COMPARISON OF DIFFERENT WAVELET BASES IN THE CASE OF WAVELETS EXPANSIONS OF RANDOM PROCESSES

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Abstract: *In the paper wavelets expansions of random processes are studied. The matter is that although it is enough information for wavelets expansions of deterministic functions, for random processes such theory is weak and it should be developed. The paper investigates uniform convergence of wavelet expansions of Gaussian random processes. The convergence is obtained under simple general conditions on processes and wavelets which can be easily verified. Applications of the developed technique are shown for several wavelet bases. So, conditions of uniform convergence for Battle-Lemarie wavelets and Meyer wavelets expansions of Gaussian random processes are presented. Another useful in various computational applications thing is the rate of convergence, especially if we are interested in the optimality of the stochastic approximation or the simulations. An explicit estimate of the rate of uniform convergence for Battle-Lemarie wavelets and Meyer wavelets expansions of Gaussian random processes is obtained and compared.*

Keywords: *random processes, wavelets expansion, uniform convergence, Battle-Lemarie wavelets, Meyer wavelets, Gaussian processes.*

ACM Classification Keywords: *G.3 Probability and Statistics - Stochastic processes*

Introduction

Wavelet analysis is an exciting effective method for solving difficult problems in mathematics, physics, economics, medicine and engineering.

The most actual issues of application of wavelet analysis related with signal processing and simulation, audio and image compression, noise removal, the identification of short-term and global patterns, spectral analysis of the signal. From a practical point of view, multiresolution analysis provides an efficient basis for the expansion of stochastic processes. Wavelet representations could be used to convert the problem of analyzing a continuous-time random process to that of analyzing a random sequence, which is much simpler. This approach is widely used in statistics to estimate a curve given observations of the curve plus some noise, in time series analysis for smoothing functional data, in simulation studies of various functionals defined on realizations of a random process, etc.

Recently, a considerable attention was given to wavelet orthonormal series representations of stochastic processes. Some results, applications, and references on convergence of wavelet expansions of random processes in various spaces can be found in [Atto et al., 2010; Bardet et al., 2010; Didier et al., 2008; Kozachenko et al., 2011, 2013; Kozachenko, Polosmak, 2008], just to mention a few.

In the paper we study uniform convergence of wavelet decompositions which is required for various practical applications (but most known results in the open literature concern the mean-square convergence of wavelets expansions). So we consider stationary Gaussian random processes $\mathbf{X}(t)$ and their approximations by sums of wavelet functions

$$\mathbf{X}_{n,\mathbf{k}_n}(t) := \sum_{|k|\le k_{0'}} \xi_{0k}\phi_{0k}(t) + \sum_{j=0}^{n-1}\sum_{|k|\le k_j} \eta_{jk}\psi_{jk}(t), \tag{1}$$

where $\mathbf{k}_n := (k_{0'}, k_0, ..., k_{n-1})$, functions $\phi_{0k}(t), \psi_{jk}(t)$ are wavelet bases (in the papper we consider Battle-Lemarie and Meyer wavelets).

In direct numerical implementations we always consider truncated series like (1), where the number of terms in the sums is finite by application reasons (this makes it possible to find an explicit estimate of the rate of uniform convergence for wavelets expansions of random processes).

The rate of convergence is very useful notion in various computational applications. But this question has been studied very little.

In this paper our focus is on the Battle-Lemarie and Meyer wavelet bases. This is done with the aim to show that all our results are not only theoretical, but they can be used in practice. Using the program Wolfram Mathematica, we get the convergence rate for Battle-Lemarie and Meyer wavelet decompositions of Gaussian random processes.

The organization of this article is the following. In the second section we introduce the necessary background from wavelet theory and a theorem on uniform convergence in probability of the wavelet expansions of stationary Gaussian random processes, obtained in [Kozachenko et al., 2011]. In the third section we give some notions about the Meyer wavelet bases and obtain conditions of uniform convergence for this wavelets. The next section contains the rate of convergence in the space $C([0,T])$ of Meyer wavelet decompositions of stationary Gaussian random processes. In the section 5 we give some notions about the Battle-Lemarie wavelet bases and obtain conditions of uniform convergence for this wavelets. The next section contains the rate of uniform convergence of Battle-Lemarie wavelet decompositions of stationary Gaussian random processes. Conclusions are made in section 7.

Wavelet Representation of Random Processes

Let $\phi(x),\ x \in \mathbf{R}$ be a function from the space $L_2(\mathbf{R})$ such that $\hat{\phi}(0) \neq 0$ and $\hat{\phi}(y)$ is continuous at 0, where $\hat{\phi}(y) = \int_{\mathbf{R}} e^{-iyx}\phi(x)dx$ is the Fourier transform of ϕ.

Suppose that the following assumption holds true: $\sum_{k\in\mathbf{Z}} |\hat{\phi}(y+2\pi k)|^2 = 1\ (a.e.)$,

there exists a function $m_0(x) \in L_2([0,2\pi])$, such that $m_0(x)$ has the period 2π and $\hat{\phi}(y) = m_0(y/2)\hat{\phi}(y/2)\ (a.e.)$. In this case the function $\phi(x)$ is called the f-wavelet.

Let $\psi(x)$ be the inverse Fourier transform of the function

$$\hat{\psi}(y) = \overline{m_0\left(\frac{y}{2}+\pi\right)} \cdot \exp\left\{-i\frac{y}{2}\right\} \cdot \hat{\phi}\left(\frac{y}{2}\right).$$

Then the function $\psi(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{iyx}\hat{\psi}(y)dy$ is called the m-wavelet.

Let $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in \mathbf{Z}.$

It is known that the family of functions $\{\phi_{0k};\psi_{jk}, j\in \mathbf{N}_0\}$ is an orthonormal basis in $L_2(\mathbf{R})$ (see, for example, [Hardle et al., 1998]).

An arbitrary function $f(x)\in L_2(\mathbf{R})$ can be represented in the form

$$f(x)=\sum_{k\in\mathbf{Z}}\alpha_{0k}\phi_{0k}(x)+\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}}\beta_{jk}\psi_{jk}(x), \tag{2}$$

$$\alpha_{0k}=\int_{\mathbf{R}}f(x)\overline{\phi_{0k}(x)}dx,\quad \beta_{jk}=\int_{\mathbf{R}}f(x)\overline{\psi_{jk}(x)}dx.$$

The representation (2) is called a wavelet representation.

The series (2) converges in the space $L_2(\mathbf{R})$ i.e. $\sum_{k\in\mathbf{Z}}|\alpha_{0k}|^2+\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}}|\beta_{jk}|^2<\infty.$

The integrals α_{0k} and β_{jk} may also exist for functions from $L_p(\mathbf{R})$ and other function spaces. Therefore it is possible to obtain the representation (2) for function classes which are wider than $L_2(\mathbf{R})$.

Let $\{\Omega, B, P\}$ be a standard probability space. Let $\mathbf{X}(t),\ t\in\mathbf{R}$ be a random process such that $\mathbf{EX}(t)=0$ for all $t\in\mathbf{R}$.

It is possible to obtain representations like (2) for random processes, if their sample trajectories are in the space $L_2(\mathbf{R})$. However the majority of random processes do not possess this property. For example, sample paths of stationary processes are not in $L_2(\mathbf{R})$ (a.s.).

We investigate a representation of the kind (2) for $\mathbf{X}(t)$ with mean-square integrals

$$\xi_{0k}=\int_{\mathbf{R}}\mathbf{X}(t)\overline{\phi_{0k}(t)}dt,\quad \eta_{jk}=\int_{\mathbf{R}}\mathbf{X}(t)\overline{\psi_{jk}(t)}dt.$$

Consider the approximants $\mathbf{X}_{n,\mathbf{k}_n}(t)$ of $\mathbf{X}(t)$ defined by (1).

Assumption S. [Hardle et al., 1998] For the function ϕ there exists a decreasing function $\Phi(x),\ x\ge 0$ such that $\Phi(0)<\infty,\ |\phi(x)|\le\Phi(|x|)$ (a.e.) and $\int_{\mathbf{R}}\Phi(|x|)dx<\infty.$

Let $\mathbf{X}(t)$ be a stationary separable centered Gaussian random process such that its covariance function $R(t,s)=R(t-s)$ is continuous. Let the f-wavelet ϕ and the corresponding m-wavelet ψ be continuous functions and the assumption S holds true for both ϕ and ψ.

Theorem 1 below guarantees the uniform convergence of $\mathbf{X}_{n,\mathbf{k}_n}(t)$ to $\mathbf{X}(t)$.

Theorem 1 [Kozachenko et al., 2011] Suppose that the following conditions hold:

1. There exist $\phi'(u),\ \widehat{\psi}''(u),$ and $\widehat{\psi}(0)=0,\ \widehat{\psi}'(0)=0;$

2. $c_\phi:=\sup_{u\in\mathbf{R}}|\hat{\phi}(u)|<\infty,\ c_{\phi'}:=\sup_{u\in\mathbf{R}}|\hat{\phi}'(u)|<\infty,\ \widehat{\psi}'(u)\in L^1(\mathbf{R}),\ c_{\psi''}:=\sup_{u\in\mathbf{R}}|\widehat{\psi}''(u)|<\infty;$

3. $\hat{\phi}(u)\to 0$ and $\widehat{\psi}(u)\to 0$ when $u\to\pm\infty;$

4. There exist $0<\gamma<\frac{1}{2}$ and $\alpha>\frac{1}{2}$ such that $\int_{\mathbf{R}}\left(\ln(1+|u|)\right)^{\alpha}|\widehat{\psi}(u)|^{\gamma}du<\infty,$

$$\int_{\mathbf{R}}\left(\ln(1+|u|)\right)^{\alpha}|\hat{\phi}(u)|^{\gamma}du<\infty;$$

5. There exists $\widehat{R}(z)$ and $\sup_{z\in\mathbf{R}}\widehat{R}(z)<\infty$;

6. $\int_{\mathbf{R}}\left|\widehat{R}'(z)\right|dz<\infty$ and $\int_{\mathbf{R}}\left|\widehat{R}^{(p)}(z)\right||z|^4dz<\infty$ for $p=0,1.$

Then $\mathbf{X}_{n,\mathbf{k}_n}(t)\to\mathbf{X}(t)$ uniformly in probability on each interval $[0,T]$ when $n\to\infty$, $k_{0'}\to\infty$ and $k_j\to\infty$ for all $j\in\mathbf{N}_0$.

Conditions of Uniform Convergence for Meyer Wavelets Decompositions of Gaussian Random Processes

Meyer wavelets $\phi(x)$ and $\psi(x)$ cab be given as inverse Fourier transforms of the functions $\hat{\phi}(y)$ and $\widehat{\psi}(y)$ respectively. The expressions of $\hat{\phi}(y)$ and $\widehat{\psi}(y)$ are following:

$$\hat{\phi}(y)=\begin{cases}\frac{1}{\sqrt{2}}\hat{h}(\frac{y}{2}), & |y|\leqslant\frac{4\pi}{3}\\ 0, & |y|>\frac{4\pi}{3}\end{cases}\tag{3}$$

where

$$\hat{h}(y)=\begin{cases}\sqrt{2}, & |y|\leqslant\frac{\pi}{3}\\ 0, & y\in[-\pi,-\frac{2\pi}{3}]\cup[\frac{2\pi}{3},\pi]\end{cases}\tag{4}$$

and

$$\widehat{\psi}(y)=\begin{cases}0, & |y|\leqslant\frac{2\pi}{3},\\ \frac{1}{\sqrt{2}}\widehat{g}(\frac{y}{2}), & \frac{2\pi}{3}\leqslant|y|\leqslant\frac{4\pi}{3}\\ \frac{1}{\sqrt{2}}e^{-i\frac{y}{2}}\hat{h}(\frac{y}{4}), & \frac{4\pi}{3}\leqslant|y|\leqslant\frac{8\pi}{3},\\ 0, & |y|>\frac{8\pi}{3}\end{cases}\tag{5}$$

where

$$\widehat{g}(y)=e^{-iy}\hat{h}^*(y+\pi).$$

The functions $\phi(x)$ and $\psi(x)$ are C^∞ because their Fourier transforms have a compact support. Wavelet $\psi(x)$ has an infinite number of vanishing moments [Mallat, 1998], so $\widehat{\psi}^{(k)}(0)=0,\,k\geq 0.$

Theorem 2 Let $\mathbf{X}(t)$ be a stationary separable centered Gaussian random process such that its covariance function $R(t,s)=R(t-s)$ is continuous. Let ϕ and ψ be Meyer wavelets. Suppose that the following conditions hold:

1. There exists $\widehat{R}(z)$ and $\sup_{z\in\mathbf{R}}\widehat{R}(z)<\infty$;

2. $\int_{\mathbf{R}}\left|\widehat{R}'(z)\right|dz<\infty$ and $\int_{\mathbf{R}}\left|\widehat{R}^{(p)}(z)\right||z|^4dz<\infty$ for $p=0,1.$

Then $\mathbf{X}_{n,\mathbf{k}_n}(t)\to\mathbf{X}(t)$ uniformly in probability on each interval $[0,T]$ when $n\to\infty$, $k_{0'}\to\infty$ and $k_j\to\infty$ for all $j\in\mathbf{N}_0$.

Proof. Statement of this Theorem follows from Theorem 1, if we take into account that assumptions 1) – 4) of Theorem 1 hold for the Meyer wavelets. Indeed, Meyer wavelet $\psi(x)$ has an infinite number of vanishing moments [Mallat, 1998], so $\widehat{\psi}^{(k)}(0)=0, k\ge 0.$ Now we can use such fact that Fourier transforms of Meyer wavelets have a compact support, so we have fulfillment of conditions 3) and 4) of Theorem 1. Another fact that Fourier transforms of Meyer wavelets is n times continuously differentiable, then assumption 2 of Theorem 1 holds true.

Convergence Rate in the Space $C[0,T]$ of the Meyer Wavelets Representations of Random Processes

In the paper [Kozachenko et al., 2013] an explicit estimate of the rate of uniform convergence for wavelets expansions of Gaussian random processes is obtained. In this section our focus is on the Meyer wavelet bases. So convergence rate in the space $C[0,T]$ for the Meyer wavelets decompositions of Gaussian random processes is studied.

Theorem 3 [Kozachenko et al., 2013] Let $X(t), t\in[0,T]$ be a separable Gaussian stationary random process. Let assumptions of Theorem 1 hold true for $X(t)$.

Then

$$P\left\{\sup_{t\in[0,T]}|\mathbf{X}(t)-\mathbf{X}_{n,\mathbf{k}_n}(t)|>u\right\}\le 2\exp\left\{-\frac{(u-\sqrt{8u\delta(\varepsilon_{\mathbf{k}_n})})^2}{2\varepsilon_{\mathbf{k}_n}^2}\right\},$$

where $u>8\delta(\varepsilon_{\mathbf{k}_n})$,

$$\varepsilon_{\mathbf{k}_n}:=\sum_{j=0}^{n-1}\frac{A}{2^{j/2}\sqrt{k_j}}+\frac{B}{\sqrt{k_{0'}}}+\frac{C}{2^{n/2}}.$$

A, B, and C are constants which depend only on the covariance function of $\mathbf{X}(t)$ and the wavelet basis. Explicit expressions for A, B, and C are given in the proof of the theorem.

From the paper [Kozachenko et al., 2013] A, B, C, are following:

$$A:=B_1^{\psi}\left(6A^{\psi}\sum_{m=1}^{\infty}\frac{1}{m^{3/2}}+4A_1^{\psi}\right)^{1/2}.$$

$$B:=B_1^{\phi}\left(6A^{\phi}\sum_{m=1}^{\infty}\frac{1}{m^{3/2}}+4A_1^{\phi}\right)^{1/2}.$$

$$C := (2+\sqrt{2})\left(3A^{\psi}\left(B_1^{\psi}\right)^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{\frac{3}{2}}}\right)^2\right.$$

$$\left.+\left(A_1^{\psi}\left(B_1^{\psi}\right)^2+\frac{c_2A^{\psi}B_1^{\psi}}{\pi}\right)\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{c_2^2A_1^{\psi}}{32\pi^2}\right)^{1/2}.$$

In the case $T = 3$ we can get:

$$B_1^{\phi} = \frac{1}{(2\pi)}\left(\int_{\mathbf{R}}\left|\hat{\phi}'(u)\right|du + T\int_{\mathbf{R}}\left|\hat{\phi}(u)\right|du\right) \approx 3.49516,$$

$$B_1^{\psi} = \frac{1}{(2\pi)}\left(\int_{\mathbf{R}}\left|\hat{\psi}'(u)\right|du + T\int_{\mathbf{R}}\left|\hat{\psi}(u)\right|du\right) < 0.01,$$

$$l = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}} = \zeta\left(\frac{3}{2}\right),\ \tilde{l} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

For the covariance function $R(\tau) = \exp\{-\frac{4}{9}\tau^2\}$ we can obtain the value of the following expressions:

$$A^{\phi} := \frac{1}{2\pi}\left(c_{\phi}^2\int_{\mathbf{R}}\left|\widehat{R'}(z)\right|dz + 2c_{\phi}c_{\phi'}\int_{\mathbf{R}}\left|\widehat{R}(z)\right|dz\right) \approx 1.1355,$$

$$A^{\psi} = \frac{c_{\psi''}^2}{2\pi}\left(\int_{\mathbf{R}}|\widehat{R}'(z)||z|^4dz + 2\int_{\mathbf{R}}|\widehat{R}(z)||z|^3dz\right) \approx 3.20112,$$

$$A_1^{\phi} := \frac{c_{\phi}^2}{2\pi}\int_{\mathbf{R}}|\widehat{R}(z)|dz \approx 0.398942,$$

$$A_1^{\psi} := \frac{c_{\psi''}^2}{2\pi}\int_{\mathbf{R}}|\widehat{R}(z)||z|^4dz \approx 0.945641,$$

$$c_2 := \int_{\mathbf{R}}\left|\widehat{\psi}(v)\right|dv < 0.01.$$

Then we can calculate constants for the expression $\varepsilon_{\mathbf{k}_n}$:

$$A \approx 0.0073456,\ B \approx 15.3922,\ C \approx 0.004424.$$

In the paper [Kozachenko et al., 2013] needed formulas for calculation of the following expression are given:

$$\delta(\varepsilon_{\mathbf{k}_n}) := \frac{\gamma}{\sqrt{2}}\left(\sqrt{\ln(T+1)} + \left(1-\frac{1}{2\alpha}\right)^{-1}\left(\frac{c}{\gamma}\right)^{\frac{1}{2\alpha}}\right),$$

where $\gamma := \min\left(\varepsilon_{\mathbf{k}_n}, \sigma\left(\frac{T}{2}\right)\right),\ \alpha > \frac{1}{2}.$

So for $\alpha = 0.6$ and $\beta = 0.52$ we can evaluate following constants:

$$c = B_0 + B_1 + B_2$$

$$B_0 := \left(q_1 + A^{\psi} Q_1 K^2\right)^{1/2} \cdot \sum_{j=0}^{\infty} \frac{(j+1)^{\alpha}}{2^{j/2}} \approx 0.107345,$$

$$B_1 := \left(q + q_1 + q_2 + A^{\psi} Q K^2\right)^{1/2} \cdot \sum_{j=0}^{\infty} \frac{(j+1)^{\alpha}}{2^{j/2}} \approx 0.108815,$$

$$B_2 := \left(q_{\phi 1} + A^{\phi} (K^{\phi})^2 Q\right)^{1/2} \approx 1308.01.$$

$$K := \pi^{-1} \Big(2^{3+\alpha-\beta} \pi^{\beta} c_{\psi'}^{\beta} \left((\ln 5)^{\alpha} c_0 + c_1\right) +$$

$$+ \pi T 2^{\alpha-1} \left((\ln 5)^{\alpha} c_2 + c_3\right) + c_{\alpha} c_2 \Big) < 0.01.$$

$$c_0 := \int_{\mathbf{R}} \left|\widehat{\psi}(v)\right|^{1-\beta} dv < \infty, \quad c_1 := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\widehat{\psi}(v)\right|^{1-\beta} dv < \infty.$$

$$c_2 := \int_{\mathbf{R}} \left|\widehat{\psi}(v)\right| dv < \infty, \quad c_3 := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\widehat{\psi}(v)\right| dv < \infty.$$

$$Q \le Q_1 = \left(\sum_{k=1}^{\infty} \frac{1}{2k^{\frac{1}{2}+\beta}}\right)^2 + c_{\delta}^{\beta} \sum_{m=1}^{\infty} \frac{1}{m^{1+\delta\beta}} \sum_{l=1}^{\infty} \frac{1}{l^{(2-\delta)\beta}} \approx 111.259,$$

$$q := \frac{2^{\alpha} A^{\psi} K((\ln 5)^{\alpha} c_2 + c_3)}{\pi} \cdot \sum_{l=1}^{\infty} \frac{1}{l^{1+\beta}} \approx 8.8 \times 10^{-6},$$

$$q_1 := \frac{A_1^{\psi} K^2}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\beta}} \approx 5.0 \times 10^{-6},$$

$$q_2 := \frac{2^{2\alpha} A_1^{\psi}}{\pi^2} \left((\ln 5)^{\alpha} c_2 + c_3\right)^2 \approx 1.2 \times 10^{-6},$$

$$q_{\phi 1} := \frac{A_1^{\phi} (K^{\phi})^2}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\beta}} \approx 8348.11.$$

$$K^{\phi} := \pi^{-1} \Big(2^{3+\alpha-\beta} \pi^{\beta} c_{\phi'}^{\beta} \left((\ln 5)^{\alpha} c_{\phi 0} + c_{\phi 1}\right) +$$

$$\pi T 2^{\alpha-1} \left((\ln 5)^{\alpha} c_{\phi 2} + c_{\phi 3}\right) + c_{\alpha} c_{\phi 2} \Big) \approx 116.087,$$

$$c_{\phi 0} := \int_{\mathbf{R}} \left|\hat{\phi}(v)\right|^{1-\beta} dv < \infty, \quad c_{\phi 1} := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\hat{\phi}(v)\right|^{1-\beta} dv < \infty,$$

$$c_{\phi 2} := \int_{\mathbf{R}} \left|\hat{\phi}(v)\right| dv < \infty, \quad c_{\phi 3} := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\hat{\phi}(v)\right| dv < \infty.$$

So, if we take into consideration this calculation, then c is following:

$$c \approx 1308.22.$$

Naturally:

$$\sigma(T) = \frac{c}{\left(\ln\left(e^{\alpha} + \frac{1}{T}\right)\right)^{\alpha}} \approx 1532.73.$$

$X_{n,k_j}(t)$ approximates process $X(t)$ with the reliability of $1-\tilde{\delta}$ and accuracy $\tilde{\varepsilon}$, if

$$\mathbf{P}\left\{\sup_{0\le t\le T} | X(t)-X_{n,k_j}(t) |> \tilde{\varepsilon}\right\}\le \tilde{\delta}.$$

Let $\tilde{\delta}=0.01,$ we can use the rule of three σ , then it can be considered as $\tilde{\varepsilon}=0.1\cdot 6\sigma$. In our case, for the covariance function $R(\tau)=\exp\{-\frac{4}{9}\tau^2\}$, we can calculate the variance $\sigma=1$, so $\tilde{\varepsilon}=0.6.$ Then, for Meyer wavelets, using the program Wolfram Mathematica, we can obtain such $\tilde{\delta}=0.01$ at $k_0=85$, $k_j=20$, $n=20$ with a slight increase k_0 , $\tilde{\delta}$ is significantly reduced.

Conditions of Uniform Convergence for Battle-Lemarie Wavelets Expansions of Gaussian Random Processes

Polynomial spline wavelets introduced by Battle and Lemarie are computed from spline multiresolution approximations. Let $\phi_m(x)$ and $\psi_m(x)$ be the inverse Fourier transforms of the functions $\hat{\phi}_m(y)$ and $\widehat{\psi}_m(y)$ respectively. The expressions of $\hat{\phi}_m(y)$ and $\widehat{\psi}_m(y)$ are following:

$$\hat{\phi}_m(y)=\frac{e^{-i\frac{y\cdot\varepsilon}{2}}}{y^{m+1}\sqrt{S_{2m+2}(y)}}, \tag{6}$$

where

$$S_n(y)=\sum_{k=-\infty}^{\infty}\frac{1}{(y+2k\pi)^n} \tag{7}$$

and $\varepsilon=1$ if m is even and $\varepsilon=0$ if m odd.

$$\widehat{\psi}_m(y)=\frac{e^{-i\frac{y}{2}}}{y^{m+1}}\sqrt{\frac{S_{2m+2}(\frac{y}{2}+\pi)}{S_{2m+2}(y)S_{2m+2}(\frac{y}{2})}}. \tag{8}$$

For the $m-$ degree spline wavelet $\psi(x)$ has $m+1$ vanishing moments [Mallat, 1998], so $\widehat{\psi}^{(k)}(0)=0, 0\le k\le m+1.$ Wavelet $\psi(x)$ has an exponential decay. Since it is a polynomial spline of degree $m,$ it is $m-1$ times continuously differentiable (see, for example, [Mallat, 1998]).

To check the assumption 1 of Theorem 1 we can use Lemma 1 from the paper [Polosmak, 2009]:

Lemma 1 [Polosmak, 2009] Let $\phi(x)$ - such function, that $\int_{\mathbf{R}} |\phi(x)| dx<\infty, \phi(x)\to 0, x\to\pm\infty,$ let the derivative $\phi'(x)$ exists, such that $\int_{\mathbf{R}} |\phi'(x)|^{\gamma} dx<\infty$ for some $0<\gamma<1.$ Let $|\phi'(x)-\phi'(y)|\le\sigma(|x-y|),$ where $\sigma=\{\sigma(u),u>0\}$ such monotone increasing function that $\sigma(0)=0.$

Then $\int_{\mathbf{R}} |\hat{\phi}(y)|\, c(y)dy < \infty$, where $\hat{\phi}(y) = \int_{\mathbf{R}} e^{-iyx}\phi(x)dx$, and $c = \{c(y), y \in \mathbf{R}\}$, $c(y) > 0$ such function, that $\int_1^\infty \frac{1}{|y|}\left(\sigma\left(\frac{\pi}{y}\right)\right)^{1-\gamma} c(y)dy < \infty.$

Remark 1 In the case of $\sigma(u) = c\,|u|^\alpha$, $0 < \alpha \le 1$, we can take $c(y) = \ln(1+|y|)^\delta$, where $\delta > \frac{1}{2}$.

Theorem 4 Let $\mathbf{X}(t)$ be a stationary separable centered Gaussian random process such that its covariance function $R(t,s) = R(t-s)$ is continuous. Let ϕ and ψ be Battle-Lemarie wavelets. Suppose that the following conditions hold:

1. There exists $\widehat{R}(z)$ and $\sup_{z\in\mathbf{R}} \widehat{R}(z) < \infty$;

2. $\int_{\mathbf{R}} \left|\widehat{R}'(z)\right| dz < \infty$ and $\int_{\mathbf{R}} \left|\widehat{R}^{(p)}(z)\right| |z|^4 dz < \infty$ for $p = 0,1.$

Then $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ uniformly in probability on each interval $[0,T]$ when $n \to \infty$, $k_{0'} \to \infty$ and $k_j \to \infty$ for all $j \in \mathbf{N}_0$.

Proof. Statement of this Theorem follows from Theorem 1, if we take into account that assumptions 1) – 4) of Theorem 1 hold for the Battle-Lemarie wavelets. Indeed, $m-$ degree ψ - wavelet Battle-Lemarie has $m+1$ vanishing moments [Mallat, 1998], so $\widehat{\psi}^{(k)}(0) = 0, 0 \le k \le m+1$. Another fact that it is $m-1$ times continuously differentiable, then, using formulas (3),(5), we have fulfillment of conditions 1) – 3) of Theorem 1. Assumption 4 follows from Lemma 1, differentiability of the Battle-Lemarie wavelets and Remark 1.

Convergence Rate in the Space $C[0,T]$ of the Battle-Lemarie Wavelets Representations of Random Processes

In the previous section it was given Theorem 3 from the paper [Kozachenko et al., 2013] in which an explicit estimate of the rate of uniform convergence for wavelets expansions of Gaussian random processes is obtained. In this section our focus is on the Battle-Lemarie wavelet bases. So convergence rate in the space $C[0,T]$ for the Battle-Lemarie wavelets decompositions of Gaussian random processes is studied. Here we obtain all constants for Theorem 3 in the case of Battle-Lemarie wavelets:

$$A := B_1^\psi \left(6A^\psi \sum_{m=1}^\infty \frac{1}{m^{3/2}} + 4A_1^\psi\right)^{1/2}.$$

$$B := B_1^\phi \left(6A^\phi \sum_{m=1}^\infty \frac{1}{m^{3/2}} + 4A_1^\phi\right)^{1/2}.$$

$$C := (2+\sqrt{2})\left(3A^\psi \left(B_1^\psi\right)^2 \left(\sum_{k=1}^\infty \frac{1}{k^{\frac{3}{2}}}\right)^2 + \left(A_1^\psi \left(B_1^\psi\right)^2 + \frac{c_2 A^\psi B_1^\psi}{\pi}\right)\sum_{k=1}^\infty \frac{1}{k^2} + \frac{c_2^2 A_1^\psi}{32\pi^2}\right)^{1/2}.$$

In the case $T = 3$ we can get:

$$B_1^{\phi} = \frac{1}{(2\pi)}\left(\int_{\mathbf{R}} \left|\hat{\phi}'(u)\right| du + T\int_{\mathbf{R}} \left|\hat{\phi}(u)\right| du \right) \approx 3.97712,$$

$$B_1^{\psi} = \frac{1}{(2\pi)}\left(\int_{\mathbf{R}} \left|\widehat{\psi}'(u)\right| du + T\int_{\mathbf{R}} \left|\widehat{\psi}(u)\right| du \right) \approx 4.64062,$$

$$l = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} = \zeta\left(\frac{3}{2}\right),\ \tilde{l} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

For the covariance function $R(\tau) = \exp\{-\frac{4}{9}\tau^2\}$ we can obtain the value of the following expressions:

$$A^{\phi} := \frac{1}{2\pi}\left(c_{\phi}^2 \int_{\mathbf{R}} \left|\widehat{R'}(z)\right| dz + 2c_{\phi}c_{\phi'} \int_{\mathbf{R}} \left|\widehat{R}(z)\right| dz \right) \approx 1.1355,$$

$$A^{\psi} = \frac{c_{\psi''}^2}{2\pi}\left(\int_{\mathbf{R}} |\widehat{R}'(z)||z|^4 dz + 2\int_{\mathbf{R}} |\widehat{R}(z)||z|^3 dz \right) \approx 3.20112,$$

$$A_1^{\phi} := \frac{c_{\phi}^2}{2\pi} \int_{\mathbf{R}} |\widehat{R}(z)| dz \approx 0.398942,$$

$$A_1^{\psi} := \frac{c_{\psi''}^2}{2\pi} \int_{\mathbf{R}} |\widehat{R}(z)||z|^4 dz \approx 0.945641,$$

$$c_2 := \int_{\mathbf{R}} \left|\widehat{\psi}(v)\right| dv \approx 7.5725.$$

Then we can calculate constants for the expression $\varepsilon_{\mathbf{k}_n}$:

$$A \approx 0.0073456,\ B \approx 17.5147,\ C \approx 129.78.$$

In the paper [Kozachenko et al., 2013] needed formulas for calculation of the following expression are given:

$$\delta(\varepsilon_{\mathbf{k}_n}) := \frac{\gamma}{\sqrt{2}}\left(\sqrt{\ln(T+1)} + \left(1 - \frac{1}{2\alpha}\right)^{-1} \left(\frac{c}{\gamma}\right)^{\frac{1}{2\alpha}} \right),$$

where $\gamma := \min\left(\varepsilon_{\mathbf{k}_n}, \sigma\left(\frac{T}{2}\right)\right),\ \alpha > \frac{1}{2}.$

So for $\alpha = 0.6$ and $\beta = 0.52$ we can evaluate following constants:

$$c = B_0 + B_1 + B_2$$

$$B_0 := \left(q_1 + A^{\psi} Q_1 K^2\right)^{1/2} \cdot \sum_{j=0}^{\infty} \frac{(j+1)^{\alpha}}{2^{j/2}} \approx 0.107345,$$

$$B_1 := \left(q + q_1 + q_2 + A^{\psi} Q K^2\right)^{1/2} \cdot \sum_{j=0}^{\infty} \frac{(j+1)^{\alpha}}{2^{j/2}} \approx 0.108815,$$

$$B_2 := \left(q_{\phi 1} + A^{\phi} (K^{\phi})^2 Q\right)^{1/2} \approx 1416.2.$$

$$K := \pi^{-1}\left(2^{3+\alpha-\beta} \pi^{\beta} c_{\psi'}^{\beta} \left((\ln 5)^{\alpha} c_0 + c_1\right) + \right.$$

$$+\pi T 2^{\alpha-1}\left((\ln 5)^{\alpha} c_2 + c_3\right) + c_{\alpha} c_2\Big) \approx 191.063.$$

$$c_0 := \int_{\mathbf{R}} \left|\widehat{\psi}(v)\right|^{1-\beta} dv < \infty, \quad c_1 := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\widehat{\psi}(v)\right|^{1-\beta} dv < \infty.$$

$$c_2 := \int_{\mathbf{R}} \left|\widehat{\psi}(v)\right| dv < \infty, \quad c_3 := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\widehat{\psi}(v)\right| dv < \infty.$$

$$Q \le Q_1 = \left(\sum_{k=1}^{\infty} \frac{1}{2k^{\frac{1}{2}+\beta}}\right)^2 + c_{\delta}^{\beta} \sum_{m=1}^{\infty} \frac{1}{m^{1+\delta\beta}} \sum_{l=1}^{\infty} \frac{1}{l^{(2-\delta)\beta}} \approx 111.259,$$

$$q := \frac{2^{\alpha} A^{\psi} K((\ln 5)^{\alpha} c_2 + c_3)}{\pi} \cdot \sum_{l=1}^{\infty} \frac{1}{l^{1+\beta}} \approx 8.8 \times 10^{-6},$$

$$q_1 := \frac{A_1^{\psi} K^2}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\beta}} \approx 5.0 \times 10^{-6},$$

$$q_2 := \frac{2^{2\alpha} A_1^{\psi}}{\pi^2} \left((\ln 5)^{\alpha} c_2 + c_3\right)^2 \approx 1.2 \times 10^{-6},$$

$$q_{\phi 1} := \frac{A_1^{\phi} (K^{\phi})^2}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\beta}} \approx 9786.24.$$

$$K^{\phi} := \pi^{-1}\Big(2^{3+\alpha-\beta} \pi^{\beta} c_{\phi'}^{\beta} \left((\ln 5)^{\alpha} c_{\phi 0} + c_{\phi 1}\right) +$$

$$\pi T 2^{\alpha-1}\left((\ln 5)^{\alpha} c_{\phi 2} + c_{\phi 3}\right) + c_{\alpha} c_{\phi 2}\Big) \approx 125.689,$$

$$c_{\phi 0} := \int_{\mathbf{R}} \left|\hat{\phi}(v)\right|^{1-\beta} dv < \infty, \quad c_{\phi 1} := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\hat{\phi}(v)\right|^{1-\beta} dv < \infty,$$

$$c_{\phi 2} := \int_{\mathbf{R}} \left|\hat{\phi}(v)\right| dv < \infty, \quad c_{\phi 3} := \int_{\mathbf{R}} \left(\ln(1+|v|)\right)^{\alpha} \left|\hat{\phi}(v)\right| dv < \infty.$$

So, if we take into consideration this calculation, then c is following:

$$c \approx 1416.41.$$

Naturally:

$$\sigma(T) = \frac{c}{\left(\ln\left(e^{\alpha} + \frac{1}{T}\right)\right)^{\alpha}} \approx 1659.48.$$

$X_{n,k_j}(t)$ approximates process $X(t)$ with the reliability of $1-\tilde{\delta}$ and accuracy $\tilde{\varepsilon}$, if

$$\mathbf{P}\left\{\sup_{0\le t\le T} |X(t) - X_{n,k_j}(t)| > \tilde{\varepsilon}\right\} \le \tilde{\delta}.$$

Let $\tilde{\delta} = 0.01$, we can use the rule of three σ, then it can be considered as $\tilde{\varepsilon} = 0.1 \cdot 6\sigma$. In our case, for the covariance function $R(\tau) = \exp\{-\frac{4}{9}\tau^2\}$, we can calculate the variance $\sigma = 1$, so $\tilde{\varepsilon} = 0.6$. Then, for Battle-Lemarie wavelets, using the program Wolfram Mathematica, we can obtain such $\tilde{\delta} = 0.01$ at $k_0 = 110$, $k_j = 20$, $n = 20$ with a slight increase k_0, $\tilde{\delta}$ is significantly reduced.

Conclusion

Conditions of uniform convergence for Meyer wavelet decompositions and Battle-Lemarie wavelet decompositions of stationary Gaussian random processes are presented. The rate of convergence in the space $C([0,T])$ of Meyer wavelet decompositions and Battle-Lemarie wavelet decompositions of stationary Gaussian random processes are obtained. We can conclude that both wavelet bases are good for expansion of stationary Gaussian random processes, but the Meyer wavelets have some advantages. For the same accuracy of the approximation in the case of the Meyer wavelets, we need fewer terms in the expansion.

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