A HIERARCHICAL APPROACH TO MULTICRITERIA PROBLEMS

Albert Voronin, Yuriy Ziatdinov, Igor Varlamov

***Abstract**: It is shown, that any multicriteria problem can be represented by a hierarchical system of criteria. Individual properties of the object (alternative) are evaluated at the bottom level of the system, using a criteria vector; and a composition mechanism is used to evaluate the object as a whole at the top level. The problem is solved by the method of nested scalar convolutions of vector-valued criteria. The methodology of the problem solving is based on the complementarity principle by N. Bohr and the theorem of incompleteness by K. Gödel.*

***Keywords**: hierarchical structure, nested scalar convolutions, multicriteria approach, decomposition; composition*

Introduction

The problem of decision making in general view can be represented by the scheme

$$\{\{x\}, Y\} \rightarrow x^*,$$

where $\{x\}$ is a set of objects (alternatives); *Y* is the function of choice (rule establishing a prefer ability on a set of alternatives); *x** is the chosen alternatives (one or more).

The function *Y* is used to solve the problem of analysis and evaluation of alternatives. On results of estimation the choice of one or a few best alternatives from the given set follows. In decision theory, there are two different approaches to evaluating objects (alternatives) subject to choice. One of them is to evaluate an object *as a whole* and to choose an alternative by comparing objects as gestalts (holistic images of objects without detailing their properties). The second approach is detailed elaboration and assessment of various object vectors of properties and making decisions after comparing these properties. If a holistic approach implies choosing *x** directly using choice function *Y*, the *vector approach* requires a mechanism to carry out decomposition of *Y* into a set (vector) of the choice functions *y*. By decomposition of the choice function *Y* is understood its equivalent representation by a certain set of other functions *y* which composition is the initial choice function *Y*.

Separation of properties of alternatives on the basis of the analysis is the decomposition leading to the hierarchical structure of properties.

Properties, for which there exist objective numerical characteristics, are called *criteria*. The approach of comparison on separate properties, at all its attraction, derivates a serious problem of return transition to required comparison of alternatives as a whole [Voronin, 2013].

Statement of the Problem

Quality of an alternative is determined by hierarchical system of vectors

$$y^{(j-1)} = \{y_i^{(j-1)}\}_{i=1}^{n^{(j-1)}}, \ j \in [2, m],$$

where $y^{(j-1)}$ is the vector of criteria on the (*j*-1)-th level of the hierarchy, by the components of which the quality of properties of alternatives for the *j*-th level is assessed; *m* is the amount of levels of the hierarchy; $n^{(j-1)}$ is the amount of estimated properties on (*j*-1)-th level of the hierarchy. The numerical values of *n* criteria $y^{(1)} = y$ of the first level of the hierarchy for the alternative are given.

The same criterion on (*j*-1)-th level can participate in the evaluation of several properties of the *j*-th level, i.e. in the hierarchy are possible cross-links. It is clear that $n^{(1)} = \sum_{i=1}^{n_1} r_i = n$ and $n^{(m)} = 1$.

Importance (significance) of each of the components of the criterion of (*j*-1)-th level in the evaluation of properties of *k*-th level is characterized by a property coefficient of the priority, their set forming the priority vectors system

$$p_{ik}^{(j-1)} = \{p_{ik}^{(j-1)}\}_{k=1}^{n^{(j)}},\ j \in [2, m].$$

It is required to find an analytical evaluation y^* and qualitative evaluation of the effectiveness of this given alternative, and from the alternatives available to choose the best.

The Method of Solution

At the study, the approach is used consisting in the creation and simultaneous co-existence of not one but many theoretical models of the same phenomenon, and some of them conceptually contradict each other. However, no one can be neglected, as each describes a property of the phenomenon and none can be taken as a single because it does not express the full range of its properties. Compare the said with the principle of complementarity, introduced into science by Niles Bohr: "... To reproduce the integrity of the phenomenon should be used mutually exclusive "complementary" classes of concepts, each of which can be used in its own, special conditions, but only when taken together, exhaust the definable information". It is the principle of complementarity that allows for separating and then linking these criteria in multicriteria evaluation. Only a full set of individual criteria (vector criterion) enables an adequate assessment of the functioning of a complex system as a manifestation of the contradictory unity of all its properties.

However, this possibility represents only a necessary but not a sufficient condition for the vector evaluation of the entire alternative as a whole.

For a complete evaluation it is necessary to go out from the lower level of the hierarchy and to rise on the following tier, i.e. to carry out an act of criteria composition. Let's compare this with the incompleteness theorem of Kurt Gödel "... In every complex enough not contradictory theory of the first order there is a statement, which by means of the theory is impossible neither to prove, nor to deny. But the self-consistency of a particular theory can be established by means of another, more powerful formal theory of the second order. But then the question of the self-consistency of this second theory arises, and so forth". We can say that Gödel's theorem is a methodological basis for the study of hierarchical structures.

With reference to our problem it means that for an adequate estimation of an alternative as a whole we should solve a task of the criteria composition on levels of hierarchy, consecutively passing from the bottom level up to top.

A scalar convolution of criteria can serve as a tool for the act of composition. The scalar convolution – it is a mathematical technique for data compressing and quantifying its integral properties by a single number.

A scalar convolution on nonlinear compromise scheme for the criteria subject to be minimized is proposed [Voronin, 2014]

$$Y[y(x)] = \sum_{k=1}^{s} p_k A_k \left[A_k - y_k(x)\right]^{-1},$$

applied in cases where the decision-maker considers as the preferred those solutions in which the values of individual criteria $y_k(x)$ are farthest from their limit values, A_k. This convolution has a number of essential advantages, which include flexibility, universality and analyticity.

The choice of a compromises scheme is made by the decision-maker and appears as explicitly conceptual.

Nested Scalar Convolutions

It is proposed for analytical evaluation of hierarchical structures to apply a method of nested scalar convolutions. The composition is performed on the "matryoshka principle": the scalar convolutions of the weighted components of vector criteria of lower level serve as the components of the vectors of higher level criteria. Scalar convolution of criteria obtained at the uppermost level is automatically considered as the expression for the analytical evaluation of effectiveness of the entire hierarchical system.

The algorithm for nested scalar convolutions is represented by an iterative sequence of operations of the weighed scalar convolutions of criteria for each level of the hierarchy from the bottom up, taking into account the priority vectors, based on the selected compromise scheme

$$\{(y^{(j-1)}, p^{(j-1)}) \rightarrow y^{(j)}\}_{j \in [2,m]} \tag{1}$$

and the searching and evaluating of effectiveness of the entire hierarchical system (alternative) as a whole is expressed by the problem of determining the scalar convolution of criteria on the top level of the hierarchy:

$$y^* = y^{(m)}.$$

When using the recurrent formula (1) important is the rational choice of the compromise scheme. For the method of nested scalar convolutions the adequate is a nonlinear compromise scheme. It is established that, without loss of generality, a premise for its use is that all the partial criteria were non-negative, were subject to minimization and were limited:

$$0 \le y_i \le A_i,\ A = \{A_i\}_{i=1}^{n},$$

where A is the vector of restrictions on the criteria of the current level of the hierarchy; *n* is the amount of them.

Preceding from (1) the expression to evaluate *k*-th property of an alternative for the *j*-th level of the hierarchy by using the nonlinear compromise scheme looks like

$$y_k^{(j)} = \sum_{i=1}^{n_k^{(j-1)}} p_{ik}^{(j-1)} [1 - y_{0ik}^{(j-1)}]^{-1}, k \in [1, n^{(j)}], \tag{2}$$

where criteria of the (*j*-1)-th level are normalized (reduced to unity). Thus, $y_{0ik}^{(j-1)}$ are the normalized vector's $y_0^{(j-1)}$ components involved in the evaluation of properties of the *k*-th alternative on the *j*-th level of the hierarchy; $n_k^{(j-1)}$ is their amount; $n^{(j)}$ is the amount of evaluated properties of the *j*-th level.

In the most simple and rather common case the multicriteria problem is formulated and solved without priorities, when decision-makers believe that all the importance parameters for all properties of alternatives are the same. In this case, a simple scalar convolution with the nonlinear trade-offs scheme in a unified form is used.

In order to formula (2) reflected the idea of the nested scalar convolutions method in accordance with the recurrent relation (1), this expression should be normalized, i.e., must be obtained a relative measure such that it were subject to be minimized, and it were the unit for it as the limit value.

The structure of the nonlinear compromise scheme enables normalizing the convolution (2) not to the maximum (which in this case is difficult), but to the minimum value of criteria convolution. Indeed, the ideal values for the criteria that are subject to be minimized are their zero points. Putting in (2)

$$y_{0ik}^{(j-1)} = 0, \forall i \in [1, n_k^{(j-1)}]$$

and taking into account the normalization $\sum_{i=1}^{n} p_i = 1$, we obtain $y_{k\min}^{(j)} = 1$.

After calculations and normalizing (reducing to unity), the final expression for the recurrent formula for calculating analytical assessments of the alternatives properties at all levels of the hierarchy becomes

$$y_{0k}^{(j)} = 1 - \{\sum_{i=1}^{n_k^{(j-1)}} p_{ik}^{(j-1)} [1 - y_{0ik}^{(j-1)}]^{-1}\}^{-1}, k \in [1, n^{(j)}], j \in [2, m].$$

Conclusion

The foregoing leads to the conclusion that any problem of vector assessment of an alternative can be represented by a hierarchical system of criteria, resulting from the decomposition of alternative properties. The lower level of the hierarchy is an object (alternative) assessment on selected properties, using initial criteria vector, and the upper level is obtained through the mechanism of the composition as a whole object evaluation. Central here is the problem of the composition of criteria for levels of the hierarchy to be solved by the method of nested scalar convolutions.

The methodological basis of an alternative properties decomposition to obtain the initial criteria vector is the Bohr's principle of complementarity. This is a *necessary* condition for vector estimation of alternatives.

The methodology of a criteria composition for levels of the hierarchy is based on the Gödel's theorem of incompleteness. This is a *sufficient* condition for vector estimation of alternatives.

We dare say that above inferences about notions of criteria decomposition and composition can be extended on the more general notions of analysis and synthesis.

Bibliography

[Voronin, 2013] Albert Voronin and Yuryi Ziatdinov, “Theory and practice of multicriteria decisions: Models, methods, realization”, Lambert Academic Publishing, 2013, [in Russian].

[Voronin, 2014] Albert Voronin, “Multicriteria Decision-Making. Lambert Academic”, Publishing, 2014.

Authors' Information

Albert Voronin *– professor, DrSc(Eng), Professor of Chair of Computer Information Technologies of National Aviation University of Ukraine; e-mail: alnv@voliacable.com*

Yuryi Ziatdinov *– professor, DrSc(Eng), Head of Chair of Computer Information Technologies of National Aviation University of Ukraine; e-mail: oberst@nau.edu.ua*

Igor Varlamov *– PhD, doctoral of National Defence University of Ukraine named after Ivan Cherniakhovsky; e-mail: igor_varlamov0@rambler.ru*