METHODS OF EVALUATION AND IMPROVEMENT OF CONSISTENCY OF EXPERT PAIRWISE COMPARISON JUDGMENTS

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***Abstract**: The notion of consistency is used to evaluate the contradiction level of expert pairwise comparison judgments and their suitability for the purpose of calculation of weights of decision alternatives. Several consistency coefficients and criteria are used to measure inconsistency of a pairwise comparison matrix (PCM). Traditional approach to increase consistency of expert information is to organize a feedback with an expert. However, it is not always possible due to financial and time limitations. The paper deals with methods of improvement (increase) PCM consistency without participation of an expert. Computer simulation is used to provide a comparative study of these methods. It is shown that taking an inadmissibly inconsistent PCM, for example, with the consistency ratio equal to CR=0.2 or CR=0.3, methods of improvement PCM consistency help to decrease inconsistency up to admissible level CR≤0.1 for n≥5. Results reveal that these methods, unfortunately, are not always effective, that is, drawing near to admissible inconsistency does not ensure closeness to the vector of real weights of decision alternatives. Also an analysis of coefficients and criteria of consistency of expert pairwise comparison judgments depending on PCM properties and level of its consistency is carried out.*

***Keywords**: pairwise comparison matrix, coefficients of consistency, admissible level of inconsistency, consistency criteria, weak consistency, methods of consistency improvement without participation of an expert, methods of identification of the most inconsistent judgment, an effective improvement of consistency*

***ACM Classification Keywords**: H.4.2. Information System Application: type of system strategy*

Introduction

Pairwise comparison methods are part of several popular decision support technologies such as the analytic hierarchy process [Saaty, 2003, 2008; Pankratova & Nedashkovskaya, 2010], the goal evaluation of decision alternatives [Totsenko, 2000; Tsyganok, 2010], PROMETHEE [Macharis et al, 2004]. These methods result in coefficients of relative importance or weights $w \in R_+^n$, $\sum_{i=1}^{n} w_i = 1$ of decision alternatives based on expert pairwise comparison matrices (PCMs) $D_{n\times n}$. For instance, PCMs $D_{n\times n}$ used in the analytic hierarchy process are positive $d_{ij} > 0$, inverse symmetrical $d_{ji} = 1/d_{ij}$ and take values from the Saaty fundamental scale [Saaty, 2008]. Calculation of weights is often based on the idea of minimization of deviation norm of expert PCM $D_{n\times n}$ from unknown consistent (theoretical) PCM $C = (w_i / w_j)$. The consistent PCM is supposed to be the best approximation of expert PCM $D_{n\times n}$. Several pairwise comparison methods are proposed to calculate weights $w \in R_+^n$. One of the most popular one is the eigenvector method (EM) [Saaty, 2003]. Depending on the choice of matrix norm other pairwise comparison methods are used, such as the least squares (LSM), the weighted least squares (WLS), the logarithmic least squares (LLSM), known also as the row geometric mean method (RGMM), the arithmetic normalization (AN) method and other (see overview in [Pankratova & Nedashkovskaya, 2010]). These methods are considered approximations of the EM method, which do not require calculation of eigenvalues and eigenvectors of matrix.

The notion of PCM consistency is used for evaluation of contradiction level of expert pairwise comparison judgments and their suitability for the purpose of calculation of weights of decision alternatives. Several consistency coefficients and criteria are proposed in [Saaty, 2008; Totsenko, 2000; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007]. During last years the attempts to analyze and compare consistency coefficients were made in [Pankratova & Nedashkovskaya, 2012, 2013; Brunelli et al, 2013]. Investigations of reliability of weights based on consistent, inconsistent and intransitive PCMs were done in [Linares, 2009; Siraj et al, 2012; Nedashkovskaya, 2014]. A PCM inconsistency is caused by usage of the Saaty fundamental scale, psychological limitations of expert, expert inaccuracies when making judgments.

Traditional approach to increase consistency of expert information is to organize a feedback with an expert that is expert is asked to overview all or the most inconsistent his/her judgments [Lipovetsky & Conklin, 2002; Siraj et al, 2012; Nedashkovskaya, 2013]. Expert is asked to repeat the overview procedure until admissible inconsistency of PCM is achieved. However, feedback with an expert is not always possible due to financial or time limitations. Another approach to improve (increase) PCM consistency consists of alteration of its elements using adjustment algorithms without participation of an expert [Xu & Da, 2003; Benítez et al, 2011; Nedashkovskaya, 2013].

Purpose of the paper is to provide a comparative study of several known methods of improvement (increasing) of PCM consistency without participation of an expert and also to provide and analysis of coefficients and criteria of consistency of expert pairwise comparison judgments depending on PCM properties and level of consistency.

1. Problem statement

Let $D_{n\times n}=\{d_{ij}\mid i,j=1,...,n\}$ be a pairwise comparisons matrix (PCM) of decision alternatives $a_1,a_2,...,a_n$ based on expert judgments, such that $d_{ij}>0$ and $d_{ji}=1/d_{ij}$ (property of inverse symmetry). The notion of consistency is used to estimate the quality of expert judgments and their suitability for reliable evaluation of decision alternatives.

PCM $D_{n\times n}$ is called **consistent (strongly consistent)** if transitivities $d_{ij}=d_{ik}d_{kj}$ are hold for all $i,j,k=1,...,n$ [Saaty, 2008].

PCM $D_{n\times n}$ is called **weak or ordinal consistent** if ordinal transitivities $(d_{ij}>1)\wedge(d_{jk}>1)\Rightarrow(d_{ik}>1)$, $(d_{ij}=1)\wedge(d_{jk}>1)\Rightarrow(d_{ik}>1)$, $(d_{ki}>1)\wedge(d_{ij}=1)\Rightarrow(d_{kj}>1)$, $(d_{ij}=1)\wedge(d_{jk}=1)\Rightarrow(d_{ik}=1)$ are hold [Siraj et al, 2012].

A weak inconsistent PCM $D_{n\times n}$ has at least one cycle defined by triplet of indexes (i,j,k), such that $(d_{ij}>1)\wedge(d_{jk}>1)\wedge(d_{ki}>1)$ or $(d_{ij}=1)\wedge(d_{jk}>1)\Rightarrow(d_{ik}\le 1)$, or $(d_{ki}>1)\wedge(d_{ij}=1)\Rightarrow(d_{kj}\le 1)$, or $(d_{ij}=1)\wedge(d_{jk}=1)\Rightarrow(d_{ik}\ne 1)$. It does not exist a ranking of decision alternatives that satisfy all elements of weak inconsistent PCM, namely $a_i>a_j$ if $d_{ij}>1$, $a_i<a_j$ if $d_{ij}<1$ and $a_i=a_j$ if $d_{ij}=1$.

Statement 1: If PCM is consistent, then it is weak consistent.

Some level of inconsistency of corresponding PCMs is admissible when solving real-life decision-making problems [Saaty, 2008; Totsenko, 2000; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007]. It is defined by several consistency criteria. Several coefficients are used to measure inconsistency of a PCM, namely consistency ratio CR [Saaty, 2008], geometric index consistency GCI [Aguaron & Moreno-Jimenez, 2003], harmonic consistency ratio HCR [Stein & Mizzi, 2007], consistency index of transitivities CI^{tr} [Peláez & Lamata, 2003] and spectral coefficient of consistency k_y [Totsenko, 2000].

Consistency criterion 1 [Saaty, 2008; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007; Pankratova & Nedashkovskaya, 2013]:

- PCM $D_{n\times n}$ is *strongly consistent* if $CR(D_{n\times n})=0$, $GCI(D_{n\times n})=0$, $HCR(D_{n\times n})=0$, and $CI^{tr}(D_{n\times n})=0$;
- PCM $D_{n\times n}$ is *admissibly inconsistent* if $CR(D_{n\times n})\le CR^{porog}$ or $GCI(D_{n\times n})\le GCI^{porog}$, or $HCR(D_{n\times n})\le HCR^{porog}$, or $CI^{tr}(D_{n\times n})\le CI^{tr\ porog}$ (depending on utilized consistency coefficient), where $CR^{porog}, GCI^{porog}, HCR^{porog}, CI^{tr\ porog}$ are threshold values of corresponding coefficients [Saaty, 2008; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007];
- PCM $D_{n\times n}$ is *inadmissibly inconsistent, that is the PCM requires an adjustment*, if the consistency coefficients exceed their threshold values;
- PCM $D_{n\times n}$ does not *contain information* if normalized coefficients $CR(D_{n\times n})\ge 1$ or $HCR(D_{n\times n})\ge 1$.

Consistency criterion 2 [Totsenko, 2000]:

- PCM $D_{n\times n}$ is *strongly consistent* if $k_y(D_{n\times n})=1$;
- PCM $D_{n\times n}$ is *admissibly inconsistent* if $k_y(D_{n\times n})\ge T_u$;
- PCM $D_{n\times n}$ is *inadmissibly inconsistent, that is the PCM requires an adjustment*, if $(k_y(D_{n\times n})\ge T_0)\wedge(k_y(D_{n\times n})<T_u)$;
- PCM $D_{n\times n}$ does not *contain information* if $k_y(D_{n\times n})<T_0$,

where T_0, T_u are threshold values that are defined, respectively, on bases of spectrum that contain minimal amount of information and spectrum of acceptable accuracy.

If PCM is inadmissibly inconsistent in terms of the consistency criterion 1 or 2, then its elements are in conflict to each other and, therefore, require adjustments in purpose of improvement of their consistency [Saaty, 2008; Totsenko, 2000; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007].

The purpose of the paper is to investigate the consistency coefficients CR, GCI, HCR, CI^{tr} and k_y, the consistency criteria 1 and 2 and methods of improvement of consistency of a PCM depending on its properties.

2. Investigation of the consistency coefficients and criteria

Different methods result in the same weights of decision alternatives $a_1, a_2, ..., a_n$ on basis of a PCM $D_{n\times n} = \{d_{ij} \mid i, j = 1, ..., n\}$ only when this PCM is strongly consistent. Otherwise the different methods result in different weights. An admissibly inconsistent PCM can be used for reliable evaluation of decision alternatives and, consequently one can trust to weights and ranking calculated on basis of this PCM [Saaty, 2008; Totsenko, 2000; Aguaron & Moreno-Jimenez, 2003; Peláez & Lamata, 2003; Stein & Mizzi, 2007]. But, as the following example shows, different consistency coefficients result in different conclusions regarding admissible inconsistency of a PCM.

Example 1: Let us consider several PCM:

$$D1 = \begin{pmatrix} 1 & 3 & 2 & 1 & 3 \\ 1/3 & 1 & 2 & 1/5 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 & 1/2 \\ 1 & 5 & 2 & 1 & 2 \\ 1/3 & 2 & 2 & 1/2 & 1 \end{pmatrix},\ D2 = \begin{pmatrix} 1 & 3 & 2 & 1 & 5 \\ 1/3 & 1 & 3 & 1/5 & 1/2 \\ 1/2 & 1/3 & 1 & 1/2 & 1/2 \\ 1 & 5 & 2 & 1 & 2 \\ 1/5 & 2 & 2 & 1/2 & 1 \end{pmatrix},\ D3 = \begin{pmatrix} 1 & 3 & 2 & 1 & 3 \\ 1/3 & 1 & 2 & 1/5 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 & 2 \\ 1 & 5 & 2 & 1 & 2 \\ 1/3 & 2 & 1/2 & 1/2 & 1 \end{pmatrix}.$$

PCMs D_1 and D_2 are weak consistent and D_3 is weak inconsistent by definition.

Consistency coefficients of the PCM D_1 are equal to CR =0.064, *GCI*=0.231, *HCR*=0.043 and CI^{tr} =0.785 and do not exceed corresponding threshold values $CR^{porog} = 0.1$, $GCI^{porog} = 0.37$, $HCR^{porog} = 0.1$ and $CI^{tr\ porog} = 1.329$. Therefore the PCM D_1 is admissibly inconsistent in terms of the consistency criterion 1. However, the vector of weights $w(D_1)$ based on D_1 using the AN method results in ranking of decision alternatives that differs from the ranking using the EM and RGMM methods (Table 1).

Consistency coefficients of the PCM D_2 lead to different results regarding the admissible inconsistency. For example the PCM D_2 is admissibly inconsistent in terms of the *HCR* coefficient and is inadmissibly inconsistent in terms of the CR , *GCI*, CI^{tr} and k_y coefficients (*HCR*=0.084, *CR*=0.125, *GCI*=0.440, CI^{tr} =1.624, k_y =0.664).

The consistency criterion 1 defines weak inconsistent PCM D_3 as admissibly inconsistent in terms of all coefficients (*CR*=0.094, *GCI*=0.321, *HCR*=0.047, CI^{tr} =1.223), so *the consistency criterion 1 does not identify violation of ordinal transitivity (a cycle)* in the PCM D_3. In terms of the consistency criterion 2 the PCM D_3 requires adjustment (k_y =0.718).

The vectors of weights based on PCMs D_1, D_2 and D_3 using the AN method result in rankings that differ from the rankings using the EM and RGMM methods (Table 1).

Table 1. The vectors of weights and rankings based on the PCMs D_1, D_2 and D_3 using the EM, RGMM and AN methods

PCM	Method	Vector of weights of decision alternatives	Ranking of decision alternatives
D_1	EM	w=(0.31, 0.106, 0.104, 0.320, 0.160)	a4>a1>a5>a2>a3
	RGMM	w=(0.314, 0.102, 0.101, 0.320, 0.162)	a4>a1>a5>a2>a3
	AN	w=(0.326, 0.090, 0.115, 0.322, 0.147)	a1>a4>a5>a3>a2
D_2	EM	w=(0.341, 0.116, 0.095, 0.306, 0.141)	a1>a4>a5>a2>a3
	RGMM	w=(0.341, 0.109, 0.092, 0.315, 0.144)	a1>a4>a5>a2>a3
	AN	w=(0.350, 0.094, 0.106, 0.332, 0.118)	a1>a4>a5>a3>a2
D_3	EM	w=(0.300, 0.115, 0.140, 0.319, 0.127)	a4>a1>a3>a5>a2
	RGMM	w=(0.316, 0.103, 0.134, 0.323, 0.124)	a4>a1>a3>a5>a2
	AN	w=(0.327, 0.090, 0.138, 0.323, 0.122)	a1>a4>a3>a5>a2

The example shows that consistency coefficients *CR*, *GCI*, *HCR* and CI^{tr} may lead to different results regarding an admissible inconsistency of a PCM. Results in terms of the consistency criteria 1 and 2 also may be different. An admissible inconsistency of a PCM in terms of the consistency criteria 1 and 2 does not guarantee the same ranking of decision alternatives using the EM, RGMM and AN methods. The consistency criterion 1 does not identify violation of ordinal transitivity (a cycle) in a PCM. Therefore it is suggested to require an additional property of weak consistency of a PCM to be carried out for the purpose of reliable evaluation of quality of this matrix.

Computer modeling of PCMs with wide range of variation of their inconsistency level shows that equality of results in terms of the consistency criteria 1 and 2 depend on properties of PCMs. For instance, if PCM is close to strong consistent then results in terms of these criteria are equal more than in 98% of experiments. If PCM is weak consistent and $n \geq 7$ the criteria 1 and 2 lead to the same results more than in 90% of experiments. In the case of weak consistent PCMs of small dimensions ($n = 3,4,5,6$) the equality of results occurs in lesser number of experiments. In most experiments the consistency criterion 2 results in necessity of adjustment of these PCMs whereas the PCMs are admissibly inconsistent in terms of the criterion 1. Results in terms of the criteria 1 and 2 are expected to be more identical for these PCMs, if the threshold T_u value in the criterion 2 will be reduced.

The small linear dependence between CR and k_y was identified in [Pankratova & Nedashkovskaya, 2013], namely the coefficient of determination R^2 in the regressions was equal to the values 0.16, 0.20, 0.28 and 0.04 depending on the level of consistency of PCMs.

3. Improvement of consistency of a PCM without participation of an expert

In the paper different methods to improve the consistency of a PCM without participation of an expert are investigated, and some recommendations to application of these methods depending on properties of PCMs are given.

3.1. Multiplicative and Additive Methods to Improve the Consistency of a PCM

Let $D_{n\times n}$ be initial PCM and CI, GCI, HCI and CI^{tr} be consistency index, geometric consistency index, harmonic consistency index and consistency index of transitivities of the PCM $D_{n\times n}$ respectively. Let us denote adjusted PCM with improved consistency by $D^*_{n\times n}$. Let CI^*, GCI^*, HCI^* and CI^{tr*} be consistency indexes of $D^*_{n\times n}$. Multiplicative and additive methods to adjust a PCM for the purpose of improving its consistency are proposed in [Xu & Da, 2003]. Generalizations of these methods are done and next two statements are proved in [Nedashkovskaya, 2013].

Statement 2 (the multiplicative method) [Nedashkovskaya, 2013]: Let us denote elements of the adjusted PCM $D^*_{n\times n}$ by $d^*_{ij} = (d_{ij})^{\alpha} (\frac{w_i}{w_j})^{1-\alpha}$, where $\alpha \in (0,1)$. Then $CI^* \leq CI$, $GCI^* \leq GCI$, $HCI^* \leq HCI$ and $CI^{tr*} \leq CI^{tr}$; and equations $CI^* = CI$, $GCI^* = GCI$, $HCI^* = HCI$ and $CI^{tr*} = CI^{tr}$ hold if $D_{n\times n}$ is consistent.

Statement 3 (the additive method) [Nedashkovskaya, 2013]: Let us denote elements of the adjusted PCM $D^*_{n\times n}$ by $d^*_{ij} = \alpha d_{ij} + (1-\alpha)(\frac{w_i}{w_j})$, if $i < j$ and $d^*_{ij} = (\alpha d_{ji} + (1-\alpha)(\frac{w_j}{w_i}))^{-1}$, if $i \geq j$, where $\alpha \in (0,1)$. Then $CI^* \leq CI$, $GCI^* \leq GCI$, $HCI^* \leq HCI$ and $CI^{tr*} \leq CI^{tr}$; and equations $CI^* = CI$, $GCI^* = GCI$, $HCI^* = HCI$ and $CI^{tr*} = CI^{tr}$ hold if $D_{n\times n}$ is consistent.

Statements 2 and 3 show that adjustment of an PCM using the multiplicative and additive methods helps to increase consistency level of this PCM in terms of coefficients $CR = \frac{CI}{MRCI}$, $HCR = \frac{HCI}{MRHCI}$, GCI and CI^{tr} , where $MRCI$ and $MRHCI$ are tabular values.

Adjustment algorithm №1

According to this algorithm, elements of a PCM $D_{n\times n}$ are iteratively altered based on the multiplicative or additive methods till its consistency level become admissible. Let us use the consistency ratio CR to measure consistency level of a PCM. The algorithm consists of the following stages [Xu & Da, 2003]:

1. Define parameter $\alpha \in (0,1)$. Calculate CR value of the PCM $D_{n\times n}$.
2. While $CR > CR^{porog}$:
   2.1. Calculate weights $w = (w_1, \ldots, w_n)^T$ based on the PCM $D_{n\times n}$.
   2.2. Calculate adjusted PCM $D^* = (d^*_{ij})$:

$$d^*_{ij} = (d_{ij})^{\alpha} \left(\frac{w_i}{w_j} \right)^{1-\alpha} \quad \text{or}$$

$$d^*_{ij} = \begin{cases} \alpha d_{ij} + (1-\alpha)\frac{w_i}{w_j}, & i = 1,2,...,n;\ j = i, i+1,...,n \\ \dfrac{1}{\alpha d_{ji} + (1-\alpha)\frac{w_j}{w_i}}, & i = 2,3,...,n;\ j = 1,2,...,i-1 \end{cases}$$

   2.3. Calculate CR value of the PCM D^*.
   2.4. $D := D^*$.

Statement 4 (convergence of the algorithm): For the described algorithm $\lim_{k\to+\infty} CR^{(k)} = 0$, where values of $CR^{(k)}$, $k = 1,2,...$ are calculated using the algorithm.

It is essential to take into account, when defining the parameter $\alpha \in (0,1)$, that greater values of α lead to smaller deviations of adjusted PCM from the initial one and, consequently, the greater number of iterations of the algorithm is needed to achieve the admissible inconsistency. It is suggested to use $0.5 \le \alpha < 1$.

3.2. Methods of Identification of the Most Inconsistent Element of a PCM

The CI method was suggested in [Lipovetsky & Conklin, 2002] and generalized in [Nedashkovskaya, 2013]. It is based on statement that the consistency coefficients (CC) CI, GCI, HCI and CI^{tr} possess their minimal values and the spectral coefficient of consistency k_y possesses its maximum value when PCM is consistent.

The CI method consists of several stages [Nedashkovskaya, 2013]:

1. Calculate the PCM $D^i_{(n-1)\times(n-1)}$ by exception i-th row and i-th column of initial PCM $D_{n\times n}$. Calculate the consistency coefficients CC of the $D^i_{(n-1)\times(n-1)}$, $\forall i = 1,...,n$.
2. Find two minimal values of consistency coefficient:

$$i^* = \arg\min_{i=1,..,n} \mathrm{CC}(D^i_{(n-1)\times(n-1)}),$$

$$j^* = \arg\min_{i=1,..,n,\, i\neq i^*} \mathrm{CC}(D^i_{(n-1)\times(n-1)})$$

Then element $d_{i^* j^*}$ is the most inconsistent in terms of consistency coefficients CI, GCI, HCI and CI^{tr}.

Find two maximum values of the coefficient k_y:

$$i^* = \arg\max_{i=1,..,n} k_y(D^i_{(n-1)\times(n-1)}),$$

$$j^* = \arg\max_{i=1,..,n,\, i\neq i^*} k_y(D^i_{(n-1)\times(n-1)}).$$

Then element $d_{i^* j^*}$ is the most inconsistent one in terms of the spectral coefficient k_y.

The Corr method is based on statement that the correlation between vectors-rows of a PCM and also the correlation between vectors-columns of a PCM becomes close to unity when consistency level of this PCM is increased. The Corr method consists of several stages [Lipovetsky & Conklin, 2002]:

1. Calculate expectation values $M\left(R_i^r\right)$ of coefficients of correlation between i-th and all other vectors-rows of a PCM, and also expectation values $M\left(R_j^c\right)$ of coefficients of correlation between j-th and all other vectors-columns of a PCM, $i, j = 1,...,n$.
2. Find the minimum value of vectors $\left\{M\left(R_i^r\right)\right\}$ and $\left\{M\left(R_j^c\right)\right\}$:

$$i^* = \arg\min_i \left\{M\left(R_i^r\right)\right\},\ j^* = \arg\min_j \left\{M\left(R_j^c\right)\right\}.$$

Then element $d_{i^*j^*}$ is the most inconsistent one.

The Xi method is based on the Xi-square criterion [Lipovetsky & Conklin, 2002]:

1. Calculate values Δ_{ij} for each element d_{ij} of a PCM: $\Delta_{ij} = \dfrac{\left(d_{ij} - t_{ij}\right)^2}{t_{ij}}$, where $t_{ij} = \left(\sum_{k=1}^{n} d_{ik}\right)\left(\sum_{l=1}^{n} d_{lj}\right) / \left(\sum_{k=1}^{n}\sum_{l=1}^{n} d_{kl}\right)$.
2. Calculate the expectation value and variance of values $\{\Delta_{i,j} \mid i, j = 1,...,n\}$. Calculate the confidence interval.
3. Find values $\Delta_{i^*j^*}$ which are outside the confidence interval. Then elements $d_{i^*j^*}$ defined by the corresponding pairs of indexes (i^*, j^*) are the most inconsistent.

The Outflow method [Siraj et al, 2012]:

1. Calculate the outflow Φ_i for each decision alternative a_i - the number of decision alternatives a_j, such that a_i outperforms a_j, namely $d_{ij} > 1$.
2. Find the maximum value of difference $\Phi_j - \Phi_i$:

$$d_{i^*j^*} : \max_{i,j}(\Phi_j - \Phi_i)\text{, if } i \neq j, d_{i,j} > 1.$$

Then element $d_{i^*j^*}$ is the most inconsistent one.

Suppose that several elements $d_{i^*j^*}$ result in maximum value of difference $\Phi_j - \Phi_i$. Then it is necessary to find an element among them which lead to more inconsistency, namely an element, which result in the maximum value of expression

$$\gamma_{ij} = \frac{1}{n-2}\sum_{k=1}^{n}(\ln d_{i,j} - \ln(d_{i,k}d_{k,j}))\text{, where } k \neq i \neq j\text{ .}$$

The Tr method is based on analysis of transitivities of a PCM [Nedashkovskaya, 2013]:

1. Calculate the set of transitivities $\Gamma = \{\Gamma_u\}$, $u = 1,...,NT$: $\Gamma_u = \{d_{ij},\, d_{jk}, d_{ik}\}$, $i,j,k = 1,...,n$, $i < j < k$, $NT = \frac{n!}{(n-3)!\,3!}$, $n \geq 3$ of a PCM and values of determinants of these transitivities:

$$Det = \{\det(\Gamma_u)\},\ \det(\Gamma_u) = \frac{d_{ij}d_{jk}}{d_{ik}} + \frac{d_{ik}}{d_{ij}d_{jk}} - 2\,.$$

2. Find values $S_{i,j} = \sum_{k=1}^{n}\left(\frac{d_{ij}d_{jk}}{d_{ik}} + \frac{d_{ik}}{d_{ij}d_{jk}} - 2\right)$ for each $i, j = 1,..., n$ and their maximum value

$$(i^*, j^*):\ \max_{i,j} S_{i,j}\,.$$

Then element $d_{i^*j^*}$ defined by the corresponding pair of indexes (i^*, j^*) is the most inconsistent one.

Adjustment algorithm №2

Algorithm of consistency improvement of a PCM is based on the CI, Corr, Xi, Outflow and Tr methods. It iteratively finds the most inconsistent elements of a PCM, while admissible inconsistency will be achieved. Let us use the consistency ratio CR to measure consistency level of a PCM. The algorithm contains several stages:

1. Calculate CR value of given PCM D.
2. While $CR > CR^{porog}$:
   - 2.1. Find the most inconsistent element $d_{i^*j^*}$ of the PCM D using one of the CI, Corr, Xi, Outflow or Tr methods.
   - 2.2. Calculate the adjusted PCM $D^* = \left(d^*_{ij}\right)$, in which only element $d_{i^*j^*}$ and its inverse symmetrical element $d_{j^*i^*} = 1/d_{i^*j^*}$ are adjusted. In accordance with the Saaty scale, element $d_{i^*j^*}$ is given a value that ensures the greatest consistency of all PCM, that is, the minimum value of CR .
   - 2.3. Calculate CR value of the PCM D^*.
   - 2.4. $D := D^*$.

Let us provide a comparative study and analysis of described adjustment algorithms and methods of identification of the most inconsistent elements of a PCM without participation of an expert.

3.3. A Comparative Study of Methods of Improvement of PCM Consistency without Participation of an Expert

Computer simulation is used to provide a comparative study of methods of improvement (increase) of PCM consistency without participation of an expert. Test sets of strongly consistent PCMs and admissibly inconsistent PCMs (denoted as D^{real}) in Saaty scale are generated in the course of the simulation. We suppose the PCMs D^{real} represent ratios of real weights. Further the PCMs D^{real} are disturbed so that their inconsistency is increased. Let PCM $D^{disturb}$ be result of disturbance of PCM D^{real}. PCMs $D^{disturb}$ are adjusted using the algorithms № 1 and 2. We analyse how much weights on basis of adjusted PCM D^{*} are closer to real weights then weights on basis of initial PCM (before the adjustment).

Improvement of consistency is considered to be effective if vector of weights w^{*} on basis of adjusted PCM D^{*} is closer to vector of real weights w^{real} then vector of weights w on basis of initial PCM $D^{disturb}$:

$$dist(w^{*}, w^{real}) < dist(w, w^{real}),$$

where $dist$ is angular distance:

$$dist(x, y) = \left\| \frac{x}{\|x\|_2} - \frac{y}{\|y\|_2} \right\|_2 .$$

The measure $dist$ is valid when evaluating effectiveness of pairwise comparison methods [20].

Sufficiently large test sets of PCMs D^{real} for each value of $n = 3,4,5,...,9$ were generated. Several technologies of disturbance of PCM D^{real} were used. In each l –th experiment disturbed PCM $D^{disturb}(l)$ was adjusted using the algorithms 1 and 2 and the multiplicative and additive methods with different values of parameter α, and also the CI, Corr, Xi, Outflow and Tr methods.

The following angular distances were calculated:

$$dist(l) = dist(w(l), w^{real}(l)) \quad (1)$$

the distance between vector of weights based on PCM $D^{disturb}$ and vector w^{real},

$$dist^{j}(l) = dist(w^{j}(l), w^{real}(l)) \quad (2)$$

the distance between vector of weights $w^{j}(l)$ based on adjusted PCM D^{*j} and vector w^{real}, where PCM D^{*j} is a result of usage of j-th adjustment method and l is a number of experiment.

Distributions of the data series $\{dist(l)\}$ (1) and $\{dist^{j}(l)\}$ (2) were, generally, close to the normal distribution. The main statistical characteristics – a sample mean and sample variance values (Table 2 – 4) were calculated. In these tables the pair $(m;\sigma)$ represents the statistical characteristics of the data series $\{dist(l)\}$ (1).

Values of distances (1) and (2) describe errors of evaluation of weights $w(l)$ and $w^{j}(l)$. Therefore, the j-th adjustment method is more effective on average if sample mean and sample variance values of data series $\{dist^{j}(l)\}$ (2) possess lesser values.

Table 2. Expectation values and standard deviations for the series (1) and (2) when PCMs $D^{disturb}$ are weak consistent and coefficient *CR* possesses values in interval $(CR^{porog}, 1.5 \cdot CR^{porog})$

n		4	5	6
Before adjustment	$(m;\sigma)$	(0.347; 0.084)	(0.342; 0.068)	(0.338; 0.058)
Adjustment algorithm № 1	$(m^{mult};\sigma^{mult})$	(0.346; 0.084)	(0.342; 0.066)	(0.340; 0.057)
	$(m^{ad};\sigma^{ad})$	(0.346; 0.084)	(0.343; 0.066)	(0.341; 0.057)
Adjustment algorithm № 2	$(m^{CI};\sigma^{CI})$	(0.361; 0.104)	(0.367; 0.075)	(0.362; 0.060)
	$(m^{Tr};\sigma^{Tr})$	(0.361; 0.104)	(0.343; 0.074)	(0.341; 0.060)
	$(m^{Oflow};\sigma^{Oflow})$	(0.352; 0.099)	(0.357; 0.074)	(0.354; 0.063)

Table 3. Expectation values and standard deviations for the series (1) and (2) when PCMs $D^{disturb}$ are weak consistent and coefficient *CR* possesses values in interval $(3.5 \cdot CR^{porog}, 5.5 \cdot CR^{porog})$

	n	4	5	6
Before adjustment	$(m;\sigma)$	(0.229; 0.092)	(0.381; 0.088)	(0.388; 0.076)
Adjustment algorithm № 1	$(m^{mult};\sigma^{mult})$	(0.235; 0.089)	(0.393; 0.082)	(0.402; 0.071)
	$(m^{ad};\sigma^{ad})$	(0.235; 0.092)	(0.398; 0.084)	(0.408; 0.071)
Adjustment algorithm № 2	$(m^{CI};\sigma^{CI})$	(0.361; 0.098)	(0.467; 0.085)	(0.469; 0.073)
	$(m^{Corr};\sigma^{Corr})$	(0.313; 0.108)	(0.444; 0.099)	(0.436; 0.087)
	$(m^{Tr};\sigma^{Tr})$	(0.361; 0.098)	(0.468; 0.083)	(0.474; 0.072)
	$(m^{Oflow};\sigma^{Oflow})$	(0.400; 0.099)	(0.497; 0.091)	(0.480; 0.077)

Table 4. Expectation values and standard deviations for the series (1) and (2) when PCMs $D^{disturb}$ are weak inconsistent with one cycle

	n	4	5	6
Before adjustment	$(m;\sigma)$	(0.551; 0.155)	(0.420; 0.156)	(0.335; 0.145)
Adjustment algorithm № 1	$(m^{mult};\sigma^{mult})$	(0.512; 0.142)	(0.367; 0.135)	(0.280; 0.119)
	$(m^{ad};\sigma^{ad})$	(0.519; 0.176)	(0.381; 0.171)	(0.297; 0.152)
Adjustment algorithm № 2	$(m^{CI};\sigma^{CI})$	(0.035; 0.045)	(0.034; 0.045)	(0.027; 0.042)
	$(m^{Corr};\sigma^{Corr})$	(0.251; 0.284)	(0.115; 0.182)	(0.067; 0.119)
	$(m^{Tr};\sigma^{Tr})$	(0.035; 0.045)	(0.034; 0.047)	(0.025; 0.041)
	$(m^{Oflow};\sigma^{Oflow})$	(0.035; 0.045)	(0.033; 0.046)	(0.025; 0.039)

Results in Tables 2 – 4 correspond to different technologies of disturbance of PCM D^{real}. Thus in Tables 2 and 3 results correspond to disturbances of all elements of PCMs D^{real}, and PCMs D^{real} were strongly consistent. Disturbances have different intensity for the purpose of investigation PCMs $D^{disturb}$ with different levels of inconsistency, that is, different levels of changing of *CR* values. Influence of weak consistency of PCM $D^{disturb}$ on result is also investigated.

Values in Table 4 correspond to strong disturbances of small number of elements of PCMs D^{real} that lead to weak inconsistent PCMs $D^{disturb}$ with cycles. Let us note that initial PCMs D^{real} were admissibly inconsistent with *CR* values $CR(D^{real}) \leq CR^{porog}$, and are weak consistent.

Results of computer simulation lead to the following conclusions:

1. Effectiveness of considered adjustment algorithms (algorithms of consistency increase of a PCM) depends on the percent of inaccurate elements in a PCM, that is, elements that are significant disturbances of real values.
2. Let us consider the case when all elements of an inadmissibly inconsistent PCMs are significantly disturbed. Then adjustments of the PCMs using the algorithms 1 and 2 without participation of an expert result in admissible inconsistent PCMs. But these adjustments do not ensure closeness to the vector of real weights (Tables 2 and 4). In particular, let us consider an inadmissibly inconsistent PCM with relatively small level of inconsistency (Table 2) and additional property of weak consistency. When these initial PCM are adjusted using the adjustment algorithm 1 then weights on basis of the adjusted PCM, on average, are practically equally distant from weights on basis of the initial PCM. Weights on basis of the adjusted PCM, on average, are slightly distant from weights on basis of the initial PCM when these initial PCM are adjusted using the algorithm 2. If initial PCM has a greater level of inconsistency (Table 3), then weights on basis of the adjusted PCM, on average, are considerably distant from weights on basis of the initial PCM.

Effectiveness of the adjustment algorithm 1 does not depend on parameter α. The algorithm 1 is more effective than the algorithm 2. Effectiveness of the adjustment algorithms 1 and 2 is decreased when level of inconsistency of initial PCM is increased. Effectiveness of the adjustment algorithms 1 and 2 is slightly decreased with the growth of *n* – the dimension of initial PCM.

Effectiveness of the adjustment algorithm 1 is significantly higher, if initial PCM has an additional property of weak consistency and $n = 4$ or $n = 5$. This influence is reduced for bigger values of n. It is worth note that number of weak consistent PCMs is decreased with the growth of *n*.

3. The adjustment algorithms 1 and 2 are effective for the PCM, which has small number of significantly disturbed elements (Table 4). For these PCMs the adjustment algorithm 2 is considerably more effective than the adjustment algorithm 1. Results in Tables 4 and 5 show that the CI, Xi, Tr and Outflow methods are more effective than the Corr method.

The Tr method has several advantages. It does not require computation of the principal eigenvalue of a PCM as opposed to the CI method. The Tr method identifies transitivity which defines a cycle in a PCM and provides more information about inconsistency of a PCM in comparison with other investigated methods.

The Xi method results in a set of the most inconsistent elements of a PCM, that is, it does not identify which one has to be corrected in the first place.

Table 5. The percent of correctly identification of the most inconsistent element of a PCM using the CI, Corr, Xi, Tr and Outflow methods

n	4	5	6
Method CI	94	94	90,4
Method Corr	56	70	73
Method Xi	99	99	98
Method Outflow	87	91	89
Method Tr	94	95	93

In some cases the Corr and Outflow methods result in incorrect identification of an outlier in a PCM. For example, the Outflow method generally results in a correct identification of a cycle when initial PCMs were close to strongly consistent. However, in these PCMs the Outflow method sometimes leads to incorrect outlier that has to be adjusted. In consequence, all allowable adjustments of this element do not eliminate a cycle in a PCM.

If PCM has several the most inconsistent elements, the CI, Corr, Xi, Tr and Outflow methods may lead to different results (example 3), that is not their drawback.

Let us illustrate these conclusions using several examples.

Example 2 illustrates ineffective application of the adjustment algorithm 1.

Let us consider the weak inconsistent PCM D. One of elements of this PCM is a significant disturbance of some real value:

$$D = \begin{pmatrix} 1 & 1 & 1/3 & 1/4 & 1/5 \\ 1 & 1 & 1/2 & 1/3 & 4 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 1 & 1 & 1 \\ 5 & 1/4 & 1 & 1 & 1 \end{pmatrix}, \quad D^{real} = \begin{pmatrix} 1 & 1 & 1/3 & 1/4 & 1/5 \\ 1 & 1 & 1/2 & 1/3 & 1/4 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 1 & 1 & 1 \\ 5 & 4 & 1 & 1 & 1 \end{pmatrix}.$$

The PCM D is based on the real PCM D^{real} and contains an expert error while defining the element $d_{2,5}$ of the PCM. Suppose that the following ranking based on the PCM D^{real} is the real ranking of five decision alternatives:

$$a_5 \succ a_4 \succ a_3 \succ a_2 \succ a_1 \tag{3}$$

Suppose that providing pairwise comparisons of the decision alternatives, expert makes an error in element $d_{2,5}$ (this element is an outlier) and give the PCM D. In accordance with the consistency criteria 1 and 2 the PCM D requires adjustment since it is inadmissibly inconsistent (*CR*=0.219, *GCI*=0.676, *HCR*=0.170, CI^{tr}=3.508 and k_y=0.546). The adjustment algorithm 1 and the multiplicative method with values $\alpha = 0.5, 0.7, 0.9$ result in the ranking $a_4 \succ a_3 \succ a_2 \succ a_5 \succ a_1$, which differs from the real ranking (3). Thus, the multiplicative method and the adjustment algorithm 1 are inefficient in given example. The similar conclusion is made when the adjustment algorithm 1 and additive method are used to improve the consistency of the PCM D.

Example 3:

a) $$D^1 = \begin{pmatrix} 1 & 1/2 & 1/5 & 1/7 & 1/9 \\ 2 & 1 & 1/2 & 1/3 & 1/4 \\ 5 & 2 & 1 & 2 & 1/2 \\ 7 & 3 & 1/2 & 1 & 1 \\ 9 & 4 & 2 & 1 & 1 \end{pmatrix},$$ b) $$D^2 = \begin{pmatrix} 1 & 1/6 & 1 & 1/6 & 1/8 \\ 6 & 1 & 3 & 1/2 & 2 \\ 1 & 1/3 & 1 & 1/9 & 1/4 \\ 6 & 2 & 9 & 1 & 1/2 \\ 8 & 1/2 & 4 & 2 & 1 \end{pmatrix}.$$

The PCMs D^1 and D^2 are weak inconsistent.

A triple of elements ($d_{3,4}$, $d_{4,5}$, $d_{3,5}$) forms a cycle in the PCM D^1, since $(d_{34} > 1) \wedge (d_{45} = 1) \wedge (d_{53} > 1)$. Element d_{34} is the most inconsistent in the PCM D^1. All described methods besides the Outflow method correctly define this element. The adjustment $d_{34} := 1/2$ results in elimination of the cycle. Inconsistency level of adjusted PCM is decreased and the PCM becomes close to strongly consistent since its consistency ratio is equal to *CR*=0.004. Therefore the element d_{34} was an outlier. The Outflow method results in element d_{53} of the PCM D^1. This result is faulty in the case of PCM D^1, since all allowable by the Saaty scale adjustments of element d_{53} help to improve the D^1 consistency not to the best advantage, namely these adjustments decrease the D^1 inconsistency only up to the value CR=0.035.

In weak inconsistent PCM D^2 cycle is formed by a triple of elements ($d_{2,5}$, $d_{5,4}$, $d_{4,2}$). The consistency ratio of the PCM is equal to *CR*=0.1. CI, Corr and Tr methods result in element $d_{4,5}$ of D^2, the Outflow method – in element $d_{2,5}$ and the Xi-squared method – in elements $d_{2,5}$, $d_{4,3}$ and $d_{5,4}$. Adjustments of elements $d_{4,5}$ and $d_{2,5}$ and their new values $d_{4,5} := 2$, $d_{2,5} := 2$ lead to elimination of the cycle and decreasing of inconsistency level of D^2 up to the same value *CR*=0.05. Thus, both elements $d_{2,5}$ and $d_{4,5}$ (and also $d_{5,4}$) are the most inconsistent in the given example.

Conclusions

The paper deals with the investigation of several known algorithms of consistency improvement (increase) of a PCM. Taking an inadmissibly inconsistent PCM, for example, with the consistency ratio equal to $CR = 0.2$ or $CR = 0.3$ these algorithms helps to decrease inconsistency up to admissible level $CR \le 0.1$ for $n \ge 5$. At the same time, drawing near to admissible inconsistency does not ensure closeness to the vector of real weights, that is, these algorithms are not always effective in this respect.

Effectiveness of considered adjustment algorithms (algorithms of increasing the consistency level of a PCM) depends on the percent of inaccurate elements in a PCM, that is, elements that are significant disturbances of real values. This percent may be estimated using the transitivities method, proposed in [Pankratova & Nedashkovskaya, 2013]. Adjustment algorithms that were investigated in the paper are effective for the PCM, which has small number of significantly disturbed elements. For these PCMs the adjustment algorithm 2 is considerably more effective than the adjustment algorithm 1.

If all elements of an inadmissibly inconsistent PCM are significantly disturbed, then the adjustment algorithms 1 and 2 without participation of expert result in admissible inconsistency of this PCM but do not always ensure closeness to the vector of real weights.

For identification of the most inconsistent elements of a PCM it is recommended to use the Tr method among other known methods. It has several advantages: does not require computation of the principal eigenvalue of a PCM as opposed to the CI method, also identifies transitivity which defines a cycle in a PCM and provides more information about inconsistency of a PCM in comparison with other investigated methods.

It was shown that well-known coefficients of consistency CR , GCI , HCR and CI^{tr} do not always correctly evaluate quality of expert judgments about pairwise comparisons of decision alternatives. For the purpose of more reliable evaluation of quality of expert judgments (PCMs) it is suggested to use an additional property of weak consistency of these PCMs.

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