimization methods [Iasemidis, 2000; Yatsenko, 2004; Pardalos, 2004; Iasemidis, 2003]. In this paper we will mainly concentrate on the theoretical problem of controllability of a system property that has shown to reflect state changes in neural systems, namely the Lyapunov exponent. Possible real world implications of controlling the Lyapunov spectrum include being able to control the convergence and divergence of this entity as quantified by the statistical measure T-index among different regions in an epileptic brain, which has been shown to be predictive of a seizure susceptible state [Yatsenko, 2004; Kaneko, 1984]. We reformulate the problem of controllability of the Lyapunov exponent as an optimization problem in which we try to estimate the control parameters by minimizing the error function calculated from the global Lyapunov exponents of a system.

The paper is organized as follows. We start by introducing the problem of control of a dynamical system and the need for optimization based techniques for choosing the optimal feedback parameters. In the next section, we introduce the problem of calculation of Lyapunov exponents and present some general feedback schemes for controlling the mean Lyapunov exponent from a system of globally coupled nonlinear maps. We also describe a constrained optimization technique to solve for the optimal feedback parameters. We then present some numerical and experimental results and finally propose a learning scheme based on optimized feedback parameter selection to simulate dynamics of an epileptic brain using a globally coupled map lattice system.

Control Of Chaos In Coupled Lattice Systems

Coupled Map Lattice. A coupled map lattice is a *N*-dimensional network of interconnected units where each unit evolves in time through a map (or recurrence equation) of the discrete form:

$$X^{k+1} = F(X^k), \tag{1}$$

where X^k denotes the field value (*N*-dimensional vector) at the indicated time *k*. In the case of a globally coupled map, with a global (mean field) coupling factor ε, the dynamics can be rewritten as:

$$x_n^{k+1} = (1-\varepsilon) f_n\left[x_n^k\right] + \frac{\varepsilon}{L}\sum_{j=1}^{L} f_j\left[x_j^k\right], \tag{2}$$

where n and j are the labels of lattice sites (j ≠ n). The term L indicates over how many neighbors we are averaging and it is sometimes referred to as coordination number. The local N-dimensional map is assumed to be chaotic. Completely synchronous chaotic states are possible with this model when corresponding N-dimensional manifolds are attracting or stable. The criterion for stability of this synchronization manifold has been derived in [Ding, 1997]. Further stability analysis of synchronized periodic orbits in coupled map lattices can be found in [Amritkar, 1991]. Varying ε and L we can change the extent of spatial correlations, from systems with local interactions to systems with long-range interactions. These systems typically exhibit spatially and/or temporally chaotic behavior, the control of which is very desirable because of its potential real-life applications. Several strategies have been proposed to control the collective spatiotemporal dynamics of such systems. In this paper we first describe adaptive feedback control strategies for coupled map lattice systems and then describe an optimization technique for choosing optimal feedback parameters.

Model Updating. The main idea behind controlling dynamical systems is to control apparently abrupt and intermittent transitions between dynamical modes of operation that are the mainstay of nonlinear chaotic systems. Some of the goals to be met while controlling spatiotemporal systems include formation of specific spatiotemporal patterns, stabilization of behavior, synchronization/desynchronization, suppression/enhancement of chaos, etc. The goal behind this adaptive feedback strategy is to control some specific property of the system. The controllers are applied in the feedback loops associated with every cell in the lattice structure, based on the internal states of the system. The control input U* to the system can be defined as follows:

$$U^{*k} = G(\psi^{*k} - \psi^k), \tag{3}$$

where *G* is the stiffness of control and *ψ** and *ψ* are the desired and estimated values of the system property respectively. The target value of the system property can either be a constant or a time varying function. In the case of a multidimensional system, *ψ* could be a global property of the system or some property of individual subsystems.

Optimization of Feedback Parameters. Next we introduce a performance function that can also be termed as an error function given below by equation (4) that calculates the error between the target value of the system property and the computed value at each time step. The goal of optimization is to minimize this error by choosing the most optimal feedback parameter for the system. If we choose the mean Lyapunov exponent $\overline{\lambda}$ as the monitored system property, then the function to be minimized is as follows:

$$\xi = \lambda^* - \overline{\lambda}^k, \tag{4}$$

where λ^* is the target value of the Lyapunov exponent and for a coupled map lattice we can define the mean Lyapunov exponent as

$$\overline{\lambda}^k = \frac{1}{L}\sum_{n=1}^{L} \lambda_n^k \tag{5}$$

Control Algorithms

Calculation of Lyapunov Exponents. When the objective is to maintain a desired level of chaoticity, a natural choice for the monitored property of the controlled system is the Lyapunov exponent. The global Lyapunov exponent for a discrete one dimensional system $x^{k+1} = f(x^k)$ can be defined by:

$$\lambda = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \ln\left| f'(x^k) \right| \tag{6}$$

In order to study the evolution of Lyapunov exponents of a coupled map lattice system as described by equation (2), we first introduce the Jacobian matrix as follows:

$$\wp^k = \begin{bmatrix} \frac{\partial x_1^{k+1}}{\partial x_1^k} & \frac{\partial x_1^{k+1}}{\partial x_2^k} & \frac{\partial x_1^{k+1}}{\partial x_3^k} & \cdots & \frac{\partial x_1^{k+1}}{\partial x_L^k} \\ \frac{\partial x_2^{k+1}}{\partial x_1^k} & \frac{\partial x_2^{k+1}}{\partial x_2^k} & \frac{\partial x_2^{k+1}}{\partial x_3^k} & \cdots & \frac{\partial x_2^{k+1}}{\partial x_L^k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_L^{k+1}}{\partial x_1^k} & \frac{\partial x_L^{k+1}}{\partial x_2^k} & \frac{\partial x_L^{k+1}}{\partial x_3^k} & \cdots & \frac{\partial x_L^{k+1}}{\partial x_L^k} \end{bmatrix}$$

The Lyapunov exponents of the system are calculated from the eigenvalues of the above matrix. If the eigenvalues of the $\wp^k$ are {Λ_1^k, Λ_2^k, Λ_3^k, ..., Λ_L^k} then the local Lyapunov exponents are given by

$$\lambda_n^k = \log|\Lambda_n^k|; (n = 1,2,...,L) \quad (8)$$

Theorem 1. Consider a system described by equation (1). Let $\wp$ denote the Jacobian of F at X, and Λi denote the ith eigenvalue of the matrix $\phi_m = \wp(X_{m-1})..\wp(X_1)\wp(X_0)$, where X = X_0 at time step t = 0. Suppose that $\|\wp'(X)\| \le M$, where M is a positive constant and that the smallest eigenvalue of $[\wp'(X)]^T \wp'(X)$ satisfies $\Lambda_{\min}([\wp'(X)]^T \wp'(X)) \ge \beta > 0$, where $\beta \le M^2$ and $X \in \Omega$. Then for any $X_0 \in \Omega$, all the Lyapunov exponents at X0 are located inside the interval [0.5 lnβ, lnN].

Additive Control. For the system described by equation (1), this control strategy is implemented by the following general dynamical equations:

$$X^{k+1} = F(X^k) + U^k, \quad (9a)$$

$$U^k = G(\lambda * - \lambda^k). \quad (9b)$$

Let us consider the logistic map, given by equation as an example:

$$x^{k+1} = \alpha^k x^k (1 - x^k) \quad (10)$$

Using equations (9) we can rewrite equation (10) as:

$$x^{k+1} = \alpha\, x^k (1 - x^k) + g(\lambda * - \lambda^k), \quad (11)$$

where g specifies the control stiffness for a single oscillator. Here the target value of the global Lyapunov exponent is a constant as opposed to a time varying function. The optimal value of the control parameter g needs to be worked out in any practical implementation of this strategy. For a coupled map lattice we use the mean global Lyapunov exponent as described by equation (5).

Multiplicative Control. Consider the system described by equation (1) with an additional controlled variable V. Here the control is implemented by changing this variable using a feedback method. Let us consider a lattice with controlled maps

$$X^{k+1} = F(X^k, U^k), \quad (12)$$

where V^k is a multiplicative control. For multiplicative control of the lattice it is necessary to study a controllability problem. Suppose that the lattice includes only a single map with a scalar multiplicative control that can be described by the following equation:

$$X^k = F(X^{k-1}, U^{k-1}) = [A + BU^{k-1}]X^{k-1}, \quad k = 1,2,3,..., \quad (13)$$

where X^k is the state vector, U^{k-1} is the scalar control, and A, B are real constant matrices of appropriate dimensions.

If $\hat{\Re}^n = \Re^n - \{0\}$ is the punctured n-space, then

$$X^k = \prod_{i=1}^{k} \left[A + U^i B\right] X^0 \equiv \gamma(X^0, U_k) \quad (14)$$

$U_k = [U^1,...,U^k] \in \Re^k$.

Definition 1. A lattice is said to be controllable on $\hat{\Re}^n$ if for any $X^1, X^2 \in \hat{\Re}^n$, there exists a positive integer *s* and finite control sequence U^s such that $X^2 = \gamma(X^1, U^s)$, where *γ* is a mapping factor.

The main result is stated in the following theorem.

Theorem 2. The lattice system given by (13) is controllable on $\hat{\Re}^n$ if there exist positive integers *P*, *Q* such that for all $X \in \hat{\Re}^n$ we have:

(a) $\left\|A^P X\right\| = \left\|X\right\|$,

(b) rank $H_Q(X) = n$,

where $H_Q(X) = [BA^{Q-1}X, ABA^{Q-2}X,..., A^{Q-1}BX]$.

The feedback control algorithm can be described by the following equations:

$$X^{k+1} = F(X^k, U^k), \quad (15a)$$

$$U^{k+1} = U^k + R(\lambda * - \lambda). \quad (15b)$$

where *R* specifies the control stiffness in this case. This scheme as with the previous one is adaptive in nature in that the parameters that determine the nature of dynamics adapt themselves to yield the desired dynamics. This type of feedback has also been termed as “dynamic feedback control” in literature [6]. We demonstrate the implementation of this control strategy in both single and coupled logistic maps, the monitored property being the mean Lyapunov exponent in the latter case.

Additive and Multiplicative Control. The third and final control strategy described in this paper is a combination of both additive and multiplicative control. The implementation of control in this case follows naturally from the above two schemes and can be described by the following dynamics:

$$\begin{aligned} X^{k+1} &= F(X^k, V^k) + G(\lambda * - \lambda^k), \\ V^{k+1} &= V^k + R(\lambda * - \lambda^k). \end{aligned} \quad (16a)$$

Heuristic Optimization Technique. The application of optimization techniques in control of dynamical systems involves minimizing the error function described in equation (4). The objective of the optimization technique in our examples is to find the optimal stiffness parameter that gives a constrained minimum of the error function. Consider the multiplicative control strategy as described in section 3.3. We define the linear inequality Rmin≤***R***≤***R***max and proceed to find iteratively, the value of ***R*** that gives the minimum of equation (4).The problem is formulated as follows:

$$\min_R \xi(R) \text{ subject to: } R_{\min} \le R \le R_{\max}, \quad (17)$$

We have used the Matlab ® optimization toolbox to solve this optimization problem. A sequential quadratic programming (SQP) method is used to solve this minimization problem. In this method, the function solves a quadratic programming (QP) sub-problem at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration. A line search is performed using a merit function similar to that proposed by [22-23]. For a more detailed explanation of the optimization function refer to Matlab® optimization toolbox documentation. multiplicative control strategies respectively.

Numerical And Experimentsl Results

Experimental studies in rodent models of epilepsy [Nair, 2004] have used EEG recordings from four to six electrodes placed in frontal and temporal regions of the animal brain. We have therefore chosen a CML model with five non-identical logistic maps. The system parameters $\alpha_1 \ldots \alpha_5$ were chosen randomly as 3.9, 3.97, 3.95, 3.965 and 3.96. The coupling term ε was varied from a value of 0.10 to 0.14 to study the dynamical behavior in both the spatial and temporal regimes. Figure 1 shows the changes in spatiotemporal patterns as we increase the value of the parameter ε. For illustration purposes we have only shown the amplitude and Lyapunov exponent profiles of the single cell (cell 1). The remaining cells exhibit a similar pattern. As we increase the value of ε gradually as shown in Figure 1D, the amplitude plot, shown in Figure 1A becomes more ordered and we can also see a drop in the Lyapunov exponents (calculated as a running mean) from the same time series, suggesting a more ordered state as illustrated in Figure 1B. Figure 1C shows the mean Lyapunov exponent profile calculated over all 5 cells in the CML. We can observe a gradual fall in the values of this global measure with increasing values of coupling.

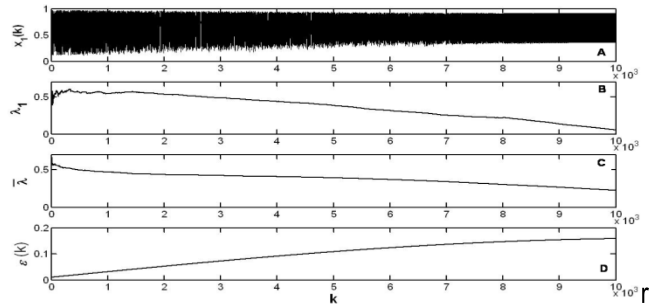

Fig. 1. (A) Amplitude spectrum as a function of time; (B) Lyapunov exponent profile of the single cell; (C) Mean Lyapunov exponent profile (L=5) estimated from a five cell CML; (D) parameter ε as a function of time.

Figure 2 illustrates the feedback control strategy also referred to as ‘dynamic feedback control’ in literature [Nair, 2004] described, for a target $\lambda^* = 0.3$. Since there can be several values of the controlled parameter α (corresponding to several different attractors) which gives the desired value of the Lyapunov exponent, the actual value of the controlled parameter takes depends on the stiffness of control, and initial conditions. The fluctuations in the controlled parameter are proportional to the value of the stiffness, converging to a single value for small stiffness while exhibiting large variations for higher values of stiffness.

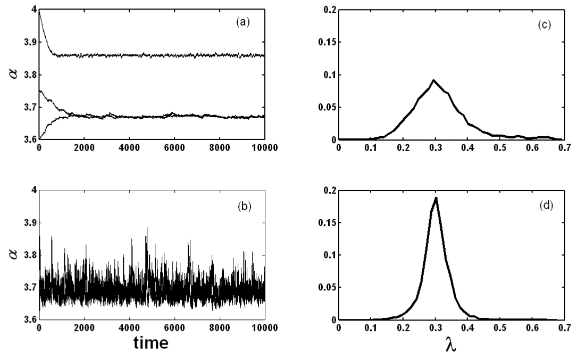

Fig. 2. Multiplicative control: of the parameter α as a function of iteration step for $\lambda^* = 0.3$, and stiffness a) g = 0.001, and b) g = 0.02. The different curves correspond to different initial α. Probability distributions of finite step Lyapunov exponents for α0 = 4.0 and stiffness (c) g = 0.001, and d) g = 0.01.

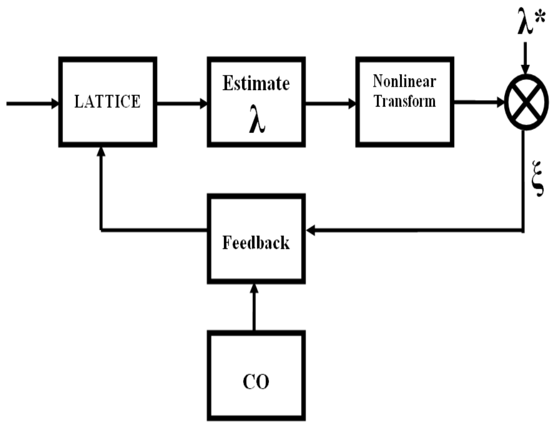

Fig.3. Proposed adaptive learning algorithm for a coupled map lattice via optimized feedback control to emulate the target dynamics of any complex network. CO refers to the constrained optimization block. ξ refers to error generated from nonlinearly transformed estimates of local Lyapunov exponents and target Lyapunov exponents.

Conclusions

In this paper, we have proposed an optimization method to control of spatiotemporal dynamic in coupled map lattice systems. We showed that a constrained optimization technique can be used for minimization of an error function to select optimal control parameters. It is also shown that feedback control can be applied to systems with hidden variables and hence may be plausible in control design for a highly complex system such as the epileptic brain where not all variables are known.

This paper presents a method to calculate finite-time Lyapunov exponents for experimental time series using numerical simulation to approximate the local Jacobian of the system at each time step. This combined numerical–experimental approach to the calculation of Lyapunov exponents is applicable to any physical system which can be numerically approximated.

Successful applications of control techniques to the brain dynamics must address in a comprehensive fashion at least the following items: (i) selection of variables for inputs, outputs and desired behaviors, (ii) construction of dynamical models of brain sites and its interactions, (iii) appropriate simplifying hypothesis to establish performance bounds.

We propose a learning algorithm in which a coupled map lattice system can be used to model the dynamical evolution of Lyapunov exponents in a complex system (Figure 3). The algorithm involves generating an error function between the target Lyapunov exponent profile of the complex system and some nonlinear transformation of estimated lattice Lyapunov exponent values. The error is used to generate an optimized feedback input to the lattice. Such a learning algorithm can be used in developing realistic model of complex system dynamics and hence make the models more useful in the study and control of such complex systems.

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Authors' Information

Vitaliy Yatsenko *– Head of Department of Remote Sensing Methods and Advanced Instrumentation, Space Research Institut of NASU-NSAU,Ukraine,03680, Kyiv, Academika Glushkova str., 40; e-mail:* vyatsenko@gmail.com

Olexandr Kochkodan *– Graduade Student, Space Research Institut of NASU-NSAU, Ukraine, 03680, Kyiv, Academika Glushkova str., 40; e-mail: olexandr.kochkodan@gmail.com*

Area of research: modeling**,** control of chas, optimization