CONVEXITY RELATED ISSUES FOR THE SET OF HYPERGRAPHIC SEQUENCES

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Abstract: *We consider $D_m(n)$, the set of all degree sequences of simple hypergraphs with n vertices and m hyperedges. We show that $D_m(n)$, which is a subset of the n-dimensional $m+1$-valued grid Ξ_{m+1}^n, is not a convex subset of Ξ_{m+1}^n; and give a characterization of the convex hull of $D_m(n)$.*

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Introduction

A hypergraph H is a pair (V, E), where V is the vertex set of H, and E, the set of hyperedges, is a collection of non-empty subsets of V. The degree of a vertex v of H, denoted by $d\ (v)$, is the number of hyperedges in H containing v. A hypergraph H is simple if it has no repeated hyperedges. A hypergraph H is r-uniform if all hyperedges contain r-vertices. 2-uniform hypergraphs (edges contain exactly 2 vertices) are simply ordinary graphs.

Let $V = \{v_1, \cdots, v_n\}$. $d(H) = (d(v_1), \cdots, d(v_n))$ is the degree sequence of hypergraph H. A sequence $d = (d_1, \cdots, d_n)$ is hypergraphic if there is a simple hypergraph H with degree sequence d. For a given m, $0 < m \leq 2^n$, let $H_m(n)$ denote the set of all simple hypergraphs $([n], E)$, where $[n] = \{1,2, \cdots, n\}$, and $|E| = m$, and $D_m(n)$ denote the set of all hypergraphic sequences of hypergraphs in $H_m(n)$.

We investigate issues related to the characterization of the set of all hypergraphic sequences. The case of graphs is easy - a simple necessary and sufficient condition for the characterization of the set of degree sequences is known by the Erdos-Gallai Theorem [Erdos,Gallai, 1960], [Harary, 1969]:

Theorem 1 (Erdos-Gallai) A decreasing sequence of non-negative integers $(d_1, \cdots, d_n)$ is the degree sequence for a simple graph if and only if:

$\sum_{i=1}^{n} d_i$ is even; (1)

$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}\ for\ k = 1, \cdots, p-1.$

In general, the characterization of degree sequences for uniform hypergraphs is an open problem when $r \geq 3$ (see [Berge, 1989], [Bill, 1988], [Bill, 1986], [BhanuSriv, 2002], [Colb, 1986] , [KocayLi, 2007]).

The characterization of $D_m(n)$, which is not easier that the case of uniform hypergraphs, - is investigated in [Sah, 2009] - [Sah, 2015], [AslGroSahWag, 2015]. The problem has its interpretation in terms of multidimensional binary cubes; it is also known as a special case in discrete tomography problem, when an additional constraint/requirement – non-repetition of rows is imposed [SahAsl, 2010], [Sah, 2013]. Structures, properties, and several related partial results were obtained in [Sah, 2009] - [Sah, 2015] for $D_m(n)$. In this research we consider convexity issues related to the set $D_m(n)$.

Convex hull of degree sequences of k-uniform hypergraphs was investigated in [Koren, 1973],[BhanuSriv, 2002], [Klivans, Reiner, 2008], [Ricky Ini Liu, 2013]. It was shown by Koren [Koren, 1973] that the inequalities in (1) define a convex polytope $D_n(2)$ of degree sequences of simple graphs, so that the sequences with even sum, lying in this polytope are exactly the degree sequences of the graphs on n vertices.

Analogous questions for k-uniform hypergraphs when $k > 2$ investigated in [Klivans, Reiner, 2008], [Ricky Ini Liu, 2013]. Klivans and Reiner [Klivans, Reiner, 2008] verified computationally that the set of degree sequences for k-uniform hypergraphs is the intersection of a lattice and a convex polytope for $k = 3$ and ≤ 8. Ricky Ini Liu [Ricky Ini Liu, 2013] show that this does not hold for $k \geq 3$ and $n \geq k + 13$.

In this paper we consider analogous convexity questions for $D_m(n)$.

Structure of $D_m(n)$

Suppose that we consider the set of all hypergraphic sequences of hypergraphs $([n], E)$, and omit the restriction of non-repetition of hyperedges. Then, every integer sequence of length n with all component values between 0 and m, can serve as degree sequence of some hypergraph with the vertex set $[n]$ and with m hyperedges.

Thus, the n-dimensional $m+1$-valued integer grid Ξ_{m+1}^n of elements: $\{(a_1, \cdots, a_n) | 0 \leq a_i \leq m \; for \; all \; i\}$ can be considered as the set of degree sequences of hypergraphs with the vertex set $[n]$ and with m hyperedges; and in this manner, $D_m(n) \subseteq \Xi_{m+1}^n$.

In this section we consider the structure of $D_m(n)$ in Ξ_{m+1}^n.

Component-wise partial order is defined on Ξ_{m+1}^n: $(a_1, \cdots, a_n) \leq (b_1, \cdots, b_n)$ if and only if $a_i \leq b_i$ for all i, and $r(a_1, \cdots, a_n) = a_1 + \cdots + a_n$ is the rank of an element $(a_1, \cdots, a_n)$. An illustration of Ξ_{m+1}^n can be given by the Hasse diagram. Figure1 illustrates the Hasse diagram of Ξ_5^3.

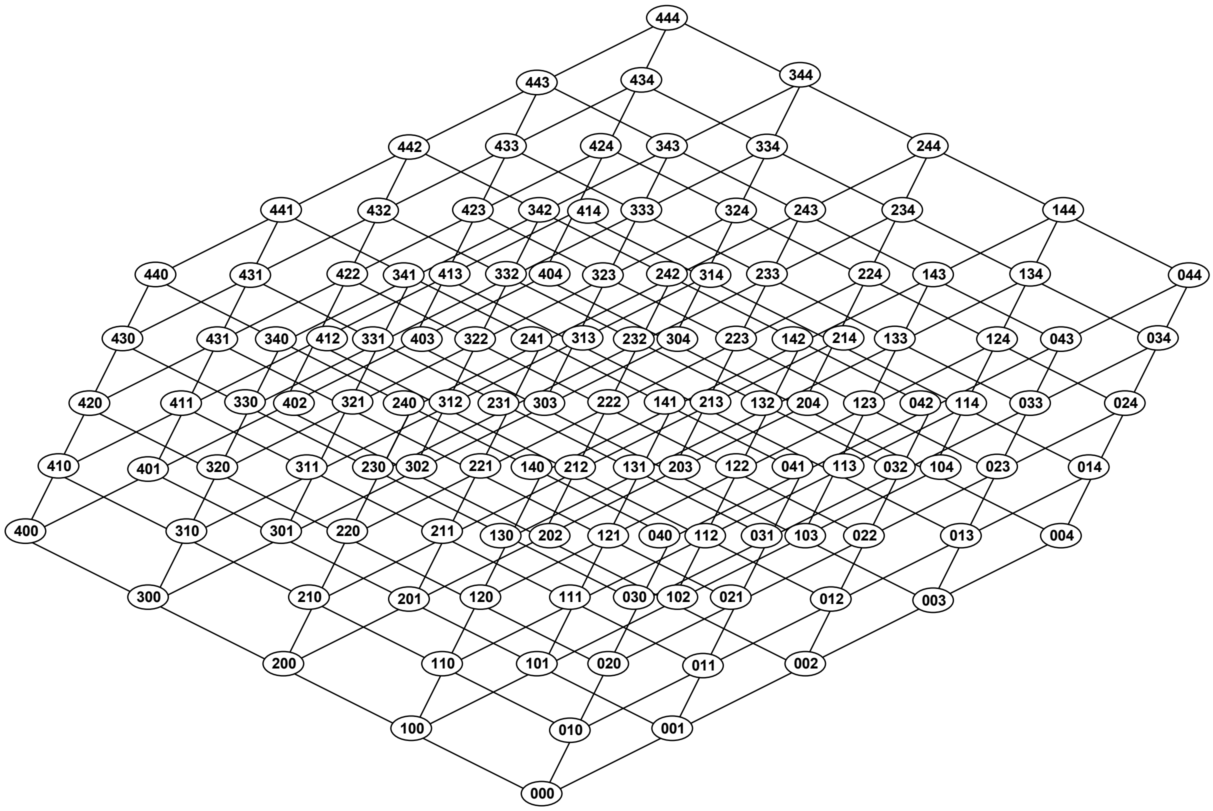

Figure 1. The Hasse diagram of Ξ_5^3

Opposite elements in $\boldsymbol{\Xi_{m+1}^n}$

A pair of elements $d, \bar{d} \in \Xi_{m+1}^n$ are called opposite if one can be obtained from the other by inversions of component values: if $d = (d_1, \cdots, d_n)$, then $\bar{d} = (m - d_1, \cdots, m - d_n)$.

Boundary elements of $\boldsymbol{D_m(n)}$.

We call $(d_1, \cdots, d_n) \in D_m(n)$ an *upper boundary /lower boundary/ element* of $D_m(n)$ if no $(a_1, \cdots, a_n) \in \Xi_{m+1}^n$ with $(a_1, \cdots, a_n) > (d_1, \cdots, d_n)$ / with $(a_1, \cdots, a_n) < (d_1, \cdots, d_n)$ / belongs to $D_m(n)$.

Let $\widehat{D}_{max}$ and $\breve{D}_{min}$ denote the sets of upper and lower boundary elements of $D_m(n)$, respectively.

Interval in $\boldsymbol{\Xi_{m+1}^n}$.

For a pair of elements $d', d'', d' \leq d''$ of Ξ_{m+1}^n, $E(d', d'')$ denotes the minimal subgrid/interval in Ξ_{m+1}^n spanned by these elements: $E(d', d'') = \{a \in \Xi_{m+1}^n | d' \leq a \leq d''\}$.

Theorem 2 ([Sah, 2009]**).** $D_m(n)$ is a union of intervals spanned by the pairs of opposite elements of $\widehat{D}_{max}$ and $\breve{D}_{min}$:

$$D_m(n) = \cup_{\widehat{D} \in \widehat{D}_{max}, \breve{D} \in \breve{D}_{min}} E(\breve{D}, \widehat{D}),$$

where $(\widehat{D}, \breve{D})$ are pairs of opposite elements.

An illustration is given in Figure 2 by the example of $D_4(3)$ in Ξ_5^3:

$\widehat{D}_{max} = \{(3,3,3), (4,2,2), (2,4,2), (2,2,4)\}$,

$\breve{D}_{min} = \{(1,1,1), (0,2,2), (2,0,4), (2,2,0)\}$,

$D_4(3) =$

$E\big((1,1,1), (3,3,3)\big) \cup E\big((0,2,2), (4,2,4)\big) \cup E\big((2,2,0), (2,2,4)\big) \cup E\big((2,0,2), (2,4,2)\big)$.

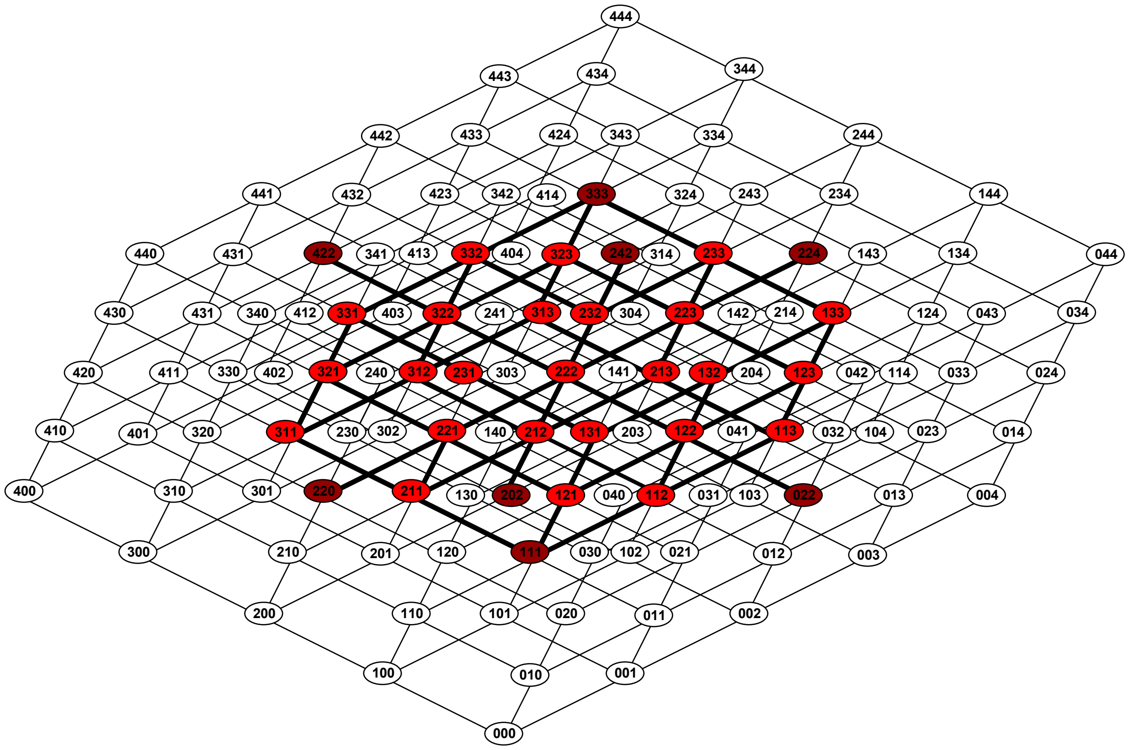

Figure 2

$D_4(3)$ in Ξ_5^3: vertices in red compose $D_4(3)$, and vertices in darker red compose sets $\widehat{D}_{max}$ and $\breve{D}_{min}$.

Non-convexity of $D_m(n)$ in Ξ_{m+1}^n

In this section we show that $D_m(n)$ is not a convex set in Ξ_{m+1}^n.

Convex set. [Birkhoff, 1948] A subset S of the poset P is *convex* whenever $a \in S$, $b \in S$ and $a \leq b$ imply $[a, b] \in S$.

It follows from the definition that each interval $E(\breve{D}, \widehat{D})$ spanned by opposite boundary elements is a convex set in Ξ_{m+1}^n.

Nevertheless we prove that $D_m(n)$ being a union of convex sets, - is not convex.

Theorem 3. $D_m(n)$ is not convex in Ξ_{m+1}^n, when $1 < m < 2^n - 1$.

We omit the details of the proof and just bring the outline. First we show that $D_m(n)$ is convex for the following values of m:

a) $m = 1$. We show that $D_m(n) = E(\widetilde{0}, \widetilde{m})$, which coincides with Ξ_{m+1}^n, and thus, is a convex set.

b) $m = 2^n$. In this case $D_m(n) = E((2^{n-1}, \cdots, 2^{n-1}), (2^{n-1}, \cdots, 2^{n-1})$ – that is 1 point of Ξ_{m+1}^n.

c) $m = 2^n - 1$. Here $D_m(n) = E((2^{n-1} - 1, \cdots, 2^{n-1} - 1), (2^{n-1}, \cdots, 2^{n-1})$ – this is an interval of Ξ_{m+1}^n, and thus, is a convex set.

Then we prove that for the following cases:

d) $1 < m \leq 2^{n-1}$

e) $2^{n-1} < m < 2^n - 1$

there always exist two comparable elements $a < b$ in $D_m(n)$, such that the spanned interval $E(a, b)$ in Ξ_{m+1}^n contain an element $c \notin D_m(n)$.

Consider an example in Figure 3.

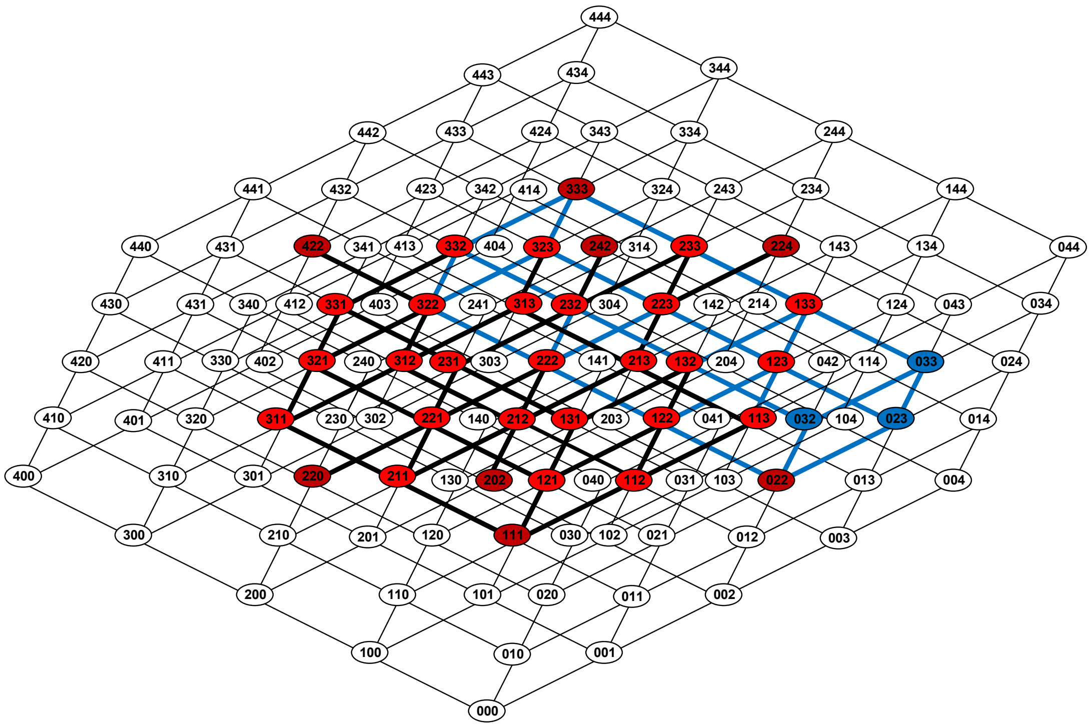

Figure 3

The elements $(0,2,2)$ and $(3,3,3)$ belong to $D_4(3)$, and $(0,2,2) < (3,3,3)$. However the elements $(0,3,2), (0,2,3), (0,3,3)$ of $\Xi 3_5^n$, which are greater than $(0,2,2)$, and less than $(3,3,3)$, - do not belong to $D_4(3)$.

Convex hull of $D_m(n)$

In this section we characterize the convex hull of $D_m(n)$.

Convex hull ([Eggleston, 1958])

Let S be a nonempty subset of R^n. Then among all convex sets containing S (these sets exist, e.g., R^n itself) there exists the smallest one, namely, the intersection of all convex sets containing S.

This set is called the *convex hull of* S (denote by: $Conv\ (S)$).

In our case we consider the intersection of $Conv(D_m(n))$ and Z^n - in other words we consider the integer points of $Conv(D_m(n))$.

Notice that Ξ_{m+1}^n itself corresponds to some convex set of R^n. $D_m(n) \subseteq \Xi_{m+1}^n$ is also contained in the mentioned convex set. We are interested in finding the smallest convex subset of Ξ_{m+1}^n, containing $D_m(n)$. We denote this set by $C_{D_m(n)}$.

Theorem 4. $C_{D_m(n)} = \bigcup_{\hat{D}\in\hat{D}_{max},\check{D}\in\check{D}_{min}} E(\check{D},\hat{D})$ (the union is by all pairs $(\hat{D},\check{D})$ and not only by opposite pairs).

We prove the theorem by showing first that the set $\bigcup_{\hat{D}\in\hat{D}_{max},\check{D}\in\check{D}_{min}} E(\check{D},\hat{D})$ is a convex set, and then - that this is the smallest convex set containing $D_m(n)$.

An illustration is in Figure 4.

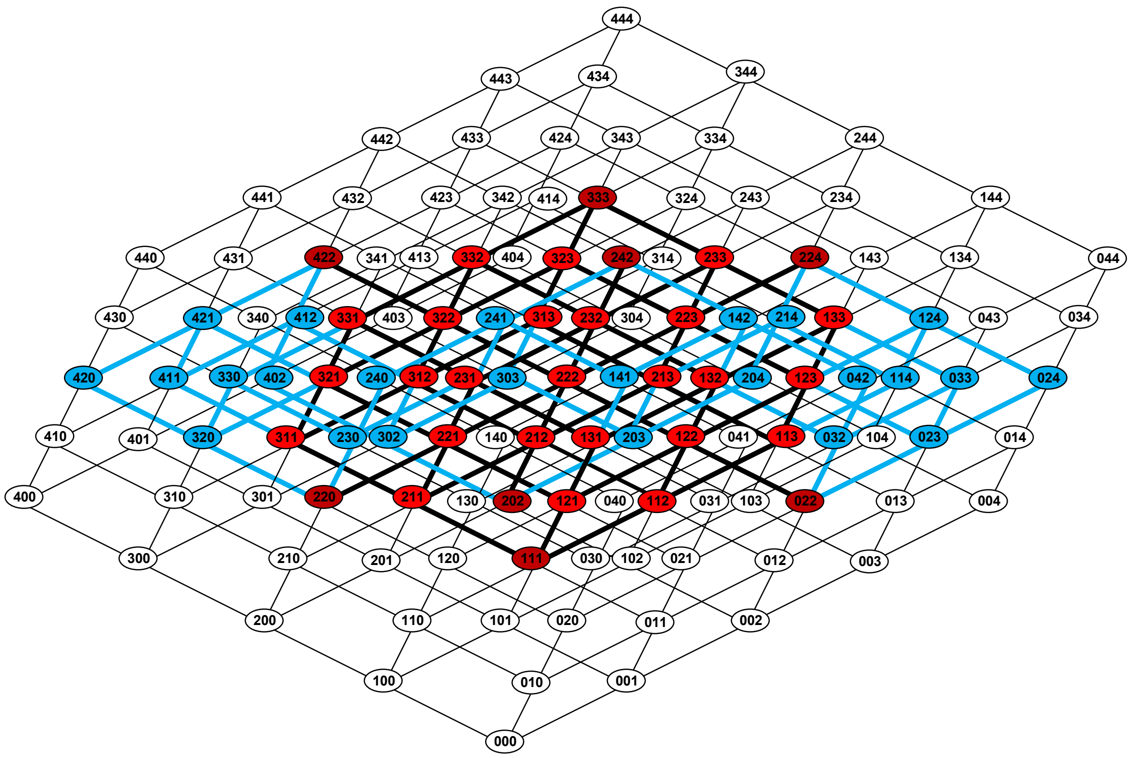

Figure 4

$C_{D_4(3)}$ in Ξ_5^3, where the elements of $D_4(3)$ are in red color, and the elements of $C_{D_4(3)}$ are in red and blue colors.

Corollary.

- The smallest convex subset of Ξ_{m+1}^n containing $D_m(n)$ is the convex hull of the set $(\widehat{D}_{max} \cup \breve{D}_{min})$.
- Each element d of $(\widehat{D}_{max} \cup \breve{D}_{min})$ is an extreme point of $C_{D_m(n)}$ since $\cup_{\widehat{D}\in\widehat{D}_{max},\breve{D}\in\breve{D}_{min}} E(\breve{D},\widehat{D}) \setminus \{d\}$ is a convex set.

Conclusion

We considered $\mathrm{D_m(n)}$, the set of all degree sequences of hypergraphs with n vertices and m hyperedges, as a subset of the n-dimensional $\mathrm{m}+1$-valued grid $\Xi_{\mathrm{m+1}}^{\mathrm{n}}$. We showed that $\mathrm{D_m(n)}$ is not a convex subset of $\Xi_{\mathrm{m+1}}^{\mathrm{n}}$, and characterized the convex hull of $\mathrm{D_m(n)}$.

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