A COMBINED EXTERIOR PENALTY FUNCTION – CONJUGATE GRADIENT ALGORITHM FOR A CLASS OF CONSTRAINED OPTIMAL CONTROL QUASILINEAR PARABOLIC SYSTEMS

M. H. Farag and T. A. Talaat

***Abstract**: In this paper, combined exterior penalty function-conjugate gradient algorithm for a constrained optimal control problem for the coefficients of a quasilinear equation is applied. The constrained optimal control problem has been converted to one of the optimization problem using a penalty function technique. One of the approaches of building the gradient of the modified functional using the solving of the adjoint problem is investigated. A special case of the optimal control problem for the considering problem to illustrate the numerical results is investigated. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. The numerical results are presented.*

***Keywords**: Optimal control, Quasilinear Parabolic Equation, Existence and Uniqueness Theorems, Adjoint system, Gradient formulae, Exterior penalty function-conjugate gradient algorithm.*

***ACM Classification Keywords**: F.2.1 Numerical algorithm and problems,G.4 Mathematical Software*

Introduction

Owing to its importance for engineering applications, the field of partial differential equations (PDE) constrained optimization has become increasingly popular [Ahmed,1996], [Farag,2012],[Iskenderov,1983]. In them, the control can occur both in the equations and in the boundary and initial conditions. These problems arise in modeling processes such as heat conduction, diffusion, filtration, in evaluating risks in financial mathematics, etc [Tagiev,2009], [Vasilev,1981], [Farag,2006]. In this paper, combined exterior penalty function-conjugate gradient algorithm for a constrained optimal control problem for the coefficients of a quasilinear equation is applied. The constrained optimal control problem has been converted to one of the optimization problem using a penalty function technique. One of the approaches of building the gradient of the modified functional using the solving of the adjoint problem is investigated. A special case of the optimal control problem for the considering problem to illustrate the numerical results is investigated. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. The numerical results are presented.

Problem Formulation

Let D be a bounded domain in E_N and $\Omega = \{(x,t) : x \in D \ , t \in (0,T]\}$, $S = \Gamma \times \ (0,T]$

and $V = \{ v : v = (v_0 \ , v_1, v_2) : v_m = (v_{0m}, v_{1m}, v_{2m} \dots) \in l_2 \ \|v_m\|_{l_2,} \le R_m , m = \overline{0,2}\}$.

Consider the following process

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\lambda_i(x,t,v_0)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} B_i(x,t,v_1)\frac{\partial u}{\partial x_i} = f(x,t,v_2),\ (x,t)\in\Omega \quad (1)$$

$$u(x,0) = \phi(x)\ ,\ x\in D \quad (2)$$

$$\left.\frac{\partial u}{\partial N}\right|_S = \left.\sum_{i=1}^{n} \lambda_i(x,t,v_0)\frac{\partial u}{\partial x_j}\cos(\nu,x_i)\right|_S = g(\zeta,t)\ ,\ (\zeta,t)\in S \quad (3)$$

are required to minimize the functional

$$f_\alpha(v) = \int_S [\,u(x,\zeta) - f_0(x,\zeta)]^2 dx\, d\zeta + \alpha\sum_{m=0}^{2}\left\|v_m - w_m\right\|_{l_2}^2 \quad (4)$$

with the following additional restrictions

$$\lambda_0 \le \lambda_i(u,v) \le \lambda_1 \quad ,\quad B_0 \le B_i(u,v) \le B_1$$
$$\mu_0 \le u(x,t) \le \mu_1\ ,\ \mu_i > 0, B_i, \lambda_i > 0\ , i = 0,1 \quad (5)$$

It is proved in [Ladyzhenskaya,1973] that, under the foregoing assumptions, a reduced problem (1) - (3) has a unique solution and $\left|\frac{\partial u}{\partial x_i}\right| \le C_1\ ,\ i=\overline{1,n}$ where C_1 is a certain constant.

Throughout this paper, we adopt the following assumptions:

A1)

$$\left|\lambda_i(u,v_0+\delta v_0) - \lambda_i(u,v_0)\right| \le S_0(x,t)\left\|\delta v_0\right\|_{l_2}, i=\overline{1,n}$$
$$\left|B_i(u,v_1+\delta v_1) - B_i(u,v_1)\right| \le S_1(x,t)\left\|\delta v_1\right\|_{l_2}, i=\overline{1,n}$$
$$\left|f(u,v_2+\delta v_2) - f(u,v_2)\right| \le S_2(x,t)\left\|\delta v_2\right\|_{l_2}$$

A2) The function $f(x,t,v_2)$ is given function continuous in v_2 on l_2 for almost all $(x,t)\in\Omega$.

A3) The functions $\lambda_i(u,v_0)\ ,\ B_i(u,v_1)$ are continuous on (u,v) and have continuous derivatives $\frac{\partial\lambda_i(u,v_0)}{\partial u}\ ,\ \frac{\partial B_i(u,v_1)}{\partial u}\ ,\ i=\overline{1,n}$ are bounded.

A4) The following operators

$$\int_\Omega \frac{\partial f(x,t,v_2)}{\partial v_2}dxdt\ ,\ \int_\Omega \frac{\partial\lambda_i(u,v_0)}{\partial u}dxdt\ ,\ \int_\Omega \frac{\partial^2\lambda_i(u,v_0)}{\partial u\,\partial v_0}dxdt\ ,$$
$$\int_\Omega \frac{\partial B_i(u,v_1)}{\partial u}dxdt\ ,\ \int_\Omega \frac{\partial^2 B_i(u,v_1)}{\partial u\,\partial v_1}dxd\ ,\ i=\overline{1,n}$$

are bounded in l_2.

The inequality constrained problem (1) - (5) is converted to a problem without inequality constrains by adding a penalty function [Farag, 1996; Yenia, 2005] to the objective function $f_\alpha(v)$ yielding the following $\Phi_{\alpha,k}(v)$ function:

$$\Phi_{\alpha,k}(v) = \int_S [\ u(x,\zeta) - f_0(x,\zeta)]^2\, dt\, d\zeta + \alpha \sum_{m-0}^{2} \|v_m - \omega_m\|_{l_2}^2$$
$$+ A_k \iint_\Omega \Psi(x,t,v) \begin{bmatrix} \{\max[\ \lambda_0 - \lambda_i(u,v);0]\}^2 + \{\max[\ B_0 - B_i(u,v);0]\}^2 + \\ \{\max[\ \lambda_i(u,v) - \lambda_1;0]\}^2 + \{\max[\ B_i(u,v) - B_1;0]\}^2 + \\ \{\max[\ u(x,t) - \mu_0;0]\}^2 + \{\max[\ \mu_1 - u(x,t);0]\}^2 \end{bmatrix} dx\, dt \quad (6)$$

Now, we give the following theorems [Farag, 2012]:

Theorem 1: *The functional $f_0(v)$ is continuous on V.*

Theorem 2: *For any $\alpha \geq 0$ the problem (1) - (4) has at least one solution.*

Theorem 3: *There exists a dense set K of l_2 such that for any $w_m \in K, m = \overline{0,m}$.*

The sufficient differentiability conditions of function (5) and its gradient formulae will be obtained by defining the Hamiltonian function $\Pi(x,\Theta,v)$ as [Tagiev, 2009]:

$$\Pi(x,\Theta,v) = -[\ \sum_{i=1}^{n} \lambda_i(x,t,v_0)\, \frac{\partial u}{\partial x_i}\frac{\partial \Theta}{\partial x_i} + \sum_{i=1}^{n} B_i(x,t,v_1)\, \frac{\partial u}{\partial x_i}\Theta - f(x,t,v_2)\Theta$$
$$+ A_k\ \{\ Z(u,v_0) + Y(u,v_1)\} + \alpha \sum_{m=0}^{2} (\ v_m - w_m)^2] \quad (6)$$

$$where \quad \begin{bmatrix} Z(u,v_0) = \sum_{i=1}^{n} [\{\max[\ \lambda_0 - \lambda_i(u,v_0);0]\}^2 + \{\max[\ \lambda_0 - \lambda_i(u,v_0);0]\}^2] \\ Y(u,v_1) = \sum_{i=1}^{n} [\{\max[\ B_0 - B_i(u,v_1);0]\}^2 + \{\max[\ B_0 - B_i(u,v_1);0]\}^2] \end{bmatrix}$$

Theorem 4: *Let the above assumptions be satisfied. Then $\Phi_{\alpha,k}(v)$ is Frechet differentiable, and its gradient satisfies the equality*

$$\frac{\partial \Phi_{\alpha,k}(v)}{\partial v} = -\frac{\partial \Pi(x,\Theta,v)}{\partial v} \equiv (-\frac{\partial \Pi(x,\Theta,v)}{\partial v_0}, -\frac{\partial \Pi(x,\Theta,v)}{\partial v_1}, -\frac{\partial \Pi(x,\Theta,v)}{\partial v_2})$$

where

$$\frac{\partial \Pi(x,\Theta,v)}{\partial v_0} = -[\sum_{i=1}^{n} \frac{\partial \lambda_i(x,v_0)}{\partial v_0} \frac{\partial u}{\partial x_i} \frac{\partial \Theta}{\partial x_i} + A_k \frac{\partial Z(u,v_0)}{\partial v_0} + 2\alpha (v_0 - w_0)$$

$$\frac{\partial \Pi(x,\Theta,v)}{\partial v_1} = -[\sum_{i=1}^{n} \frac{\partial B_i(x,v_1)}{\partial v_1} \frac{\partial u}{\partial x_i} \frac{\partial \Theta}{\partial x_i} + A_k \frac{\partial Y(u,v_1)}{\partial v_1} + 2\alpha (v_1 - w_1)$$

$$\frac{\partial \Pi(x,\Theta,v)}{\partial v_2} = -[\frac{\partial f_i(x,2_1)}{\partial v_2}\Theta + 2\alpha (v_2 - w_2)]$$

COMBINED EXTERIOR PENALTY FUNCTION – CONJUGATE GRADIENT ALGORITHM

A) Statement of the control problem and Modified function

Let D be a bounded domain in E_N and $\Omega = \{(x,t) : x \in D , t \in (0,T)\}$. Consider the following process

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(u,v)\frac{\partial u}{\partial x}\right) + B(u,v)\frac{\partial u}{\partial x} = f(x,t,v), \ (x,t) \in \Omega$$

$$u(x,0) = \phi(x) \quad , x \in D$$

$$\lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=0} = g_0(t), \ \lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=l} = g_1(t) \ , \ 0 \le t \le T$$

On the set $V = \{v : v=(v_1,v_2 , v_3 ,\cdots, v_N) \in E_N, \|v\|_{E_N} \le R, R>0 \}$ under the above conditions and additional restrictions

$$\lambda_0 \le \lambda(u,v) \le \lambda_1 \quad , \quad B_0 \le B(u,v) \le B_1$$

$$\mu_0 \le u(x,t) \le \mu_1 \ , \ \mu_i > 0 , B_i , \lambda_i > 0 , i = 0,1$$

are required to minimize the functional

$$f_\alpha(v) = \beta_0 \int_0^T [u(0,t) - f_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - f_1(t)]^2 dt + \alpha \|v - \omega\|^2_{E_N}$$

on the where $f_i(t) \in L_2(0,T) , i = 0,1$ are given functions.

The above inequality constrained problem is converted to a problem without inequality constrains by adding a penalty function [Rao, 1984] to the objective function $f_\alpha(v)$ yielding the following $\Phi_{\alpha,k}(v)$ function:

$$\Phi_{\alpha,k}(v) = \beta_0 \int_0^T [u(0,t) - f_0(t)]^2 dt + \beta_1 \int_0^T [u(l,t) - f_1(t)]^2 dt + \alpha \|v - \omega\|^2_{E_N}$$

$$+ A_k \iint_\Omega \Psi(x,t,v) \begin{bmatrix} \{\max[\lambda_0 - \lambda(u,v);0]\}^2 + \{\max[B_0 - B(u,v);0]\}^2 + \\ \{\max[\lambda(u,v) - \lambda_1;0]\}^2 + \{\max[B(u,v) - B_1;0]\}^2 + \\ \{\max[u(x,t) - \mu_0;0]\}^2 + \{\max[\mu_1 - u(x,t);0]\}^2 \end{bmatrix} dx\, dt$$

where $\Psi(x,t,v)$ is the adjoint state of the above PDE and from the results the section 1 we can compute the gradient $\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}$.

B) Numerical Algorithm

With the gradient obtained $\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}$, the following conjugate gradient method [Rao,1984] combined with the penalty function method [Yenia, 2005] can be developed for the optimal control values of $v \in V$. The outlined of the algorithm for solving the optimal control problem are as follows:

Step 1: Choose an initial control $v^{(0)}, \varepsilon > 0,\ A_0, \varepsilon_1 > 0$. If $\left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(0)}} = 0$, then $v^{(0)}$ is the optimal solution of the problem.

Step 2: Set the first searching direction $S^{(0)} = -\left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(0)}}$

Step 3: Set $S^{(0)} = v^{(0)} + \alpha^{(0)} S^{(0)}$, with $S^{(0)}$ being the optimal step length in the searching direction $S^{(0)}$. Set $IT = 1$ and go to step 4.

Step 4: Find $\left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(IT)}}$,by solving the state and adjoint systems and then, set

$$S^{(IT)} = -\left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{u=u^{(IT)}} + \theta^{(IT)} S^{(IT-1)}, \text{ with } \theta^{(IT)} = \frac{\left\langle \left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(IT)}}, \left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(IT)}} \right\rangle_{E_N}}{\left\langle \left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(IT-1)}}, \left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^{(IT-1)}} \right\rangle_{E_N}}$$

Step 5: Compute the optimum step length $\alpha^{(It)}$ in the searching direction $S^{(IT)}$ and update $v^{(IT)}$ by $v^{(IT)} = v^{(IT)} + \alpha^{(IT)} S^{(IT)}$.

Step 6: Test the optimality of $u^{(IT+1)}$. If $v^{(IT+1)}$ is optimum, stop the process. Otherwise, set $IT = IT + 1$, $A^{(IT)} = \varepsilon_1 A^{(IT-1)}$ and go to Step 4.

Numerical Results and Dissection

The problem in the above section is considered as one of the identification problems on definition of unknown coefficients of parabolic quasilinear equation type. The numerical results were carried out for the following examples:

1) $y = x + t, \lambda(y,u) = \dfrac{y}{1+y}$, $B(y,u) = \tan^{-1}(y)$, $x \in [0,0.8]$, $t \in [0,0.001]$.

2) $y = x + t, \lambda(y,u) = \tan^{-1}(y)$, $B(y,u) = \dfrac{y^2}{1-y^2}$, $x \in [0,0.9]$, $t \in [0,0.001]$

3) $y = \sin(3\pi x + t)$, $\lambda(y,u) = \tan^{-1}(y)$, $B(y,u) = \log(1+y)$, $x \in [0,0.7]$, $t \in [0,0.001]$

The Numerical study has given the following results:

1) Knowing the computed optimal control values v^* obtained by the above numerical algorithm, we can calculate the approximate values of the unknown coefficients $\lambda(u,v) = \sum_{k=1}^{NT/2} v_{2k-1} u^{2k-1}$, $B(u,v) = \sum_{k=1}^{NT/2} v_{2k} u^{2k}$, each one can be represented in a series according to every example.

2) For example 1, in Figure 1 the curves are denoted by $L(u,v^*)\ NC = 4, L(u,v^*)\ NC = 6, \cdots$ are the approximate values of $\lambda(u,v)$ with v^* . Obviously, by increasing the number of controls NC, the approximate values of the coefficient $\lambda(u,v^*)$ are agree with the exact values. Also, in Figures 1 the curves are denoted by $ERR\ L4, ERR\ L6, \cdots$ are the absolute errors of $\lambda(u,v)$. It is clear that the absolute errors are decreased by increasing the number of controls NC.

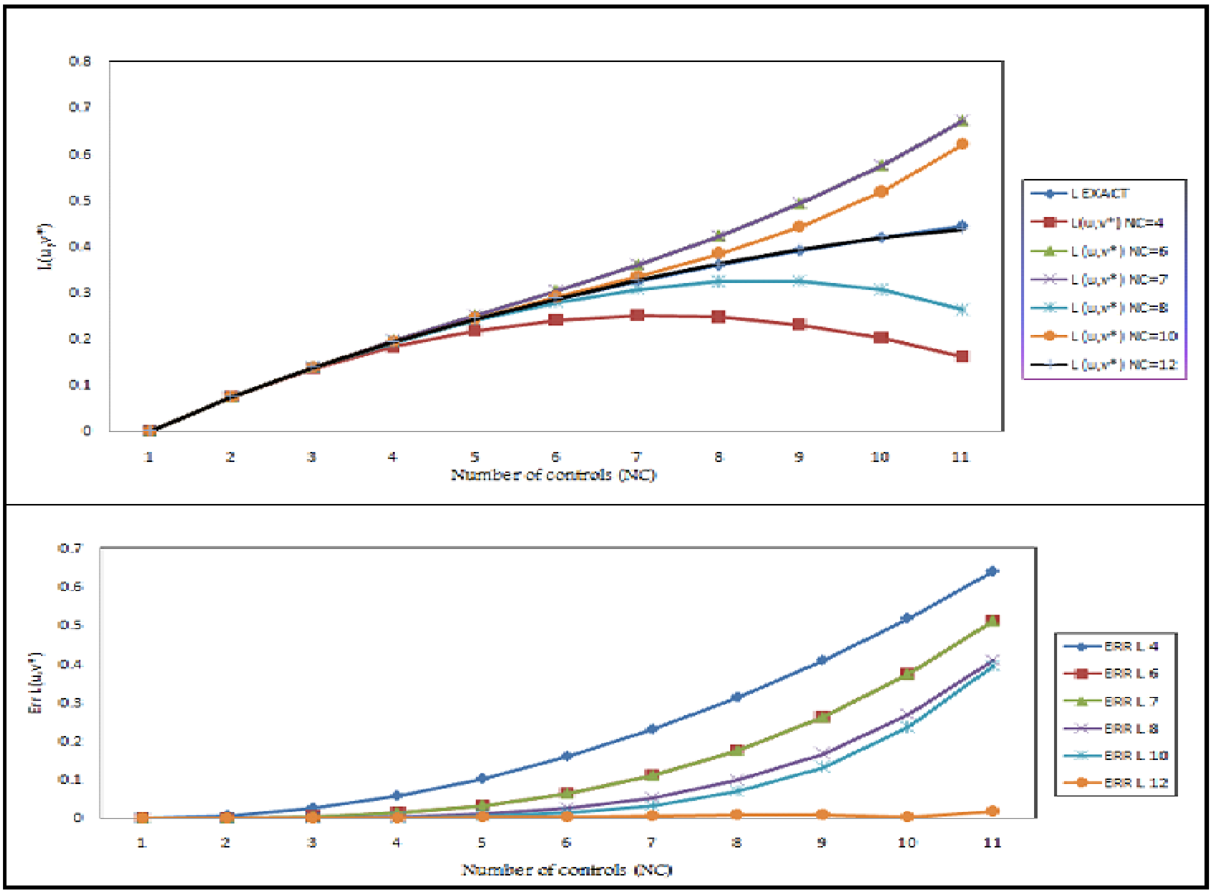

Figure 1

3) For example 2, the curves of the initial and computed optimal control by the above numerical algorithm versus number of controls (NC) are displayed in Figure 2.

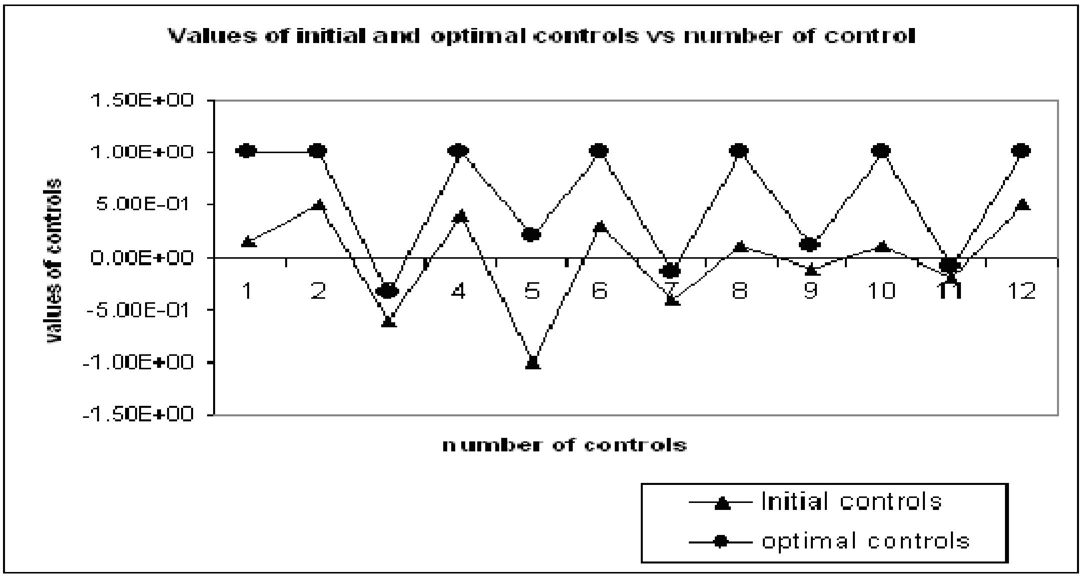

Figure 2

4) For examples 4, Figure 3 shows the values of the components of the gradient modified function $\left.\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}\right|_{v=v^*}$ at every iteration (IT) versus number of controls (NC).

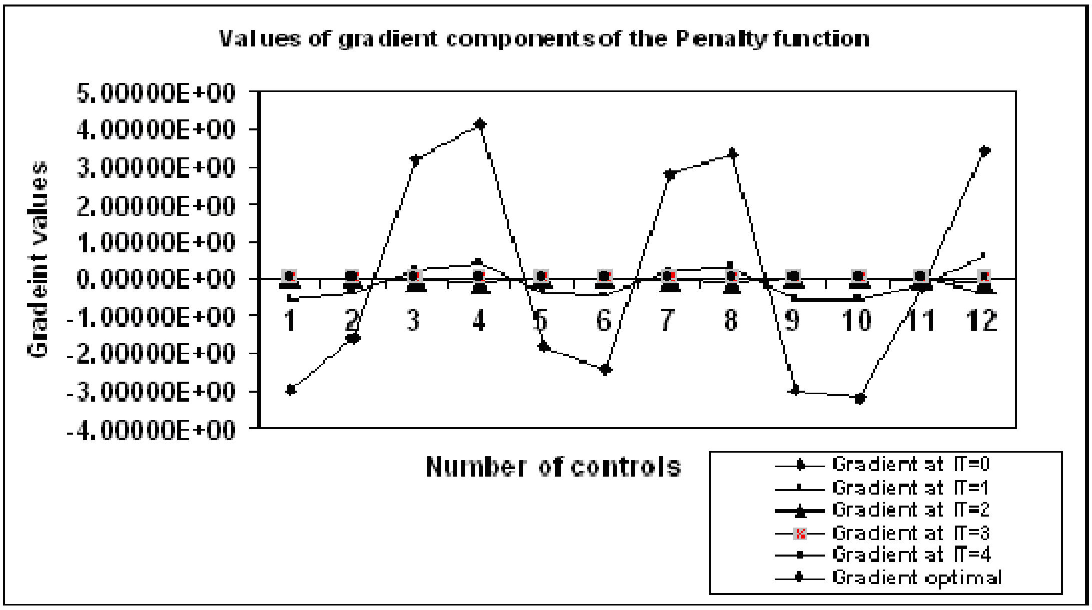

Figure 3

5) For example 2, the maximum absolute errors of $\lambda(u,v)$ and $B(u,v)$ versus the number of controls NT are displayed in table (1) which are calculated by the following formulae

$$MAE_{\lambda} = \frac{\max \left| \lambda_{EXACT} - \lambda^{*}_{\frac{NT}{2}} \right|}{\lambda_{EXACT}}, \quad MAE_{\mathbf{B}} = \frac{\max \left| B_{EXACT} - B^{*}_{\frac{NT}{2}} \right|}{B_{EXACT}}.$$

It is clear that the maximum absolute errors are decreased by increasing the number of controls.

Numbers of Controls	Max Absolute Error of L(u,v*)	Max Absolute Error of B(u,v*)
3	1.4026E+00	1.1887E+00
5	8.3558E-01	1.5795E-01
8	4.6555E-01	3.7993E-02
12	2.6045E-01	5.4946E-03

Table 1

6) For example 3, Figure 4 we display the exact and the approximate solutions $u(x,t,v^{*})$ of the state equation (29) - (31) at the computed optimal control values v^{*}.

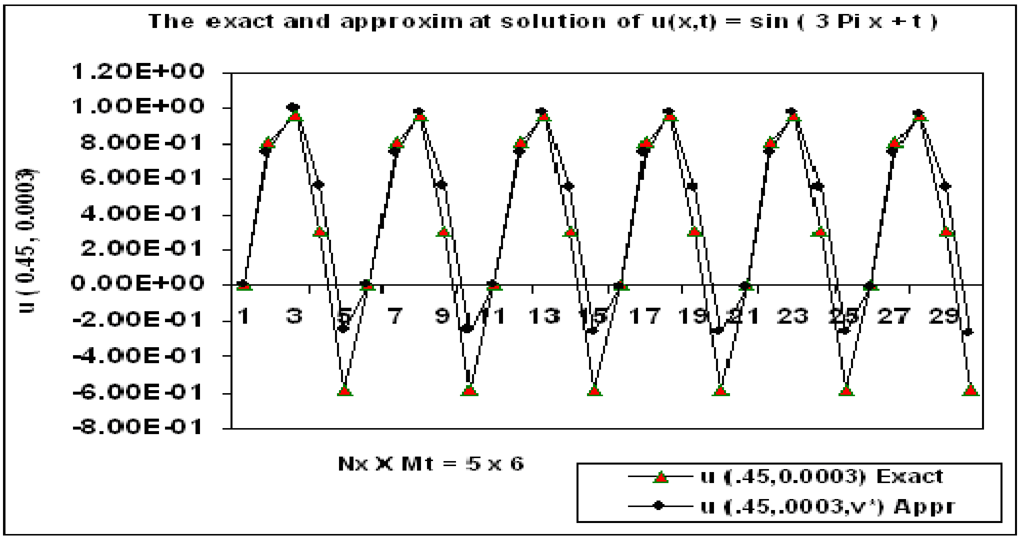

Figure 4

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Authors' Information

M. H. FARAG *– Professor of Mathematics (Numerical Analysis and Optimal control PDEs), Mathematics Department; Faculty of Science; Minia University; Minia, Egypt*

e-mail: farag5358@yahoo.com

T. A. TALAAT *– M.Sc Student, Mathematics Department; Faculty of Science; Minia University; Minia, Egypt*

e-mail: talaat.2008@yahoo.com