MEMBRANE STRUCTURE SIMPLIFICATION

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Abstract: *Idempotent operators are one of the possible criterions for simplification trees. These operators act over internal nodes of trees. Moreover, they transform structural equivalent sub-trees, which have the same root into a single copy. This copy will be placed into the common root reducing –simplifying- the tree.*

This process can be also applied to membrane structures; in this case idempotent operator will be applied to non elementary membranes, and the process will produce the simplest possible membrane structure. The main idea of this work is to apply this kind of operator to binary membrane structures to identify possible recurrent sub-trees and to study possible application of this concept to hardware design implementations.

Keywords: *Membrane structures, Data structures, Trees, Simplification*

ACM Classification Keywords: *E.1 Data Structures - Trees*

Initial concepts

In literature, membrane structures are usually defined as strings or words built over the alphabet {[,]}. However, it can be also find alternative definitions as trees [Păun 2002]. Using this alternative definition, we will define an idempotent operator acting over trees and we will try to characterize binary membrane structures after the simplification procedure studying the average size and the variance of such structures.

In this paragraph we are going to study binary membrane structure simplification based on idempotent operators. First, we will provide some classical definitions based on trees, which we will need for defining the concept of membrane simplification and one simplification algorithm.

Let $\mathcal{M}_{\mathcal{B}}$ be a binary family membrane with two types of leaves. This family can be recursively defined by the equation:

$$\mathcal{M}_{\mathcal{B}} = \lambda + x + \circ(\mathcal{M}_{\mathcal{B}}, \mathcal{M}_{\mathcal{B}}) \tag{1}$$

where symbols λ and x are the two types of leaves and $\circ$ is a binary operator acting over pairs of binary membranes. In this case, the size of a binary membrane structure $\mu \in \mathcal{M}_{\mathcal{B}}$ (represented by $|\mu|$) is the number of non-elementary membranes.

Definition 1: *The probability p(μ) of a membrane structure* $\mu \in \mathcal{M}_{\mathcal{B}}$ *is:*

$$p(\mu) = \begin{cases} \dfrac{1}{2} & if \in \{\lambda, x\} \\ \dfrac{p(u)p(v)}{1+|u|+|v|} & if\ \mu = \circ(u, v) \end{cases} \tag{2}$$

Definition 2: *Let* $\mathcal{S}$ *be any set. It is said that a binary operator* $\circ$ *defined over* $\mathcal{S}$ *is idempotent if and only if for every* $a \in \mathcal{S}$,, $a \circ a \equiv a$

Definition 3: *If* $\circ$ *is an idempotent operator defined over the set of binary membrane structures* $\mathcal{M}_{\mathcal{B}}$*. Given* $\mu \in \mathcal{M}_{\mathcal{B}}$*, it is defined the simplified membrane of μ, represented by simp(μ), the membrane obtained starting in μ applying iteratively the rule* $\circ(u, u) = u$ *whenever appears the sub-tree* $\circ(u, u)$*.*

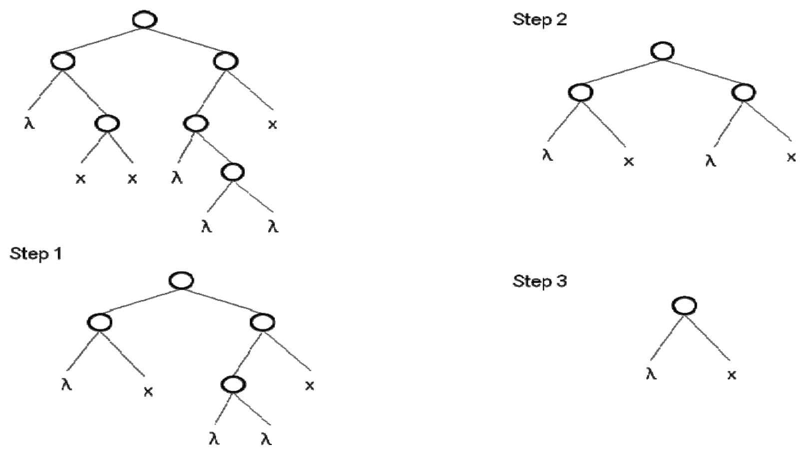

Figure 1: Simplification process of a binary membrane structure with idempotent rule

```
function simp(μ:Mb):Mb;
Local u,v :Mb;
If |μ|=0 then simp := μ;
Else
   u:= simp(μ.izq); v:=simp(μ.der);
   if eq(u,v) then simp := u;
                else simp:=o(u,v);fi;fi;
end(simp);

function eq(u,v: Mb):boolean;
if |u| = 0 or |v|=0 then eq:=(u.info = v.info);
   else if eq(u.izq = v.izq)
   then eq:=eq(u.der=v.der);
   else eq:=false;fi;fi;
end (eq)
```

Figure 2: Simplification algorithm with idempotent rule

Average and variance of simplified binary membrane structure size

A binary membrane structure $\mu \in \mathcal{M}_b$ can be simplified applying definition 3 and the simplification algorithm. Our first goal is to study the average size of $\mu \in \mathcal{M}_b$ where |μ| = n (expressed as function of n) when it is assumed the probability model of definition 1. That is:

$$s_n = \sum_{\mu \in \mathcal{M}_b, |\mu|=n} |\, simp(\mu)\,| \,.\, p(\mu) \tag{3}$$

Using the generatrix function technique, it is defined the associated power series:

$$S(z) = \sum_{\mu \in \mathcal{M}_b} |\, simp(\mu)\,| \,.\, p(\mu) . z^{|\mu|} = \sum_{n \geq 0} s_n z^n \tag{4}$$

Now it is needed to look for direct recurrences to characterize the series through some functional equation.

Let $\mathcal{S}$ be the set of unreductable elements of $\mathcal{M}_b$, that is, the set of binary membrane structures such that simp(μ)=μ.

$$\mathcal{I} = \lambda + x + \circ(\mathcal{I},\mathcal{I}) - \{\circ(\mu,\mu) : \mu \in \mathcal{I}\} \tag{5}$$

Then for each $\mu \in \mathcal{I}$ let $\mathcal{M}_\mu = \{\mu' \in \mathcal{M}_\varepsilon : simp(\mu') = \mu\}$ and let

$$M_\mu(z) = \sum_{\mu' \in \mathcal{M}_\mu} p(\mu')z^{|\mu'|} \tag{6}$$

be the generatrix function associated to membrane structures that simplify to μ. Then,

$$S(z) = \sum_{n \geq 0} s_n z^n = \sum_{\mu \in \mathcal{I}} |\mu| M_\mu(z) \tag{7}$$

Next step is to characterize the succession $\{M_\mu\}_{\mu \in \mathcal{I}}$. Firstly, it is easy to see that a membrane simplifies to a leave $l \in \{\lambda, x\}$ if and only if all their leaves have the same label, hence next equation defines the set of membranes in $\mathcal{M}_\varepsilon$ that simplify to the leaf *l*.

$$\mathcal{M}_\ell = l + \circ(\mathcal{M}_\ell, \mathcal{M}_\ell) \tag{8}$$

Now from (2) and (8) it is obtained that the following generatrix function

$$M_l(z) = \sum_{\mu \in \mathcal{M}_l} p(\mu)z^{|\mu|} \tag{9}$$

satisfies the differential equation

$$M'_l(z) = M_l^2(z), \quad M_l(0) = \frac{1}{2}, \tag{10}$$

and its solution is

$$M_l(z) = \frac{1}{2 - z} \tag{11}$$

On the other hand, let $\circ(u,v)$ be an element of $\mathcal{I}$, then $u,v \in \mathcal{I}$ and $u \neq v$. The fact that $\mu \in \mathcal{M}_{\circ(u,v)}$ implies that either the left sub-tree of μ simplifies to u and the right one to v, or both sub-trees simplify to $\circ(u,v)$ producing $\circ(\circ(u,v),\circ(u,v))$ and this tree reduces to $\circ(u,v)$. Therefore, it is possible to recursively define the set $\mathcal{M}_{\circ(u,v)}$ by

$$\mathcal{M}_{\circ(u,v)} = \circ(\mathcal{M}_u, \mathcal{M}_v) + \circ(\mathcal{M}_{\circ(u,v)}, \mathcal{M}_{\circ(u,v)}) \tag{12}$$

Expressing (12) in terms of a generatrix function

$$M_{\circ(u,v)}(z) = \sum_{\mu \in \mathcal{M}_{\circ(u,v)}} p(\mu)z^{|\mu|} = \sum_{\substack{\mu = \circ(u,v) \in \mathcal{M} \\ simp(\mu_1) = u \\ simp(\mu_2) = v}} p(\mu)z^{|\mu|} + \sum_{\substack{\mu = \circ(u,v) \in \mathcal{M}_\varepsilon \\ simp(\mu_1) = simp(\mu_2) = \circ(u,v)}} p(\mu)z^{|\mu|} \tag{13}$$

Deriving with respect to *z* it is obtained.

$$M'_{\circ(u,v)}(z) = M_u(z)M_v(z) + M^2_{\circ(u,v)}(z), \quad M_{\circ(u,v)}(0) = 0 \tag{13}$$

This inductive definition of the $M_\mu(z)$ series for each $\mu \in \mathcal{I}$ permit us to characterize the *S(z)* series trough a differential equation

Lemma 1: *Generatrix function S(z) satisfies the differential equation*

$$S'(z) = \frac{1}{(1-z)^2} + \frac{2S(z)}{1-z} - \sum_{\mu \in \mathcal{I}} (|\mu| + 1) M_\mu^2(z), \quad S(0) = 0 \tag{14}$$

<u>*Proof:*</u>

Let us to consider the generatix function of $\mathcal{M}_\varepsilon$.

$$M(z) = \sum_{\mu \in \mathcal{M}_\varepsilon} p(\mu)z^{|\mu|}$$

We have

$$M(z)=\sum_{n\geq 0}\sum_{|\mu|=n} p(\mu)z^{|\mu|}=\sum_{n\geq 0} z^n=\frac{1}{1-z}$$

On the other hand $\{\mathfrak{M}_\mu\}_{\mu\in\mathcal{I}}$ is a partition of $\mathfrak{M}_s$ and then

$$M(z)=\sum_{\mu\in\mathcal{I}} M_\mu(z)$$

To evaluate *S(z)* we introduce the two variables series

$$F(z,y)=\sum_{\mu\in\mathcal{I}} y^{|\mu|}M_\mu(z) \tag{15}$$

That satisfies the following identities

$$S(z)=\left[\frac{\partial F(z,y)}{\partial y}\right]_{y=1} \text{ and } S'(z)=\left[\frac{\partial^2 F(z,y)}{\partial z\partial y}\right]_{y=1}$$

Now considering (14) and the fact that $\circ(u,v)\in\mathcal{I}$ if $u, v\in\mathcal{I}$ and $u\neq v$ it is obtained the following partial differential equation

$$\frac{\partial^2 F(z,y)}{\partial z\partial y}=F^2(z,y)+2yF(z,y)\frac{\partial F(z,y)}{\partial y}-\sum_{\mu\in\mathcal{I}}\left[(2|\mu|+1)y^{2|\mu|}-|\mu|y^{|\mu|-1}\right]M_\mu^2(z)$$

That in *y=1* give us the expected result.■

Hence the problem is reduced to study the function

$$P(z)=\sum_{\mu\in\mathcal{I}}(|\mu|+1)M_\mu^2(z) \tag{16}$$

In particular, we are interested in finding out an appropriate functional bound for each one of the $M_\mu(z)$. In order to get this goal, we need some previous results.

Lemma 2: *For each* $\mu\in\mathcal{I}$ *and* $0\leq z\leq 2e^{-\frac{2}{\sqrt{3}}arctg\sqrt{3}}$ *we have that* $M_\mu(z)=\frac{1}{2-z}$

Proof: We will proceed by induction over |μ|. If |μ|=0 the result is evident.

Let now be |μ|=∘(u,v) and let $J=\left[0,\quad 2-2e^{-\frac{2}{\sqrt{3}}arctg\sqrt{3}}\right]$. Having in mind (13) and applying the induction hypothesis, we have that for each $z\in J$,

$$M'_\mu(z)\leq\frac{1}{(2-z)^2}+M_\mu^2(z),\quad M_\mu(0)=0.$$

Considering the differential equation $u'(z)=\frac{1}{(2-z)^2}+u^2(z),\quad u(0)=0$. Applying the Opial lemma, we have that $M_\mu(z)\leq u(z)$ for any z≥0 whenever u(z) exists. Solution of the previous equation is

$$u(z)=\frac{g(z)}{2-z},\quad g(z)=\frac{1}{2}\left[1-\sqrt{3}\frac{1-h(z)\sqrt{3}}{h(z)+\sqrt{3}}\right],\quad h(z)=\tan\left(\frac{\sqrt{3}}{2}\ln\frac{2}{2-z}\right). \tag{17}$$

Finally, with a very simple calculation it is proof $g(z)\leq 1$ that when $z\in J$. ■

Lemma 3:

$$\forall \mu \in \mathscr{I} - \{\lambda, x\} : M_\mu(1) < \frac{29}{50}$$

Proof: The proof will be done by induction over |μ|. When |μ|=1 from (13) and proof of lemma 2, it can be concluded that $M_\mu(1) \approx 0.566 < 29/50$, due to that in this case $M_\mu(1)=g(1)$ where $g(z)$ is given in (17). If $|\mu| \geq 2$ then at least one of the two sub-tree in μ is not a leave. Let us suppose that $|u| \geq 1$. It is known that $M_\mu(z)$ is a non negative function and it is monotonically increasing in [0, 1] whatever will be $\mu \in \mathscr{I}$, from (13) and lemma 2, we have that for each $z \in [0, 1]$,

$$M_\mu(z) \leq M_v(z)\int_0^z M_u(s)ds + \int_0^z M_t^2(s)ds \leq \int_0^z M_u(s)ds + \int_0^z M_t^2(s)ds. \tag{18}$$

Now applying the trapezium rule [Burden 1985] in the integral $\int_0^z M_\mu(s)ds$ we have that for each $z \in [0, 1]$,

$$\int_0^z M_u(s)ds = z\frac{M_u(0)+M_u(z)}{2} - \frac{z^3}{12}M_u''(\xi) \leq \frac{z}{2}M_u(z), \quad 0 < \xi < z, \tag{19}$$

Because $M_u(z)$ is a series with positive coefficient we have that $M''_u(z)<0$ for each $\xi \in (0, z)$.
Inequalities (18) and (19) guarantees that $M_\mu(z)$ satisfies in [0, 1] the differential inequality

$$M_\mu(z) \leq \frac{z}{2}M_u(1) + \int_0^z M_\mu^2(s)ds. \tag{20}$$

Now considering that by induction hypothesis it is $M_\mu(1)<29/50$, we have that for each $z \in [0, 1]$

$$M_\mu(z) \leq 0.29z + \int_0^z M_\mu^2(s)ds. \tag{21}$$

And finally, considering the differential equation $w'(z)=0.29+w^2(z)$, $w(0)=0$, which has as solution $w(z) = \sqrt{0.29}\tan(z\sqrt{0.29})$,it is deduced that $M_\mu(1) \leq w(1) \approx 0.321 < \frac{29}{50}$ ■

Lemma 4:

$$\forall \mu = \circ(u,v) \in \mathscr{I}^* \text{ and } z \in [0,1]: \quad M_\mu(z) \leq \sqrt{2}\int_0^z M_u(s)M_v(s)ds.$$

Proof: It |μ|≥1, from lemma 3 and trapezium rule, we have that for each $z \in [0, 1]$,

$$\int_0^z M_\mu^2(s)ds \leq M_\mu(z)\int_0^z M_\mu(s)ds \leq \frac{29}{50}z\frac{M_\mu(0)+M_\mu(0)}{2} \leq 0.29M_\mu(z)$$

The result is immediate considering (13).■
These previous results give us the desired bound for $M_\mu(z)$.

Lemma 5:

$$\forall \mu \in \mathscr{I}^* \text{ and } z \in [0,1]: \quad M_\mu(z) \leq 2^{|\mu|}p(t)\ln^{|\mu|}\frac{2}{2-z\sqrt{2}}.$$

Proof: The proof will be done by induction over |*μ*|. If |*μ*|=*1* we have that $M_{\circ(\lambda,x)}(z)=M_{\circ(x,\lambda)}(z)=\frac{g(z)}{2-z}\leq\frac{1}{2}\alpha(z)$ where *g(z)* is like in (17), and $\alpha(z)=\ln\frac{2}{2-z\sqrt{2}}$ .Let us suppose now that *μ=∘(u, v)* with |*μ*|>1. We will distinguish two cases: firstly the case in which one sub-tree is a leave, for instance *v.* In this case we have that |*u*|=|*μ*|-*1* and |*v*|=0. Applying (11) and the induction hypothesis, we have that:

$$M_\mu(z)\leq\sqrt{2}\int_0^z M_u(s)M_v(s)ds\leq$$

$$\sqrt{2}\int_0^z 2^{|\mu|-1}p(u)\alpha^{|\mu|-1}(s)\frac{1}{2-s}ds=$$

$$\sqrt{2}\int_0^z 2^{|\mu|}p(u)p(v)\alpha^{|\mu|-1}(s)\frac{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{2-\frac{s}{2}}ds\leq$$

$$\int_0^z 2^{|\mu|}\,|\,\mu\,|\,p(\mu)\frac{\frac{1}{\sqrt{2}}}{1-\frac{s}{\sqrt{2}}}\alpha^{|\mu|-1}(s)ds=$$

$$2^{|\mu|}p(\mu)\alpha^{|\mu|}(z)$$

In the second case both sub-trees are not leaves, that is, |*u*|≥1 and |*v*|≥1. Applying again the lemma 4 and induction hypothesis, we have that

$$M_\mu(z)\leq\sqrt{2}\int_0^z M_u(s)M_v(s)ds\leq$$

$$\sqrt{2}\int_0^z 2^{|u|+|v|}p(u)p(v)\alpha^{|u|+|v|}(s)ds=$$

$$\sqrt{2}\int_0^z 2^{|\mu|-1}\,|\,\mu\,|\,p(\mu)\frac{\frac{1}{\sqrt{2}}}{1-\frac{s}{\sqrt{2}}}\alpha^{|\mu|-1}(s)\frac{1-\frac{s}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}ds\leq$$

$$\int_0^z 2^{|\mu|}\,|\,\mu\,|\,p(\mu)\frac{\frac{1}{\sqrt{2}}}{1-\frac{s}{\sqrt{2}}}\alpha^{|\mu|-1}(s)ds=$$

$$2^{|\mu|}p(\mu)\alpha^{|\mu|}(z)$$ ■

This result permits us to conclude that

$$P(z)\leq\frac{2}{(2-z)^2}+\sum_{\mu\in\mathcal{T}^*}(|\,\mu\,|+1)2^{2|\mu|}p^2(\mu)\ln^{2|\mu|}\frac{2}{2-z\sqrt{2}}\qquad(21)$$

for each $z\in[0, 1]$ whenever the series be convergent, but this fact does not implies that *P(z)* be convergent in the interval *[0, 1]*. To guarantee this convergence, we consider the generatrix function

$$I(z)=\sum_{\mu\in\mathcal{I}} p(\mu)z^{|\mu|} \tag{22}$$

associated to the family $\mathcal{I}$ of unreductable binary membrane structures (5). Considering the inductive definition of this family, it is easy to verify that *I(z)* satisfy the differential equation

$$I'(z)=I^2(z)-\sum_{\mu\in\mathcal{I}} p^2(\mu)z^{2|\mu|},\quad I(0)=1 \tag{23}$$

From this differential equation it is easy to see that inside its convergence circle, *I(z)* verifies the differential inequality

$$I'(z)\le I^2(z)-\frac{1}{2},\quad I(0)=1,\quad z\ge 0 \tag{24}$$

Let *L(z)* be the solution of the differential equation (24) replacing "≤" by "=". Then we have

$$L(z)=\frac{1}{\sqrt{2}}\frac{(1+\sqrt{2})^2 e^{-z\sqrt{2}}+1}{(1+\sqrt{2})^2 e^{-z\sqrt{2}}-1}. \tag{25}$$

Hence, for each $0\le z\le \rho$ is $I(z)\le L(z)$, been $\rho=\sqrt{2}\ln(1+\sqrt{2})\approx 1.246$ the convergence radius of *L(z)*.

Let $\alpha(z)=\ln\frac{2}{2-z\sqrt{2}}$ and the constants $K=1.24$ and $M=4\alpha^2(1)/K\approx 4.864<4.87$. These constants will be very useful to proof that *P(z)* is an analytic function in an appropriate domain.

Lemma 6: *The set* $\{M^{|\mu|}p(\mu)\}_{\mu\in\mathcal{M}}$ *is bounded by a constant* M^*.

Proof: A simple calculation proof that $M^{|\mu|}p(\mu)\le M^*=5.41$ for each $\mu\in\mathcal{M}_b$ with $|\mu|\le 26$, considering for each size the membrane structure almost complete well balanced corresponding. Let now be $\mu=\circ(u,v)$ with $|\mu|\ge 27$. Applying the induction hypothesis, we have that $M^{|\mu|}p(\mu)=M^{|u|}p(u)\,M^{|v|}p(v)\le M^{*2}M/27\le M^*$.■

From (21) and lemma 6 is immediate that for each $z\in[0,1]$,

$$P(z)\le\frac{2}{(2-z)^2}+\sum_{\mu\in\mathcal{I}^*}(|\mu|+1)p^2(\mu)M^{|\mu|}K^{|\mu|}$$

$$\le\frac{2}{(2-z)^2}+M^*\sum_{\mu\in\mathcal{I}^*}(|\mu|+1)p^2(\mu)K^{|\mu|},$$

since α(z) is growing in [0, 1]. Consequently, *P(z)* uniformly converges in the disk $|z|\le 1$ because $|P(z)|\le P(|z|)$ and $K<\rho$. At this point it is possible to extend the convergence of *P(z)* to a disk of radius bigger than 1.

Corollary: *There exists ε>0 such that the series P(z) converges in* $|z|\le 1+\varepsilon$.

Proof: For continuity, is enough to choose $\varepsilon>0$ in shuch a way that satisfies the following conditions at the same time:

a. $\frac{u(1+\varepsilon)}{1-\varepsilon}<\frac{29}{50}$, *where u(z) is the defined function in (17)*

b. $\frac{29}{50}(1+\varepsilon)\le 2-\sqrt{2}.$

c. $f(1+\varepsilon) > 0,$ where $f(z) = \frac{1}{2}\alpha(z) - u(z)$

d. *f'(τ)=0* for only one single τ ∈*[0, 1+ε].*

e. $\frac{4\alpha^2(1+\varepsilon)}{K} < 4.87.$

These requirements come from the previous development.

Then, *P(z)* is an analytic function in the disk |z|<1+ε by the Weierstrass (Appendix 1) theorem. Solving the equation of lemma 1, we have that

$$S(z) = \frac{1}{(1-z)^2}\left[z - \int_0^z (1-s)^2 P(s)ds\right]$$

and consequently, *S(z)* satisfies every hypothesis to apply the Darboux theorem (Appendix 2), from which the following result is deduced

Theorem 1: *Under the probabilistic model of (2), the average size s_n of membrane structures resulting from simplify with idempotent rule random membrane structures of size n is*

$$s_n = \gamma n + O(1)$$

where constant γ is given by $\gamma = 1 - \int_0^1 (1-s)^2 P(s)ds \approx 0.752.$

Now we will study the variance of this random variable.

$$v_n = \sum_{\mu\in\mathcal{M}_\varepsilon, |\mu|=n} (|\,simp(\mu)\,| - s_n)^2 p(\mu) = \sum_{\mu\in\mathcal{M}_\varepsilon, |\mu|=n} (|\,simp(\mu)\,|)^2 p(\mu) - s_n^2. \quad (26)$$

Let *H(z)* be the generatrix function associated to the second moment, that is,

$$H(z) = \sum_{\mu\in\mathcal{M}_\varepsilon} |\,simp(\mu)\,|^2\; p(\mu) z^{|\mu|} = \sum_{\mu\in\mathcal{I}} |\,\mu\,|^2\; M_\mu(z).$$

This function is characterized by the following lemma.

Lemma 7: *Generatrix function H(z) satisfies the following differential equation*

$$H'(z) = S'(z) + \frac{2S(z)}{1-z} + 2S^2(z) + \frac{2H(z)}{1-z} - A(z), \quad H80) = 0,$$

where S(z) is defined in (4) and $A(z) = \sum_{\mu\in\mathcal{I}} 3\,|\,\mu\,|\,(|\,\mu\,|+1) M_\mu^2(z)$.

Proof: Considering the series *F(z,y)* defined in (15). We have $H(z) = \left[\frac{\partial^2 F(z,y)}{\partial y^2}\right]_{y=1} + S(z)$ and $H'(z) = \left[\frac{\partial^3 F(z,y)}{\partial z \partial y^2}\right]_{y=1} + S'(z)$. Now is enough to see that $S(z) = \left[\frac{\partial F(z,y)}{\partial y}\right]_{y=1}$, $F(z,1) = \frac{1}{1-z}$ and using (13),

$$\frac{\partial^3 F(z,y)}{\partial z \partial y^2} = 4F(z,y)\frac{\partial F(z,y)}{\partial y} + 2y\left(\frac{\partial F}{\partial y}\right)^2 + 2yF(z,y)\frac{\partial^2 F(z,y)}{\partial y^2}$$
$$- \sum \left[2\,|\,\mu\,|\,(2\,|\,\mu\,|+1) y^{2|\mu|-1} - |\,\mu\,|\,(|\,\mu\,|-1) y^{|\mu|-2}\right] M_\mu^2(z). \qquad \blacksquare$$

Solution of the previous equation is

$$H(z) = S(z) + \frac{2q^2(z)}{(1-z)^3} + \frac{N(z)}{(1-z)^2}, \tag{27}$$

where $q(z) = z - \int_0^z (1-s)^2 P(s)ds,$ *P(s)* is the defined series in (16) and

$$N(z) = 4\int_0^z (1-s)q(s)P(s)ds - \int_0^z (1-s)^2 A(s)ds.$$

Functions *q(z)* and *N(z)* are both analytic in a disk with radius bigger than 1, hence Darboux theorem is applicable, providing the following asymptotic equivalent for the coefficients of the functions appearing in (27):

$$[z^n]S(z) = \gamma(n+1) - 1 + O(1/n),$$

$$[z^n]\frac{2q^2(z)}{(1-z)^3} = \gamma^2(n^2+3n) - 4\gamma n + O(1),$$

$$[z^n]\frac{N(z)}{(1-z)^2} = \alpha n + O(1),$$

been *γ=q(1)≈0.752* and *α=N(1)≈0.457*. Now it is possible to quantify the variance of the simplified membrane structures.

Theorem 2: *Under the probabilistic model of (2), the variance of the size of simplified membrane structures with idempotent rule is* $v_n = \xi n + O(1)$, *where* $\xi = \alpha + \gamma^2 - \gamma \approx 0.271$.

Appendix

This paragraph will be enunciated the Weierstrass and Darboux theorems which are used in the previous section of this paper.

Appendix 1: *Weierstrass theorem [Henrici 1977, Marsden 1987]*

Let $\{f_n(z)\}_{n\geq 0}$ *a sequence of analytic functions defined on a region* $\Omega \subset C$. *We have that*

- *If* $f_n \to f$ *uniformly on every closed disk in Ω, then f is analytic. Moreover,* $f'_n \to f'$ *pointwise in Ω and uniformly on every closed disc included in Ω.*
- *If* $g(z) = \sum_{n\geq 0} f_n(z)$ *converges uniformly on every closed disk in Ω, then g(z) is analytic in Ω and* $g'(z) = \sum_{n\geq 0} f'_n(z)$ *pointwise in Ω and uniformly in every closed disk included in Ω.*

Appendix 2: *Darboux theorem [Henrici 1977]*

Let f(z) be an analytic function in the disk |z|<ρ and let us suppose that it has only one single singularity on its convergence circle in z=ρ. Let us also suppose that f(z) admits, in an environment of z=ρ, a local development in the form

$$f(z) = \left(1 - \frac{z}{\rho}\right)^{-s} g(z) + h(z)$$

for some functions g(z) and h(z) analytics in an environment of z=ρ, with g(ρ)≠0, and for some real number $s \notin \{0,-1,-2,\ldots\}$. *Then, when n→∞,*

$$[z^n]f(z) \approx \rho^{-n} \sum (-1)^n g_k \binom{k-s}{n},$$

where g_k are the coefficients of $g(z)=\sum_{k\ge 0} g_k (1-\frac{z}{\rho})^k$ *and the symbol ≈ denotes asymptotic equivalence.*

Since the first series term is preponderant, theorem conclusion is written many times as

$$[z^n]f(z) \approx \rho^{-n} n^{s-1} \frac{g(\rho)}{\Gamma(s)}\left[1+O\left(\frac{1}{n}\right)\right],$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt, \quad \mathrm{Re}\, z>0,$$

Is the Euler's gamma functions and $e=\exp(1)=\sum_{k\ge 0}\frac{1}{k!}$

Conclusions

This paper study a simplification process on binary membrane structures. The study establishes that the average size and variance of simplified membrane structures are both linear on the number of non elementary membranes.

The study of non reducible membrane structures can be useful to determine hardware modules that could be implemented a priory and they can be used in general membrane systems implementations.

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