SENSITIVITY ANALYSIS OF A DECISION-MAKING PROBLEM USING THE ANALYTIC HIERARCHY PROCESS

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Abstract: *The paper deals with the methodology of complex sensitivity analysis of solution given by one of the popular multiple-criteria decision-making methods, namely the Analytic Hierarchy Process. This methodology includes evaluation of sensitivity of hierarchy elements local ranking to changes in an expert pairwise comparison judgments and evaluation of sensitivity of global ranking of decision alternatives to changes in weights of hierarchy elements. The sensitivity analysis is illustrated on a problem of evaluation of renewable energy technologies for an eco-house in Ukraine.*

Keywords: *the analytic hierarchy process, uncertainty of expert pairwise comparison judgments, sensitivity analysis, stability intervals, critical expert pairwise comparison judgments, critical elements of a hierarchy.*

ACM Classification Keywords: *H.4.2. INFORMATION SYSTEM APPLICATION: type of system strategy*

Introduction

In the paper one of the multiple-criteria decision-making methods – the analytic hierarchy process (AHP) – is investigated. The AHP results in relative weights or priorities of decision alternatives, is based on a hierarchical model of decision factors, criteria, goals and uses expert judgments of pairwise comparison of elements of a hierarchy as initial information [1, 2]. This method is applied in many areas, such as economy, industry, social sphere, ecology, politics, military science while solving such problems as: choice and evaluation of decision alternatives and decision factors, resource allocation, analysis of benefits-costs-opportunities-risks, forecasting, analytical planning, construction and evaluation of scenarios of development and other [1 - 5].

Expert pairwise comparison judgments contain uncertainty. Therefore the question of reliability of results given by the AHP arises. To evaluate reliability of obtained results it is reasonable to find dependency between results of the AHP and inaccuracies of initial data – expert judgments. In practice a sensitivity analysis of solutions obtained by the AHP, is often carried out using graphical methods, which are proposed by T.L. Saaty and implemented in the decision-support system SuperDecisions [7]. These methods are also implemented in decision-support systems Decision Lens [8], MakeItRational [9] and

LogicalDecisions [10]. In the graphical methods a decision-maker or analyst changes a local weight of some element of a hierarchy and observes changes in global weights of decision alternatives.

The AHP is successfully used while solving different decision-making problems [1–5]. In repetitive problems the graphical methods, which are implemented in the decision-support system SuperDecisions [7], are enough to use. However, a more complete, complex sensitivity analysis has to be done while solving such decision problems as evaluation and choosing of scenarios of development and of decision alternatives on a level of big companies, branches of industry and a country as a whole, resource allocation problems and planning complex target-oriented programs, and also when making decisions concerning innovation development. While solving these problems a complex sensitivity analysis has to be integrated in each stage of decision-making, included in continuous cyclic process of problem solution.

One of the approaches to a complex sensitivity analysis in the AHP is to investigate changes of calculated global ranking of decision alternatives while varying weights of hierarchy elements and changing a hierarchical model structure [11].

Change of global ranking of decision alternatives when adding or removing an alternative, so called rank reversal, was studied in [1, 12, 13]. In these papers it was shown that rank reversal may occur in different aggregation rules of the AHP, namely in the distributive, ideal, multiplicative, max-min rules and in the rule of group consideration of binary preferences of the alternatives. Probabilities of appearance of several types of rank reversals in the aggregation rules were estimated [12, 13]. Thus, the AHP is sensitive to changes of a hierarchical model structure.

In this paper a complex methodology of sensitivity analysis of results obtained by the AHP is proposed. This methodology includes:

- evaluation of sensitivity of a local ranking of decision alternatives to changes in expert pairwise comparison judgments,
- evaluation of sensitivity of a global ranking of decision alternatives to changes in weights of hierarchy elements.

As a result, stability intervals are defined which allow to find so called critical elements of the decision-making problem. These are critical expert pairwise comparison judgments that are sensitive to changes of a local ranking of decision alternatives, and critical hierarchy elements (decision criteria, goals) – elements that are characterized by the least changes of their weights that lead to changes of a global ranking of decision alternatives.

Problem statement

Let H be an analytic hierarchy with $p+1$ levels. Level L_0 of the hierarchy has one element — the main goal of decision-making, the last level L_p contains decision alternatives. Hierarchy levels, that are between L_0 and L_p, contain possible factors (criteria, goals) that influence the decision. Denote number of elements on a L_k-th level as N_{L_k}, $L_k \in \lfloor L_0 ; L_p \rfloor$.

$\hat{A}_r^{L_k L_{k-1}}$ is a pairwise comparison matrix (PCM) of elements of L_k-th level in terms of r-th element of L_{k-1}-th level, $r \in \lfloor 1; N_{L_{k-1}} \rfloor$, constructed on the basis of expert judgments.

$\hat{w}_{lr}^{L_k L_{k-1}}$ is a local weight of l-th element of L_k-th level in terms of r-th element of L_{k-1}-th level, $l \in \lfloor 1; N_{L_k} \rfloor$, $r \in \lfloor 1; N_{L_{k-1}} \rfloor$. Weight vector $\hat{w}_r^{L_k L_{k-1}} = \left\{ \hat{w}_{lr}^{L_k L_{k-1}} \mid l \in \left[1; N_{L_k} \right] \right\}$ calculates on the basis of the PCM $\hat{A}_r^{L_k L_{k-1}}$ using the eigenvector method, the row geometric mean method and others [1, 2].

$\hat{w}_l^{L_k}$ is a global weight of l-th element of L_k-th level, $l \in \lfloor 1; N_{L_k} \rfloor$. In the analytic hierarchy process vector of global weights can be calculated using the distributive or multiplicative aggregation rules [1, 2]. Vector of global weights of decision alternatives $\hat{w}^{L_p} = \left\{ \hat{w}_i^{L_p} \mid i \in \left[1; N_{L_p} \right] \right\}$ is a result of the analytic hierarchy process.

It is necessary to provide a complex sensitivity analysis of rankings obtained using the AHP to inaccuracy and subjectivity of expert judgments:

- to evaluate sensitivity of a local ranking of decision alternatives to changes in expert pairwise comparison judgments (elements of a PCM $\hat{A}_r^{L_k L_{k-1}}$);
- to evaluate sensitivity of a global ranking of decision alternatives to changes in weights of hierarchy elements;
- to find critical and stable expert pairwise comparison judgments;
- to find critical and stable elements of L_k-th hierarchy level, $L_k \in \lfloor L_1 ; L_{p-1} \rfloor$.

The problem solving. Sensitivity analysis of a local ranking of decision alternatives when changing expert pairwise comparison judgments

Let us consider calculation of local weights of hierarchy elements, for example, decision alternatives $a_1, a_2, ..., a_n$ in terms of their common feature (an element of a parent hierarchy level). Suppose $D = \{(d_{ij}) \mid i, j = 1, ..., n\}$ is a PCM constructed on the basis of expert pairwise comparison judgments.

Using the Row Geometric Mean Method (RGMM), nonnormalized local weights $v_1, v_2, ..., v_n$ of decision alternatives are calculated as follows:

$$v_i = (\prod_{l=1}^{n} d_{il})^{1/n}, \ i = 1,...,n. \quad (1)$$

We are interested in how much ranking of decision alternatives, built on the basis of calculated local weights is insensitive to changes of expert judgments (PCM elements). Let us investigate two cases:

1. whether the best alternative remains unchanged,
2. whether an overall ranking of alternatives remains unchanged.

A stability interval of expert pairwise comparison judgments concerning change of ranking of decision alternatives is an interval within the bounds of which an expert judgment may be changed so that a local ranking of alternatives remains unchanged.

Denote $[\underline{d_{ij}}, \overline{d_{ij}}]$ a stability interval for an expert judgment d_{ij}.

Without loss of generality suppose that decision alternatives are renumbered in order of importance decreasing, that is the ranking of alternatives is

$$a_1 \succ a_2 \succ ... \succ a_n, \quad (2)$$

where a_1 and a_n are the best (the most important) and the worst decision alternatives, respectively.

In terms of weights (2) means that $v_i > v_j$ for $i < j$.

Let us find for each expert judgment a stability interval concerning change of alternatives ranking, when the RGMM is used for weights calculation.

A case when the best alternative remains unchanged. At first consider a case when change of a PCM element does not lead to change of the best decision alternative a_1.

Suppose the PCM element d_{1j}, $j \neq 1$ is changed within the bounds of interval $[\underline{d_{1j}}, \overline{d_{1j}}]$. Then in accordance with the RGMM (1), weights of decision alternatives a_1 and a_j are changed. Denote these new weights $v_1' = [\underline{v_1}, \overline{v_1}]$ and $v_j' = [\underline{v_j}, \overline{v_j}]$, where

$$\underline{v_1} = \left(\frac{\underline{d_{1j}}}{d_{1j}}\right)^{1/n} \cdot v_1 \text{ and } \overline{v_1} = \left(\frac{\overline{d_{1j}}}{d_{1j}}\right)^{1/n} \cdot v_1, \tag{3}$$

$$\underline{v_j} = \left(\frac{\underline{d_{j1}}}{d_{j1}}\right)^{1/n} \cdot v_j = \left(\frac{\underline{d_{1j}}}{d_{1j}}\right)^{1/n} \cdot v_j \text{ and } \overline{v_j} = \left(\frac{\overline{d_{j1}}}{d_{j1}}\right)^{1/n} \cdot v_j = \left(\frac{d_{1j}}{\underline{d_{1j}}}\right)^{1/n} \cdot v_j. \tag{4}$$

We want to find an interval $[\underline{d_{1j}}, \overline{d_{1j}}]$, such that the best decision alternative does not change, i.e. $v_1' > v_j'$, $j \neq 1$ and $v_1' > v_k'$, $k \neq j \neq 1$. This is equivalent to implementation of the following two conditions:

$$\underline{v_1} > \overline{v_j}, \ \underline{v_1} > v_k, \tag{5}$$

where $k \neq j \neq 1$.

Substitute expressions (3) and (4) in (5) and find the following constraints for the left bound of a stability interval:

$$\underline{d_{1j}} > d_{1j} \cdot \left(\frac{v_j}{v_1}\right)^{n/2} \text{ and } \underline{d_{1j}} > d_{1j} \cdot \left(\frac{v_k}{v_1}\right)^{n} \tag{6}$$

$$\text{or } \underline{d_{1j}} > d_{1j} \cdot \left(\frac{\prod_{l=1}^{n} d_{jl}}{\prod_{l=1}^{n} d_{1l}}\right)^{1/2} \text{ and } \underline{d_{1j}} > d_{1j} \cdot \frac{\prod_{l=1}^{n} d_{kl}}{\prod_{l=1}^{n} d_{1l}}, \text{ where } k \neq j \neq 1.$$

There are no constrains on the right bound of a stability interval, so let us assign it the maximum permissible value, namely, the largest value in the Saaty scale used by an expert when making an assessment: $\overline{d_{1j}} = 9$. Comparing right parts of inequality (6) one can formulate the following statement for a stability interval calculation.

Statement 1: A stability interval $[\underline{d_{1j}}, \overline{d_{1j}}]$ for an expert judgment d_{1j}, $j \neq 1$, such that the best decision alternative a_1 remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:

$\underline{d_{1j}} > d_{1j} \cdot (\frac{v_j}{v_1})^{n/2}$, if $v_1 \cdot v_j \ge (v_k)^2$, $k \ne j \ne 1$,

$\underline{d_{1j}} > d_{1j} \cdot (\frac{v_k}{v_1})^{n}$, if $v_1 \cdot v_j < (v_k)^2$, $k \ne j \ne 1$,

$\overline{d_{1j}} = 9$.

In practice the $\underline{d_{1j}}$ value is the nearest value of the Saaty scale that satisfies inequalities of the Statement 1.

Consider a case when any PCM element d_{kj}, $k \ne j \ne 1$ is changed. It is necessary to find a stability interval $[\underline{d_{kj}}, \overline{d_{kj}}]$ for this element. According to the RGMM (1), change of d_{kj} leads to change of weights of decision alternatives a_k and a_j. Denote these new weights $v'_k = [\underline{v_k}, \overline{v_k}]$, $v'_j = [\underline{v_j}, \overline{v_j}]$ and calculate their left and right bounds using the RGMM:

$$\underline{v_k} = (\frac{\underline{d_{kj}}}{d_{kj}})^{1/n} \cdot v_k \text{ and } \overline{v_k} = (\frac{\overline{d_{kj}}}{d_{kj}})^{1/n} \cdot v_k, \quad (7)$$

$$\underline{v_j} = (\frac{\underline{d_{jk}}}{d_{jk}})^{1/n} \cdot v_j = (\frac{\underline{d_{kj}}}{d_{kj}})^{1/n} \cdot v_j \text{ and } \overline{v_j} = (\frac{\overline{d_{jk}}}{d_{jk}})^{1/n} \cdot v_j = (\frac{d_{kj}}{\underline{d_{kj}}})^{1/n} \cdot v_j . \quad (8)$$

The best decision alternative does not change if inequalities $v'_1 > v'_j$, $j \ne 1$ and $v'_1 > v'_k$, $k \ne j \ne 1$ are satisfied. This is equivalent to implementation of the following conditions:

$$v_1 > \overline{v_j} \text{ and } v_1 > \overline{v_k}, \quad (9)$$

where $k \ne j \ne 1$.

Substitute expressions (7) and (8) in (9) and find the following constraints for the left and right bounds of a stability interval:

$$\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_1})^n \text{ and } \overline{d_{kj}} < d_{kj} \cdot (\frac{v_1}{v_k})^n \quad (10)$$

or $\underline{d_{kj}} > d_{kj} \cdot \prod_{l=1}^{n} d_{jl} / \prod_{l=1}^{n} d_{1l}$ and $\overline{d_{kj}} < d_{kj} \cdot \prod_{l=1}^{n} d_{1l} / \prod_{l=1}^{n} d_{kl}$, where $k \neq j \neq 1$.

Statement 2: A stability interval $[\underline{d_{kj}}, \overline{d_{kj}}]$ for an expert judgment d_{kj}, $k \neq j \neq 1$, such that the best decision alternative a_1 remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:

$$\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_1})^n, \ \overline{d_{kj}} < d_{kj} \cdot (\frac{v_1}{v_k})^n .$$

It should be noted that PCM elements take values from the Saaty scale (namely, values from the set $\{1/9,...,9\}$). Therefore in practice values $\underline{d_{kj}}$ and $\overline{d_{kj}}$ are the nearest values of this scale that satisfy the corresponding inequalities of the Statement 2.

A case when an overall ranking of alternatives remains unchanged. Now consider a case when change of a PCM element leads to steady overall ranking (2) of decision alternatives. Similarly to the previous case, it is necessary to analyze separately change of an element d_{1j}, $j \neq 1$ and an element d_{kj}, $k \neq j \neq 1$.

Consider change of d_{1j}, $j \neq 1$ within an interval $[\underline{d_{1j}}, \overline{d_{1j}}]$. To save the overall ranking (2), it is necessary to impose the following additional constraints besides mentioned above constraints $v_1' > v_j'$, $j \neq 1$ and $v_1' > v_k'$, $k \neq j \neq 1$:

$$v_j' > v_k' \text{ when } j < k \quad (11)$$

$$v_k' > v_j' \text{ when } k < j \quad (12)$$

where $k \neq j \neq 1$.

Inequalities (11) and (12) are equivalent to the following:

$$\underline{v_j} > v_k \text{ when } j < k$$

$$v_k > \overline{v_j} \text{ when } k < j$$

or using (3) and (4):

$$\overline{d_{1j}} < d_{1j} \cdot (\frac{v_j}{v_k})^n \text{ when } j < k \tag{13}$$

$$\underline{d_{1j}} > d_{1j} \cdot (\frac{v_j}{v_k})^n \text{ when } k < j \tag{14}$$

Taking into account conditions $v_1' > v_j'$, $j \neq 1$ and $v_1' > v_k'$, $k \neq j \neq 1$, that lead to (6), we obtain statement for a stability interval calculation.

Statement 3: A stability interval $[\underline{d_{1j}}, \overline{d_{1j}}]$ for an expert judgment d_{1j}, $j \neq 1$, such that the overall ranking $a_1 \succ a_2 \succ ... \succ a_n$ of decision alternatives remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:

3.1. $\underline{d_{1j}} > d_{1j} \cdot (\frac{v_j}{v_1})^{n/2}$, if $v_1 \cdot v_j \geq (v_k)^2, j \neq 1, j < k$

$\underline{d_{1j}} > d_{1j} \cdot (\frac{v_k}{v_1})^n$, if $v_1 \cdot v_j < (v_k)^2$ $, j \neq 1, j < k$

$\overline{d_{1j}} < d_{1j} \cdot (\frac{v_j}{v_k})^n$, if $j \neq 1$, $j < k$.

3.2. $\underline{d_{1j}} > d_{1j} \cdot (\frac{v_j}{v_k})^n$, if $v_1 \cdot v_j \geq (v_k)^2, j \neq 1, j > k$

$\underline{d_{1j}} > d_{1j} \cdot (\frac{v_k}{v_1})^n$, if $v_1 \cdot v_j < (v_k)^2$ $, j \neq 1, j > k$

$\overline{d_{1j}} = 9$, if $j \neq 1$, $j > k$.

Consider change of an element d_{kj} , $k \neq j \neq 1$ within an interval $[\underline{d_{kj}}, \overline{d_{kj}}]$. Similarly to previous case, to save the overall ranking (2), additional conditions (11) and (12) are added, which in this case take a form:

$$\underline{v_j} > \overline{v_k} \text{ when } j < k$$

$$\underline{v_k} > \overline{v_j} \text{ when } k < j$$

Using (7) and (8) we get:

$$\overline{d_{kj}} < d_{kj} \cdot (\frac{v_j}{v_k})^{n/2} \text{ when } j < k$$

$$\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_k})^{n/2} \text{ when } k < j$$

Taking into account $v_1' > v_j'$, $j \neq 1$ and $v_1' > v_k'$, $k \neq j \neq 1$, which in this case lead to constraints (10), we obtain statement for calculation of a stability interval.

Statement 4: A stability interval $[\underline{d_{kj}}, \overline{d_{kj}}]$ for an expert judgment d_{kj} , $k \neq j \neq 1$, such that the overall ranking $a_1 \succ a_2 \succ ... \succ a_n$ of decision alternatives remains unchanged, when the RGMM is used for weights calculation, satisfies the inequalities:

4.1. $\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_1})^n$ if $j \neq 1$, $j < k$

$\overline{d_{kj}} < d_{kj} \cdot (\frac{v_j}{v_k})^{n/2}$ if $(v_1)^2 \geq v_j v_k$ $, j \neq 1, j < k$

$\overline{d_{kj}} < d_{kj} \cdot (\frac{v_1}{v_k})^n$ if $(v_1)^2 < v_j v_k$ $, j \neq 1, j < k$

4.2. $\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_k})^{n/2}$ if $(v_1)^2 \geq v_j v_k$ $, j > k, k \neq 1$

$\underline{d_{kj}} > d_{kj} \cdot (\frac{v_j}{v_1})^n$ if $(v_1)^2 < v_j v_k$ $, j > k, k \neq 1$

$\overline{d_{kj}} < d_{kj} \cdot (\frac{v_1}{v_k})^n$ if $j > k$, $k \neq 1$.

As was mentioned above, in practice the bounds of stability intervals $[\underline{d_{1j}}, \overline{d_{1j}}]$ and $[\underline{d_{kj}}, \overline{d_{kj}}]$ in the Statements 3 and 4 are the nearest values of the Saaty scale that satisfy the corresponding inequalities of the statements.

Sensitivity analysis of a global ranking of decision alternatives when changing weights of elements of an hierarchy

Let us consider a multiple-criteria problem of calculation of global weights of decision alternatives on the basis on a hierarchy of criteria. A method of sensitivity analysis described in this section is a generalization of the method proposed in [6]. Without loss of generality suppose that decision alternatives are renumbered such that

$$\hat{w}_i^{L_p} \geq \hat{w}_j^{L_p}, \ i \in [1; N_{L_p}], \ j \in [1; N_{L_p}] \text{ when } i < j.$$

Denote $\Delta_{i,j,l}^{L_k}$, $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, $l \in [1; N_{L_k}]$, $L_k \in [L_1; L_{p-1}]$ value of an **absolute change of weight** $\hat{w}_l^{L_k}$ that leads to change of global ranking between i-th and j-th elements of L_p-th level (i-th and j-th decision alternatives). That is, a new weight of l-th element of L_k-th level equals $\hat{w}_l'^{L_k} = \hat{w}_l^{L_k} - \Delta_{i,j,l}^{L_k}$, $\hat{w}_l'^{L_k} > 0$, and $\hat{w}_i'^{L_p} < \hat{w}_j'^{L_p}$ holds when $i < j$, where $\hat{w}_i'^{L_p}$ is a new global weight of i-th element of L_p-th level.

Denote $\delta_{i,j,l}^{L_k}$, $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, $l \in [1; N_{L_k}]$, $L_k \in [L_1; L_{p-1}]$ value of a **relative change of weight** $\hat{w}_l^{L_k}$ that leads to change of global ranking between i-th and j-th elements of L_p-th level (i-th and j-th decision alternatives). That is, a new weight equals $\hat{w}_l'^{L_k} = \hat{w}_l^{L_k} - \dfrac{\delta_{i,j,l}^{L_k} \hat{w}_l^{L_k}}{100}$, $\hat{w}_l'^{L_k} > 0$. The values of absolute and relative changes of weight $\hat{w}_l^{L_k}$ of l-th element in L_k-th level are in the following relation: $\delta_{i,j,l}^{L_k} = \dfrac{\Delta_{i,j,l}^{L_k}}{\hat{w}_l^{L_k}} 100\,\%$.

l-th element of L_k-th level **is stable** if any permissible changes of its weight do not lead to changes of global rank of any decision alternative.

Degree of criticality $C_l^{L_k}$ of l-th element of L_k-th level is a value of the least relative change of its weight $\hat{w}_l^{L_k}$ that leads to change of global ranking of decision alternatives: $C_l^{L_k} = \min\limits_{\substack{i,j\in[1;N_{L_p}],\\ i<j}} \left\{ | \delta_{i,j,l}^{L_k} | \right\}.$

Sensitivity $S_l^{L_k}$ of l-th element of L_k-th level is a reciprocal value to the degree of criticality of this element: $S_l^{L_k} = \dfrac{1}{C_l^{L_k}}$. $S_l^{L_k}$ is assigned a zero value if l-th element of L_k-th level is stable.

Less values of the degree of criticality $C_l^{L_k}$ mean that it is easier to change a ranking of decision alternatives. So less values of the degree of criticality $C_l^{L_k}$ indicate that less change of weight $\hat{w}_l^{L_k}$ is sufficient for a change of ranking of decision alternatives. Therefore "the easier" change of ranking of decision alternatives results in larger value of sensitivity $S_l^{L_k}$ of l-th element of L_k-th level.

Critical element of L_k-th level is an element of L_k-th level which has the least value $\left|\delta_{i,j,l}^{L_k}\right|$, that is l_{crit}-th element of L_k-th level is critical if $| \delta_{i,j,l_{crit}}^{L_k} | = \min\limits_{l\in[1;N_{L_k}]} \left\{ \delta_{i,j,l}^{L_k} | \right\}$, $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$.

Values of relative change $\delta_{i,j,l}^{L_k}$ when the distributive and multiplicative aggregation rules are used for calculation of global weights of hierarchy elements can be found using the following statements 7 and 8.

Statement 7: A value $\delta_{i,j,l}^{L_k}$ of relative change of weight $\hat{w}_l^{L_k}$ that is necessary for a change of global ranking between i-th and j-th elements of L_p-th level, $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, $l \in [1; N_{L_k}]$, $L_k \in [L_1; L_{p-1}]$, when $i < j$ and the distributive aggregation rule is used for calculation of global weights satisfies the inequality [11]:

$$\delta_{i,j,l}^{L_k} < \delta_{i,j,l}^{L_k\,porog}, \text{ if } \hat{w}_{jl}^{L_p L_k} > \hat{w}_{il}^{L_p L_k},$$

$$\delta_{i,j,l}^{L_k} > \delta_{i,j,l}^{L_k\,porog}, \text{ if } \hat{w}_{jl}^{L_p L_k} < \hat{w}_{il}^{L_p L_k},$$

where the threshold value $\delta_{i,j,l}^{L_k\,porog}$ of $\delta_{i,j,l}^{L_k}$ is calculated as follows:

$$\delta_{i,j,l}^{L_k porog} = \Delta_{i,j,l}^{L_k porog} \frac{100}{\hat{w}_l^{L_k}} (\%) , \tag{15}$$

$$\Delta_{i,j,l}^{L_k porog} = \frac{\hat{w}_j^{L_p} - \hat{w}_i^{L_p}}{\hat{w}_{jl}^{L_p L_k} - \hat{w}_{il}^{L_p L_k}} \tag{16}$$

under conditions:

1) $\hat{w}_i^{L_p} \geq \hat{w}_j^{L_p}$ when $i < j$;

2) $\hat{w}_l^{L_k} > \Delta_{i,j,l}^{L_k porog}$ (that is equivalent to $\delta_{i,j,l}^{L_k porog} < 100\%$).

Corollary: l-th element of L_k-th level, $l \in [1; N_{L_k}]$, is stable if $\hat{w}_l^{L_k} \leq \Delta_{i,j,l}^{L_k porog}$ holds when $i < j$ for all $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, where threshold value $\Delta_{i,j,l}^{L_k porog}$ of absolute change $\Delta_{i,j,l}^{L_k}$ of weight $\hat{w}_l^{L_k}$ of l-th element in L_k-th level is calculated using the formula (16).

Corollary: If $\hat{w}_{jl}^{L_p L_k} \leq \hat{w}_{il}^{L_p L_k}$ holds for **all** $l \in [1; N_{L_k}]$, that is j-th element of L_p-th level does not dominate i-th element of L_p-th level in terms of **all** elements of L_k-th level, $L_k \in [L_1; L_{p-1}]$, then any changes of weights of L_k-th level elements do not lead to changes of global ranking between these elements of L_p-th level.

Statement 8: A value $\delta_{i,j,l}^{L_k}$ of relative change of weight $\hat{w}_l^{L_k}$ that is necessary for a change of global ranking between i-th and j-th elements of L_p-th level, $i,j = \overline{1, N_{L_p}}$, $l = \overline{1, N_{L_k}}$, $L_k = \overline{L_1, L_{p-1}}$, when $i < j$ and the multiplicative aggregation rule is used for calculation of global weights satisfies the inequality [1]:

$\delta_{i,j,l}^{L_k} < \delta_{i,j,l}^{L_k porog}$, if $\hat{w}_{jl}^{L_p L_k} > \hat{w}_{il}^{L_p L_k}$;

$\delta_{i,j,l}^{L_k} > \delta_{i,j,l}^{L_k porog}$, if $\hat{w}_{jl}^{L_p L_k} < \hat{w}_{il}^{L_p L_k}$,

where the threshold value $\delta_{i,j,l}^{L_k porog}$ of $\delta_{i,j,l}^{L_k}$ is calculated as follows:

$$\delta_{i,j,l}^{L_k porog} = \Delta_{i,j,l}^{L_k porog} \cdot \frac{100}{\hat{w}_l^{L_k}} (\%) ,$$

where $\Delta_{ijl}^{L_{p-k} порог} = \dfrac{\ln(w_i^{L_p} / w_j^{L_p})}{\ln\left(\prod_{j_1=1}^{N_{L_{p-1}}} \left(\frac{a_{ij_1}^{L_p L_{p-1}}}{a_{jj_1}^{L_p L_{p-1}}}\right)^{\prod_{j_2=1}^{N_{L_{p-2}}} (a_{j_1 j_2}^{L_{p-1} L_{p-2}})^{\cdot^{\cdot^{\cdot^{\prod_{j_{k-1}=1}^{N_{L_{p-k+1}}} (a_{j_{k-2} j_{k-1}}^{L_{p-k+2} L_{p-k+1}})^{a_{j_{k-1} l}^{L_{p-k+1} L_{p-k}}}}}}}} \right)}$

under conditions:

1) $\hat{w}_i^{L_p} \geq \hat{w}_j^{L_p}$ when $i < j$;

2) $\hat{w}_l^{L_k} > \Delta_{i,j,l}^{L_k porog}$ (that is equivalent to $\delta_{i,j,l}^{L_k porog} < 100\%$).

A case of change of local weights of decision alternatives. In this subsection we will find an interval of changes of a local weight $\hat{w}_{ir}^{L_p L_{p-1}}$ of i-th decision alternative in terms of r-th element of a parent hierarchy level, that do not lead to changes of global ranking between i-th and j-th alternatives. This allows to define how critical (sensitive) every decision alternative is in terms of selected element of a parent hierarchy level, i.e. to find a value of the least change of a local weight of decision alternative that results in change of global ranking of decision alternatives.

Denote $\delta_{i,j,r}^a$, $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, $r \in [1; N_{L_{p-1}}]$ value of a relative change of local weight $\hat{w}_{ir}^{L_p L_{p-1}}$ of i-th element of L_p-th level (i-th decision alternative) in terms of r-th element of a parent L_{p-1}-th level that leads to change of global ranking between i-th and j-th elements of L_p-th level (i-th and j-th decision alternatives). That is, a new weight of i-th element of L_p-th level in terms of r-th element of L_{p-1}-th level equals $\hat{w}_{ir}'^{L_p L_{p-1}} = \hat{w}_{ir}^{L_p L_{p-1}} - \dfrac{\delta_{i,j,r}^a \hat{w}_{ir}^{L_p L_{p-1}}}{100}$, $\hat{w}_{ir}'^{L_p L_{p-1}} > 0$, and $\hat{w}_i'^{L_p} < \hat{w}_j'^{L_p}$ holds when $i < j$, where $\hat{w}_i'^{L_p}$ is a new global weight of i-th element of L_p- th level.

i-th element of L_p-th level **is stable** in terms of r-th element of a parent L_{p-1}-th level if any permissible changes of a local weight $\hat{w}_{ir}^{L_p L_{p-1}}$ of this element do not lead to changes of global rank of any decision alternative.

Degree of criticality C^a_{ir} of i-th element of L_p-th level (i-th decision alternative) in terms of r-th element of a parent L_{p-1}-th level is the minimum of values $\left|\delta^a_{i,j,r}\right|$ that lead to change of global rank of this decision alternative: $C^a_{ir} = \min\limits_{\substack{j\in[1;N_{L_p}] \\ j\neq i}} \left\{\left|\delta^a_{i,j,r}\right|\right\}$, $i\in[1;N_{L_p}]$, $r\in[1;N_{L_{p-1}}]$.

Sensitivity S^a_{ir} of i-th element of L_p-th level (i-th decision alternative) in terms of r-th element of a parent L_{p-1}-th level is a reciprocal value to the degree of criticality of this element: $S^a_{ir} = \dfrac{1}{C^a_{ir}}$, $i\in[1;N_{L_p}]$, $r\in[1;N_{L_{p-1}}]$. S^a_{ir} is assigned a zero value if i-th element of L_p-th level is stable in terms of r-th element of L_{p-1}-th level.

Critical element of L_p-th level (**critical alternative***)* is an element of L_p-th level which has the least degree of criticality, that is i_{crit}-th element of L_p-th level is critical in terms of r-th element of a parent L_{p-1}-th level, if $C^a_{i_{crit}r} = \min\limits_{i\in[1;N_{L_p}]}\left\{\min\limits_{r_1\in[1;N_{L_{p-1}}]}\left\{C^a_{ir_1}\right\}\right\}$.

A value of relative change $\delta^a_{i,j,r}$ of local weight $\hat{w}_{ir}^{L_pL_{p-1}}$ of i-th alterative in terms of r-th element of L_{p-1}-th level can be found using the following statement 9.

Statement 9: A value $\delta^a_{i,j,r}$ of relative change of a local weight $\hat{w}_{ir}^{L_pL_{p-1}}$ that is necessary for a change of global ranking between i-th and j-th elements of L_p-th level (i-th and j-th decision alternatives), $i\in[1;N_{L_p}]$, $j\in[1;N_{L_p}]$, $r\in[1;N_{L_{p-1}}]$, when the multiplicative aggregation rule is used for calculation of global weights satisfies the inequality:

$$\delta^a_{i,j,r} > \delta^{a\ porog}_{i,j,r}, \text{ if } i<j,$$

$$\delta^a_{i,j,r} < \delta^{a\ porog}_{i,j,r}, \text{ if } i>j,$$

where the threshold value $\delta^{a\ porog}_{i,j,r}$ of $\delta^a_{i,j,r}$ is calculated as follows:

$$\delta^{a\ porog}_{i,j,r} = \left(1-\left(\frac{\hat{w}_j^{L_p}}{\hat{w}_i^{L_p}}\right)^{1/\hat{w}_r^{L_{p-1}}}\right)\cdot 100\,(\%) \tag{17}$$

under conditions:

1) $\hat{w}_i^{L_p} \geq \hat{w}_j^{L_p}$ when $i<j$;

2) $\delta_{i,j,r}^{a\ porog} < 100\%$.

Corollary: i -th element of L_p -th level is stable in terms of r -th element of L_{p-1} -th level if $\delta_{i,j,r}^{a\ porog} > 100\%$ holds for all $i \in [1; N_{L_p}]$, $j \in [1; N_{L_p}]$, where threshold value $\delta_{i,j,r}^{a\ porog}$ is calculated using the formula (17).

Sensitivity analysis of a global ranking of decision alternatives in a problem of evaluation of renewable energy technologies for an eco-house in Ukraine

Let us consider a multiple-criteria decision-making problem of evaluation of renewable energy technologies for an eco-house and solve it using the analytic hierarchy process (AHP). Several technologies of renewable energy for an eco-house are selected for investigation by a decision-maker:

- geothermal thermal pump (a_1);
- biofuel production (a_2);
- solar plant (a_3).

To evaluate these technologies (alternatives) a decision-maker develops the following four criteria:

- accessibility (c_1);
- economic efficiency during the use of a technology (c_2);
- initial costs (c_3);
- costs during the use of a technology (c_4).

Criteria weights, local weights of decision alternatives in terms of each criterion and global weights of the alternatives using the distributive and multiplicative aggregation rules of the AHP are shown in the Table 1.

Table 1: Solution of the problem using the distributive and multiplicative aggregation rules of the AHP

	c_1 (0.094)	c_2 (0.509)	c_3 (0.243)	c_4 (0.154)	Global weights	
					Distributive aggregation rule	Multiplicative aggregation rule
a_1	0.090	0.649	0.065	0.114	0.372	0.312
a_2	0.455	0.279	0.361	0.405	0.335	0.436
a_3	0.455	0.072	0.574	0.481	0.293	0.252

Sensitivity analysis was done separately for the global rankings of the decision alternatives using the distributive and multiplicative aggregation rules.

A case of the distributive aggregation rule. The global ranking of decision alternatives in this case is: $a_1 \succ a_2 \succ a_3$, and alternative a_1 is the optimal one. The criterion c_2 is the most important, its weight equals 0.509. Let us calculate a threshold value of relative change of this criterion weight that leads to changing of the global ranking, for example, between alternatives a_1 and a_2. This value is calculated as follows:

$$\delta_{1,2,2}^{porog} = \frac{0.335 - 0.372}{0.279 - 0.649} \cdot \frac{1}{0.509} = 0.198 .$$

A positive value of $\delta_{1,2,2}^{porog}$ means that the criterion c_2 weight has to be decreased to change the ranking between alternatives a_1 and a_2. The relative value of this decreasing equals 19.8%.

$\delta_{1,2,2} > \delta_{1,2,2}^{porog} = 0.198$, since $w_{22} < w_{12}$. Thus, an interval of relative change of the criterion c_2 weight that leads to changing of the global ranking between a_1 and a_2 is $\delta_{1,2,2} \in$ (0.198; 1.000).

For example, suppose that decision-maker preferences are changed and the criterion c_2 weight is decreased up to the value 0.407 (that is on 20%). The criteria weights after renormalization are $w_1^C = 0.105$, $w_2^C = 0.453$, $w_3^C = 0.271$ and $w_4^C = 0.171$. Then the global weights of the decision alternatives are: $w_1^{glob} = 0.341$, $w_2^{glob} = 0.341$, $w_3^{glob} = 0.318$, and the alternative a_2 becomes as important as the a_1.

Relative changes of all criteria weights that lead to changing of the global ranking between different pairs of alternatives are given in the Table 2.

According to the definition, a critical criterion for changing of optimal alternative defines as the minimum by absolute value in rows of the Table 2, that correspond to the optimal alternative a_1. This minimum value equals 19.8% and corresponds to the criterion c_2 and alternatives a_1 and a_2. Decreasing of the criterion c_2 weight on 19.8% leads to changing of the optimal alternative, and a_2 becomes optimal.

The criterion c_2 is the most sensitive in this problem, the next less sensitive criteria are c_3, c_4 and c_1 (Table 3).

Table 2: Threshold values $\delta_{i,j,l}^{porog}$ (case of the distributive aggregation rule)

Pair of alternatives (i, j)	$\delta_{i,j,l}^{porog}$, %			
	c_1	c_2	c_3	c_4
(1,2)	-108.7*	19.8	-51.8	-83.2
(1,3)	-230.8	27.0	-64.0	-140.1
(2,3)	-	39.8	-81.0	-358.0

* Negative value of $\delta_{i,j,l}^{porog}$ means that the criterion C_l weight has to be increased to change ranking between alternatives a_i and a_j.

Table 3: Degrees of criticality $CritVal$ and sensitivity $SensVal$ for the criteria (case of the distributive aggregation rule)

Criterion	$CritVal$, %	$SensVal$
c_1	108.7	0.009
c_2	19.8	0.051
c_3	51.8	0.019
c_4	83.2	0.012

A case of the multiplicative aggregation rule. The global weights of the decision alternatives in this case equal $w_1^{glob} = 0.312$, $w_2^{glob} = 0.436$, $w_3^{glob} = 0.252$ (see Table 1). So the global ranking of the alternatives is $a_2 \succ a_1 \succ a_3$, and a_2 is the best (optimal) one.

Relative changes of criteria weights that lead to changing of the global ranking are shown in the Table 4. For example, the value $\delta_{1,3,2}^{porog}$ of relative change of the criterion c_2 weight that leads to changing of the ranking between alternatives a_1 and a_3 is calculated as follows:

$$\delta_{1,3,2}^{porog} = \frac{\ln(0.312) - \ln(0.252)}{\ln(0.649) - \ln(0.072)} \cdot \frac{1}{0.509} = 0.193 \quad \Rightarrow \delta_{1,3,2} \in \text{ (19.3\%; 100\%).}$$

Indeed the relative decrease of the criterion c_2 weight, for example, on 20%, results in a new weight $(w_2^C)' = 0.509 - 0.2 \cdot 0.509 = 0.407$. Then global weights of the alternatives are $w_1^{glob} = 0.277$, $w_2^{glob} = 0.443$, $w_3^{glob} = 0.280$, and alternative a_3 becomes more important than alternative a_2 .

Table 4: Threshold values $\delta_{i,j,l}^{porog}$ (case of the multiplicative aggregation rule)

Pair of alternatives (i, j)	$\delta_{i,j,l}^{porog}$, %			
	c_1	c_2	c_3	c_4
(1,2)	-	-77.8	80.3	-
(1,3)	-141.7	19.3	-40.8	-97.3
(2,3)	-	79.8	-488.3	-

A critical criterion for changing of optimal alternative defines as the minimum by absolute values in rows of the Table 4, that correspond to the optimal alternative a_2 . This minimum value equals 77.8% and corresponds to the criterion c_2 and alternatives a_1 and a_2 . Increase of the criterion c_2 weight more that on 77.8% results in changing of the optimal alternative, and a_1 becomes optimal. The criterion c_2 is also critical for changing the global ranking between any two considered alternatives: relative change of its weight that equal 19.3%, is enough for changing the global ranking between nonoptimal alternatives a_1 and a_3 .

The criterion c_2 is the most sensitive in this problem, the next less sensitive criteria are c_3 , c_4 and c_1 (Table 5).

Table 5: Degrees of criticality $CritVal$ and sensitivity $SensVal$ for the criteria (case of the multiplicative aggregation rule)

Criterion	$CritVal$, %	$SensVal$
c_1	141.7	0.007
c_2	19.3	0.052
c_3	40.8	0.025
c_4	97.3	0.010

Conclusion

The paper deals with the methods of complex sensitivity analysis of solution given by the Analytic Hierarchy Process. These methods include evaluation of sensitivity of a local ranking of hierarchy elements as to changes in an expert pairwise comparison judgments and evaluation of sensitivity of a global ranking of decision alternatives as to changes of weights of hierarchy elements.

Formulas for calculation of stability intervals of expert pairwise comparison judgments as to change of a local ranking are obtained. Within these intervals change of the expert judgments does not lead to change of the best decision alternative or an overall ranking of alternatives. The obtained formulas for the stability intervals calculation may be used when the Row Geometric Mean Method is applied to find local weights. The method of sensitivity analysis of multiple-criteria problem solution using the AHP is also considered. This method results in stability intervals of a global ranking of decision alternatives in terms of changing of hierarchy elements weights.

The stability intervals allow to find so called critical elements of a decision-making problem. Critical expert pairwise comparison judgments can be found that are sensitive to changes of a local ranking of decision alternatives. Also critical hierarchy elements, i.e. decision criteria, decision goals etc. can be determined – elements that are characterized by the least changes of their weights necessary for changes of a global ranking of decision alternatives.

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