SENSITIVITY ANALYSIS OF A DECISION-MAKING PROBLEM USING THE ANALYTIC HIERARCHY PROCESS

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Abstract: The paper deals with the methodology of complex sensitivity analysis of solution given by one of the popular multiple-criteria decision-making methods, namely the Analytic Hierarchy Process. This methodology includes evaluation of sensitivity of hierarchy elements local ranking to changes in an expert pairwise comparison judgments and evaluation of sensitivity of global ranking of decision alternatives to changes in weights of hierarchy elements. The sensitivity analysis is illustrated on a problem of evaluation of renewable energy technologies for an eco-house in Ukraine.

Keywords: the analytic hierarchy process, uncertainty of expert pairwise comparison judgments, sensitivity analysis, stability intervals, critical expert pairwise comparison judgments, critical elements of a hierarchy.

ACM Classification Keywords: H.4.2. INFORMATION SYSTEM APPLICATION: type of system strategy

Introduction

In the paper one of the multiple-criteria decision-making methods – the analytic hierarchy process (AHP) – is investigated. The AHP results in relative weights or priorities of decision alternatives, is based on a hierarchical model of decision factors, criteria, goals and uses expert judgments of pairwise comparison of elements of a hierarchy as initial information [1, 2]. This method is applied in many areas, such as economy, industry, social sphere, ecology, politics, military science while solving such problems as: choice and evaluation of decision alternatives and decision factors, resource allocation, analysis of benefits-costs-opportunities-risks, forecasting, analytical planning, construction and evaluation of scenarios of development and other [1 - 5].

Expert pairwise comparison judgments contain uncertainty. Therefore the question of reliability of results given by the AHP arises. To evaluate reliability of obtained results it is reasonable to find dependency between results of the AHP and inaccuracies of initial data – expert judgments. In practice a sensitivity analysis of solutions obtained by the AHP, is often carried out using graphical methods, which are proposed by T.L. Saaty and implemented in the decision-support system SuperDecisions [7]. These methods are also implemented in decision-support systems Decision Lens [8], MakeltRational [9] and
LogicalDecisions [10]. In the graphical methods a decision-maker or analyst changes a local weight of some element of a hierarchy and observes changes in global weights of decision alternatives.

The AHP is successfully used while solving different decision-making problems [1–5]. In repetitive problems the graphical methods, which are implemented in the decision-support system SuperDecisions [7], are enough to use. However, a more complete, complex sensitivity analysis has to be done while solving such decision problems as evaluation and choosing of scenarios of development and of decision alternatives on a level of big companies, branches of industry and a country as a whole, resource allocation problems and planning complex target-oriented programs, and also when making decisions concerning innovation development. While solving these problems a complex sensitivity analysis has to be integrated in each stage of decision-making, included in continuous cyclic process of problem solution.

One of the approaches to a complex sensitivity analysis in the AHP is to investigate changes of calculated global ranking of decision alternatives while varying weights of hierarchy elements and changing a hierarchical model structure [11].

Change of global ranking of decision alternatives when adding or removing an alternative, so called rank reversal, was studied in [1, 12, 13]. In these papers it was shown that rank reversal may occur in different aggregation rules of the AHP, namely in the distributive, ideal, multiplicative, max-min rules and in the rule of group consideration of binary preferences of the alternatives. Probabilities of appearance of several types of rank reversals in the aggregation rules were estimated [12, 13]. Thus, the AHP is sensitive to changes of a hierarchical model structure.

In this paper a complex methodology of sensitivity analysis of results obtained by the AHP is proposed. This methodology includes:

- evaluation of sensitivity of a local ranking of decision alternatives to changes in expert pairwise comparison judgments,
- evaluation of sensitivity of a global ranking of decision alternatives to changes in weights of hierarchy elements.

As a result, stability intervals are defined which allow to find so called critical elements of the decision-making problem. These are critical expert pairwise comparison judgments that are sensitive to changes of a local ranking of decision alternatives, and critical hierarchy elements (decision criteria, goals) – elements that are characterized by the least changes of their weights that lead to changes of a global ranking of decision alternatives.
Problem statement

Let \( H \) be an analytic hierarchy with \( p + 1 \) levels. Level \( L_0 \) of the hierarchy has one element — the main goal of decision-making, the last level \( L_p \) contains decision alternatives. Hierarchy levels, that are between \( L_0 \) and \( L_p \), contain possible factors (criteria, goals) that influence the decision. Denote number of elements on a \( L_k \)-th level as \( N_{L_k} \), \( L_k \in [L_0; L_p] \).

\( \hat{A}_{r}^{L_k L_{k-1}} \) is a pairwise comparison matrix (PCM) of elements of \( L_k \)-th level in terms of \( r \)-th element of \( L_{k-1} \)-th level, \( r \in \{1; N_{L_{k-1}}\} \), constructed on the basis of expert judgments.

\( \hat{w}_{lr}^{L_k L_{k-1}} \) is a local weight of \( l \)-th element of \( L_k \)-th level in terms of \( r \)-th element of \( L_{k-1} \)-th level, \( l \in \{1; N_{L_k}\}, r \in \{1; N_{L_{k-1}}\} \). Weight vector \( \hat{w}_{l}^{L_k L_{k-1}} = \left\{ \hat{w}_{lr}^{L_k L_{k-1}} \mid l \in \{1; N_{L_k}\} \right\} \) calculates on the basis of the PCM \( \hat{A}_{r}^{L_k L_{k-1}} \) using the eigenvector method, the row geometric mean method and others [1, 2].

\( \hat{w}_{l}^{L_k} \) is a global weight of \( l \)-th element of \( L_k \)-th level, \( l \in \{1; N_{L_k}\} \). In the analytic hierarchy process vector of global weights can be calculated using the distributive or multiplicative aggregation rules [1, 2].

Vector of global weights of decision alternatives \( \hat{w}_{i}^{L_p} = \left\{ \hat{w}_{il}^{L_p} \mid i \in \{1; N_{L_p}\} \right\} \) is a result of the analytic hierarchy process.

It is necessary to provide a complex sensitivity analysis of rankings obtained using the AHP to inaccuracy and subjectivity of expert judgments:

- to evaluate sensitivity of a local ranking of decision alternatives to changes in expert pairwise comparison judgments (elements of a PCM \( \hat{A}_{r}^{L_k L_{k-1}} \));
- to evaluate sensitivity of a global ranking of decision alternatives to changes in weights of hierarchy elements;
- to find critical and stable expert pairwise comparison judgments;
- to find critical and stable elements of \( L_k \)-th hierarchy level, \( L_k \in [L_1; L_{p-1}] \).

The problem solving. Sensitivity analysis of a local ranking of decision alternatives when changing expert pairwise comparison judgments

Let us consider calculation of local weights of hierarchy elements, for example, decision alternatives \( a_1, a_2, ..., a_n \) in terms of their common feature (an element of a parent hierarchy level). Suppose \( D = \{ (d_{ij}) \mid i, j = 1, ..., n \} \) is a PCM constructed on the basis of expert pairwise comparison judgments.
Using the Row Geometric Mean Method (RGMM), nonnormalized local weights \( v_1, v_2, \ldots, v_n \) of decision alternatives are calculated as follows:

\[
v_i = \left( \prod_{i=1}^{n} d_{ij} \right)^{1/n}, \quad i = 1, \ldots, n.
\]

We are interested in how much ranking of decision alternatives, built on the basis of calculated local weights is insensitive to changes of expert judgments (PCM elements). Let us investigate two cases:

1. whether the best alternative remains unchanged,
2. whether an overall ranking of alternatives remains unchanged.

**A stability interval of expert pairwise comparison judgments** concerning change of ranking of decision alternatives is an interval within the bounds of which an expert judgment may be changed so that a local ranking of alternatives remains unchanged.

Denote \([d_{ij}, \overline{d_{ij}}]\) a stability interval for an expert judgment \(d_{ij}\).

Without loss of generality suppose that decision alternatives are renumbered in order of importance decreasing, that is the ranking of alternatives is

\[
a_i \succ a_2 \succ \ldots \succ a_n,
\]

where \(a_1\) and \(a_n\) are the best (the most important) and the worst decision alternatives, respectively.

In terms of weights (2) means that \(v_i > v_j\) for \(i < j\).

Let us find for each expert judgment a stability interval concerning change of alternatives ranking, when the RGMM is used for weights calculation.

**A case when the best alternative remains unchanged.** At first consider a case when change of a PCM element does not lead to change of the best decision alternative \(a_i\).

Suppose the PCM element \(d_{ij}, j \neq 1\) is changed within the bounds of interval \([d_{ij}, \overline{d_{ij}}]\). Then in accordance with the RGMM (1), weights of decision alternatives \(a_i\) and \(a_j\) are changed. Denote these new weights \(v'_i = [v_i, \overline{v}_i]\) and \(v'_j = [v_j, \overline{v}_j]\), where
\[ v_i = \left( \frac{d_{ij}}{d_{ij}} \right)^{1/n} \cdot v_i \quad \text{and} \quad \overline{v}_i = \left( \frac{d_{ij}}{d_{ij}} \right)^{1/n} \cdot \overline{v}_i, \]  
\[ v_j = \left( \frac{d_{ij}}{d_{ij}} \right)^{1/n} \cdot v_j \quad \text{and} \quad \overline{v}_j = \left( \frac{d_{ij}}{d_{ij}} \right)^{1/n} \cdot \overline{v}_j. \]

We want to find an interval \([d_{ij}, \overline{d}_{ij}]\), such that the best decision alternative does not change, i.e., \( v'_i > v'_j, \quad j \neq 1 \) and \( v'_i > v'_k, \quad k \neq j \neq 1 \). This is equivalent to implementation of the following two conditions:

\[ v_i > \overline{v}_j, \quad v_i > v_k, \]  
(5)

where \( k \neq j \neq 1 \).

Substitute expressions (3) and (4) in (5) and find the following constraints for the left bound of a stability interval:

\[ \frac{d_{ij}}{d_{ij}} > d_{ij} \cdot \left( \frac{v_j}{v_i} \right)^{n/2} \quad \text{and} \quad \frac{d_{ij}}{d_{ij}} > d_{ij} \cdot \left( \frac{v_k}{v_i} \right)^n. \]  
(6)

or \( d_{ij} > d_{ij} \left( \prod_{i=1}^{n} d_{ij} \right)^{1/2} \) and \( d_{ij} > d_{ij} \cdot \left( \prod_{i=1}^{n} d_{ij} \right) \), where \( k \neq j \neq 1 \).

There are no constraints on the right bound of a stability interval, so let us assign it the maximum permissible value, namely, the largest value in the Saaty scale used by an expert when making an assessment: \( \overline{d}_{ij} = 9 \). Comparing right parts of inequality (6) one can formulate the following statement for a stability interval calculation.

**Statement 1:** A stability interval \([d_{ij}, \overline{d}_{ij}]\) for an expert judgment \( d_{ij}, \quad j \neq 1 \), such that the best decision alternative \( a_i \) remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:
\[ d_{ij} > d_{ij} \cdot \left( \frac{v_j}{v_i} \right)^{n/2}, \text{ if } v_i \cdot v_j \geq (v_k)^2, \ k \neq j \neq 1, \]
\[ d_{ij} > d_{ij} \cdot \left( \frac{v_k}{v_i} \right)^n, \text{ if } v_i \cdot v_j < (v_k)^2, \ k \neq j \neq 1, \]
\[ \overline{d_{ij}} = 9. \]

In practice the \( d_{ij} \) value is the nearest value of the Saaty scale that satisfies inequalities of the Statement 1.

Consider a case when any PCM element \( d_{kj}, \ k \neq j \neq 1 \) is changed. It is necessary to find a stability interval \([d_{kj}, \overline{d_{kj}}]\) for this element. According to the RGMM (1), change of \( d_{kj} \) leads to change of weights of decision alternatives \( a_k \) and \( a_j \). Denote these new weights \( v'_k = [\overline{v}_k, \underline{v}_k], \ v'_j = [\underline{v}_j, \overline{v}_j] \) and calculate their left and right bounds using the RGMM:

\[ \underline{v}_k = \left( \frac{d_{kj}}{d_{kj}} \right)^{1/n} \cdot v_k \text{ and } \overline{v}_k = \left( \frac{d_{kj}}{d_{kj}} \right)^{1/n} \cdot \underline{v}_k, \quad (7) \]
\[ \underline{v}_j = \left( \frac{d_{kj}}{d_{kj}} \right)^{1/n} \cdot v_j \text{ and } \overline{v}_j = \left( \frac{d_{kj}}{d_{kj}} \right)^{1/n} \cdot v_j = \left( \frac{d_{kj}}{d_{kj}} \right)^{1/n} \cdot \underline{v}_j. \quad (8) \]

The best decision alternative does not change if inequalities \( v'_k > v'_j, \ j \neq 1 \) and \( v'_k > v'_j, \ k \neq j \neq 1 \) are satisfied. This is equivalent to implementation of the following conditions:

\[ v_i > \overline{v}_j \text{ and } v_i > \overline{v}_k, \quad (9) \]

where \( k \neq j \neq 1 \).

Substitute expressions (7) and (8) in (9) and find the following constraints for the left and right bounds of a stability interval:
\[ d_{kj} > d_{kj} \cdot \left( \frac{v_j}{v_1} \right)^n \quad \text{and} \quad d_{kj} < d_{kj} \cdot \left( \frac{v_k}{v_1} \right)^n \] (10)

or
\[ d_{kj} > \prod_{l=1}^n d_{jl} / \prod_{l=1}^n d_{kl} \quad \text{and} \quad d_{kj} < \prod_{l=1}^n d_{jl} / \prod_{l=1}^n d_{kl} , \text{where} \ k \neq j \neq 1 . \]

**Statement 2:** A stability interval \([d_{kj}, \overline{d}_{kj}]\) for an expert judgment \(d_{kj}, k \neq j \neq 1\), such that the best decision alternative \(a_1\) remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:

\[ d_{kj} > d_{kj} \cdot \left( \frac{v_j}{v_1} \right)^n \quad \overline{d}_{kj} < d_{kj} \cdot \left( \frac{v_k}{v_1} \right)^n \]

It should be noted that PCM elements take values from the Saaty scale (namely, values from the set \(\{1/9, \ldots, 9\}\)). Therefore in practice values \(d_{kj}\) and \(\overline{d}_{kj}\) are the nearest values of this scale that satisfy the corresponding inequalities of the Statement 2.

**A case when an overall ranking of alternatives remains unchanged.** Now consider a case when change of a PCM element leads to steady overall ranking (2) of decision alternatives. Similarly to the previous case, it is necessary to analyze separately change of an element \(d_{1j}, j \neq 1\) and an element \(d_{kj}, k \neq j \neq 1\).

Consider change of \(d_{1j}, j \neq 1\) within an interval \([d_{1j}, \overline{d}_{1j}]\). To save the overall ranking (2), it is necessary to impose the following additional constraints besides mentioned above constraints \(v_j' > v_j'\), \(j \neq 1\) and \(v_k' > v_j'\), \(k \neq j \neq 1\):

\[ v_j' > v_k' \text{ when } j < k \] (11)

\[ v_k' > v_j' \text{ when } k < j \] (12)

where \(k \neq j \neq 1\).
Inequalities (11) and (12) are equivalent to the following:

$$v_j > v_k \quad \text{when} \quad j < k$$

$$v_k > \overline{v_j} \quad \text{when} \quad k < j$$

or using (3) and (4):

$$\bar{d}_{ij} < d_{ij} \cdot \left(\frac{v_j}{v_k}\right)^n \quad \text{when} \quad j < k$$  \tag{13}$$

$$\overline{d}_{ij} > d_{ij} \cdot \left(\frac{v_j}{v_k}\right)^n \quad \text{when} \quad k < j$$  \tag{14}$$

Taking into account conditions $v_j' > v_j, \ j \neq 1$ and $v_k' > v_k, \ k \neq 1$, that lead to (6), we obtain statement for a stability interval calculation.

**Statement 3:** A stability interval $[\underline{d}_{ij}, \overline{d}_{ij}]$ for an expert judgment $d_{ij}, \ j \neq 1$, such that the overall ranking $a_1 > a_2 > ... > a_n$ of decision alternatives remains unchanged, when the RGMM is used for weights calculation, satisfies the conditions:

3.1. $\underline{d}_{ij} > d_{ij} \cdot \left(\frac{v_j}{v_i}\right)^{n/2}, \ \text{if} \ v_i \cdot v_j \geq (v_k)^2, \ j \neq 1, \ j < k$

$$d_{ij} > d_{ij} \cdot \left(\frac{v_j}{v_i}\right)^n, \ \text{if} \ v_i \cdot v_j < (v_k)^2, \ j \neq 1, \ j < k$$

$$\bar{d}_{ij} < d_{ij} \cdot \left(\frac{v_j}{v_k}\right)^n, \ \text{if} \ j \neq 1, \ j < k.$$  

3.2. $\underline{d}_{ij} > d_{ij} \cdot \left(\frac{v_j}{v_k}\right)^n, \ \text{if} \ v_j \cdot v_j \geq (v_k)^2, \ j \neq 1, \ j > k$

$$d_{ij} > d_{ij} \cdot \left(\frac{v_k}{v_i}\right)^n, \ \text{if} \ v_i \cdot v_j < (v_k)^2, \ j \neq 1, \ j > k$$

$$\bar{d}_{ij} = 9, \ \text{if} \ j \neq 1, \ j > k.$$
Consider change of an element $d_{kj}, k \neq j \neq 1$ within an interval $[d_{kj}, \overline{d_{kj}}]$. Similarly to previous case, to save the overall ranking (2), additional conditions (11) and (12) are added, which in this case take a form:

$$v_j > \overline{v}_k \text{ when } j < k$$
$$v_k > \overline{v}_j \text{ when } k < j$$

Using (7) and (8) we get:

$$\overline{d_{kj}} < d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n/2} \text{ when } j < k$$
$$d_{kj} > d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n/2} \text{ when } k < j$$

Taking into account $v'_j > v'_k, j \neq 1$ and $v'_j > v'_k, k \neq j \neq 1$, which in this case lead to constraints (10), we obtain statement for calculation of a stability interval.

**Statement 4:** A stability interval $[\underline{d_{kj}}, \overline{d_{kj}}]$ for an expert judgment $d_{kj}, k \neq j \neq 1$, such that the overall ranking $a_1 > a_2 > ... > a_n$ of decision alternatives remains unchanged, when the RGMM is used for weights calculation, satisfies the inequalities:

4.1. $d_{kj} > d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n/2}$ if $j \neq 1, j < k$

$$\overline{d_{kj}} < d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n/2} \text{ if } (v_j)^2 \geq v_j v_k \text{, } j \neq 1, j < k$$

$$\overline{d_{kj}} < d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n} \text{ if } (v_j)^2 < v_j v_k \text{, } j \neq 1, j < k$$

4.2. $d_{kj} > d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n/2}$ if $j \neq k, k \neq 1$

$$d_{kj} > d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n} \text{ if } (v_j)^2 \geq v_j v_k \text{, } j > k, k \neq 1$$

$$d_{kj} > d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n} \text{ if } (v_j)^2 < v_j v_k \text{, } j > k, k \neq 1$$

$$\overline{d_{kj}} < d_{kj} \cdot \left(\frac{v_j}{v_k}\right)^{n} \text{ if } j > k, k \neq 1.$$
As was mentioned above, in practice the bounds of stability intervals \([d_{ij}, d_{ij}^{*}]\) and \([d_{ij}^{*}, d_{ij}^{**}]\) in the Statements 3 and 4 are the nearest values of the Saaty scale that satisfy the corresponding inequalities of the statements.

**Sensitivity analysis of a global ranking of decision alternatives when changing weights of elements of an hierarchy**

Let us consider a multiple-criteria problem of calculation of global weights of decision alternatives on the basis on a hierarchy of criteria. A method of sensitivity analysis described in this section is a generalization of the method proposed in [6]. Without loss of generality suppose that decision alternatives are renumbered such that

\[
\hat{w}_i^{L_p} \geq \hat{w}_j^{L_p}, \quad i \in [1; N_{L_p}], \quad j \in [1; N_{L_p} \] \text{ when } i < j.
\]

Denote \(\Delta_{i,j; L}, \quad i \in [1; N_{L_p}], \quad j \in [1; N_{L_p}], \quad l \in [1; N_{L_k}], \quad L_k \in [L_1; L_{p-1}]\) value of an absolute change of weight \(\hat{w}_i^{L_k}\) that leads to change of global ranking between \(i\)-th and \(j\)-th elements of \(L_p\)-th level (\(i\)-th and \(j\)-th decision alternatives). That is, a new weight of \(l\)-th element of \(L_k\)-th level equals

\[
\hat{w}_i^{L_k} = \hat{w}_i^{L_k} - \Delta_{i,j; L}, \quad \hat{w}_i^{L_k} > 0, \quad \text{and} \quad \hat{w}_i^{L_p} < \hat{w}_j^{L_p}\]

holds when \(i < j\), where \(\hat{w}_i^{L_p}\) is a new global weight of \(i\)-th element of \(L_p\)-th level.

Denote \(\delta_{i,j; L, k}, \quad i \in [1; N_{L_p}], \quad j \in [1; N_{L_p}], \quad l \in [1; N_{L_k}], \quad L_k \in [L_1; L_{p-1}]\) value of a relative change of weight \(\hat{w}_i^{L_k}\) that leads to change of global ranking between \(i\)-th and \(j\)-th elements of \(L_p\)-th level (\(i\)-th and \(j\)-th decision alternatives). That is, a new weight equals \(\hat{w}_i^{L_k} = \hat{w}_i^{L_k} - \frac{\delta_{i,j; L, k} \hat{w}_i^{L_k}}{100}, \quad \hat{w}_i^{L_k} > 0\). The values of absolute and relative changes of weight \(\hat{w}_i^{L_k}\) of \(l\)-th element in \(L_k\)-th level are in the following relation: \(\delta_{i,j; L, k} = \frac{\Delta_{i,j; L, k}}{\hat{w}_i^{L_k}} 100\%\).

\(l\)-th element of \(L_k\)-th level is **stable** if any permissible changes of its weight do not lead to changes of global rank of any decision alternative.
Degree of criticality $C_{i}^{L_k}$ of $l$-th element of $L_k$-th level is a value of the least relative change of its weight $\hat{w}_i^{L_k}$ that leads to change of global ranking of decision alternatives:

$$C_{i}^{L_k} = \min_{i,j \in [1;N_L] \land i < j} \left\{ | \delta_{i,j}^{L_k} | \right\}.$$

Sensitivity $S_{i}^{L_k}$ of $l$-th element of $L_k$-th level is a reciprocal value to the degree of criticality of this element: $S_{i}^{L_k} = \frac{1}{C_{i}^{L_k}}$. $S_{i}^{L_k}$ is assigned a zero value if $l$-th element of $L_k$-th level is stable.

Less values of the degree of criticality $C_{i}^{L_k}$ mean that it is easier to change a ranking of decision alternatives. So less values of the degree of criticality $C_{i}^{L_k}$ indicate that less change of weight $\hat{w}_i^{L_k}$ is sufficient for a change of ranking of decision alternatives. Therefore “the easier” change of ranking of decision alternatives results in larger value of sensitivity $S_{i}^{L_k}$ of $l$-th element of $L_k$-th level.

Critical element of $L_k$-th level is an element of $L_k$-th level which has the least value $| \delta_{i,j}^{L_k} |$, that is $l_{crit}$-th element of $L_k$-th level is critical if $| \delta_{i,j}^{L_k} | = \min_{l \in [1;N_L], i \in [1;N_{L_p}], j \in [1;N_{L_p}]} \left\{ | \delta_{i,j}^{L_k} | \right\}.$

Values of relative change $\delta_{i,j}^{L_k}$, when the distributive and multiplicative aggregation rules are used for calculation of global weights of hierarchy elements can be found using the following statements 7 and 8.

**Statement 7**: A value $\delta_{i,j}^{L_k}$ of relative change of weight $\hat{w}_i^{L_k}$ that is necessary for a change of global ranking between $i$-th and $j$-th elements of $L_p$-th level, $i \in [1;N_{L_p}]$, $j \in [1;N_{L_p}]$, $L \in [L_1;L_{p-1}]$, when $i < j$ and the distributive aggregation rule is used for calculation of global weights satisfies the inequality [11]:

$$\delta_{i,j}^{L_k} < \delta_{i,j}^{L_p \text{porg}}, \text{ if } \hat{w}_j^{L_p L_k} > \hat{w}_i^{L_p L_k},$$

$$\delta_{i,j}^{L_k} > \delta_{i,j}^{L_p \text{porg}}, \text{ if } \hat{w}_j^{L_p L_k} < \hat{w}_i^{L_p L_k},$$

where the threshold value $\delta_{i,j}^{L_p \text{porg}}$ of $\delta_{i,j}^{L_k}$ is calculated as follows:
\[ \delta_{i,j,l} = \Delta_{i,j,l} \frac{100}{\hat{w}_i^L} (\%) , \tag{15} \]
\[ \Delta_{i,j,l} = \frac{\hat{w}_j^L}{\hat{w}_j^L} - \frac{\hat{w}_j^L}{\hat{w}_j^L} \tag{16} \]

under conditions:

1) \( \hat{w}_i^L \geq \hat{w}_j^L \) when \( i < j \);

2) \( \hat{w}_i^L > \Delta_{i,j,l} \) (that is equivalent to \( \delta_{i,j,l} < 100\%) \).

**Corollary:** \( l \)-th element of \( L_k \)-th level, \( l \in [1; N_{L_k}] \), is stable if \( \hat{w}_i^L \leq \Delta_{i,j,l} \) holds when \( i < j \) for all \( i \in [1; N_{L_k}] \), \( j \in [1; N_{L_p}] \), where threshold value \( \Delta_{i,j,l} \) of absolute change \( \Delta_{i,j,l} \) of weight \( \hat{w}_i^L \) of \( l \)-th element in \( L_k \)-th level is calculated using the formula (16).

**Corollary:** If \( \hat{w}_{jl}^{L_k} \leq \hat{w}_{il}^{L_k} \) holds for all \( l \in [1; N_{L_k}] \), that is \( j \)-th element of \( L_p \)-th level does not dominate \( i \)-th element of \( L_p \)-th level in terms of all elements of \( L_k \)-th level, \( L_k \in [L_1; L_{p-1}] \), then any changes of weights of \( L_k \)-th level elements do not lead to changes of global ranking between these elements of \( L_p \)-th level.

**Statement 8:** A value \( \delta_{i,j,l} \) of relative change of weight \( \hat{w}_j^L \) that is necessary for a change of global ranking between \( i \)-th and \( j \)-th elements of \( L_p \)-th level, \( i, j = 1, N_{L_p} \), \( i = 1, N_{L_j} \), \( L_k = L_1, L_{p-1} \), when \( i < j \) and the multiplicative aggregation rule is used for calculation of global weights satisfies the inequality [1]:

\[ \delta_{i,j,l} < \delta_{i,j,l} \] \( \delta_{i,j,l} \) \( \delta_{i,j,l} \)
\[ \delta_{i,j,l} > \delta_{i,j,l} \] \( \delta_{i,j,l} \) \( \delta_{i,j,l} \)

where the threshold value \( \delta_{i,j,l} \) of \( \delta_{i,j,l} \) is calculated as follows:

\[ \delta_{i,j,l} = \Delta_{i,j,l} \frac{100}{\hat{w}_i^L} (\%) , \]
where
\[ \Delta_{ij}^{L_p L_{p-1}} = \ln \left( \frac{w_i^{L_p}}{w_j^{L_p}} \right) \]

under conditions:

1) \( \hat{w}_i^{L_p} \geq \hat{w}_j^{L_p} \) when \( i < j \);

2) \( \hat{w}_i^{L_p} > \Delta_{i,j}^{L_p \text{parog}} \) (that is equivalent to \( \delta_{i,j}^{L_p \text{parog}} < 100\% \)).

**A case of change of local weights of decision alternatives.** In this subsection we will find an interval of changes of a local weight \( \hat{w}_i^{L_p L_{p-1}} \) of \( i \)-th decision alternative in terms of \( r \)-th element of a parent hierarchy level, that do not lead to changes of global ranking between \( i \)-th and \( j \)-th alternatives. This allows to define how critical (sensitive) every decision alternative is in terms of selected element of a parent hierarchy level, i.e. to find a value of the least change of a local weight of decision alternative that results in change of global ranking of decision alternatives.

Denote \( \delta_{i,j}^{a} \), \( i \in [1; N_{L_p}] \), \( j \in [1; N_{L_p}] \), \( r \in [1; N_{L_{p-1}}] \) value of a relative change of local weight \( \hat{w}_i^{L_p L_{p-1}} \) of \( i \)-th element of \( L_p \)-th level (\( i \)-th decision alternative) in terms of \( r \)-th element of a parent \( L_{p-1} \)-th level that leads to change of global ranking between \( i \)-th and \( j \)-th elements of \( L_p \)-th level (\( i \)-th and \( j \)-th decision alternatives). That is, a new weight of \( i \)-th element of \( L_p \)-th level in terms of \( r \)-th element of \( L_{p-1} \)-th level equals \( \hat{w}_i^{L_p L_{p-1}} = w_i^{L_p L_{p-1}} + \frac{\delta_{i,j}^{a} w_i^{L_p L_{p-1}}}{100} \), \( \hat{w}_i^{L_p L_{p-1}} > 0 \), and \( \hat{w}_i^{L_p} < \hat{w}_j^{L_p} \) holds when \( i < j \), where \( \hat{w}_i^{L_p} \) is a new global weight of \( i \)-th element of \( L_p \)-th level.

\( i \)-th element of \( L_p \)-th level is **stable** in terms of \( r \)-th element of a parent \( L_{p-1} \)-th level if any permissible changes of a local weight \( \hat{w}_i^{L_p L_{p-1}} \) of this element do not lead to changes of global rank of any decision alternative.
Degree of criticality $C_{ir}^a$ of $i$-th element of $L_p$-th level ($i$-th decision alternative) in terms of $r$-th element of a parent $L_{p-1}$-th level is the minimum of values $|\delta_{i,j,r}^a|$ that lead to change of global rank of this decision alternative: $C_{ir}^a = \min_{j \in [1; N_{L_{p}}], \ r \in [1; N_{L_{p-1}}]} \left\{ |\delta_{i,j,r}^a| \right\}, \ i \in [1; N_{L_p}], \ r \in [1; N_{L_{p-1}}]$.

Sensitivity $S_{ir}^a$ of $i$-th element of $L_p$-th level ($i$-th decision alternative) in terms of $r$-th element of a parent $L_{p-1}$-th level is a reciprocal value to the degree of criticality of this element: $S_{ir}^a = \frac{1}{C_{ir}^a}$, $i \in [1; N_{L_p}], \ r \in [1; N_{L_{p-1}}]$. $S_{ir}^a$ is assigned a zero value if $i$-th element of $L_p$-th level is stable in terms of $r$-th element of $L_{p-1}$-th level.

Critical element of $L_p$-th level (critical alternative) is an element of $L_p$-th level which has the least degree of criticality, that is $i_{cr}$-th element of $L_p$-th level is critical in terms of $r$-th element of a parent $L_{p-1}$-th level, if $C_{i_{cr}nr}^a = \min_{i \in [1; N_{L_p}]} \left\{ \min_{r \in [1; N_{L_{p-1}}]} \left\{ C_{ir}^a \right\} \right\}$.

A value of relative change $\delta_{i,j,r}^a$ of local weight $\hat{w}_{ir}^{L_pL_{p-1}}$ of $i$-th alternative in terms of $r$-th element of $L_{p-1}$-th level can be found using the following statement 9.

**Statement 9:** A value $\delta_{i,j,r}^a$ of relative change of a local weight $\hat{w}_{ir}^{L_pL_{p-1}}$ that is necessary for a change of global ranking between $i$-th and $j$-th elements of $L_p$-th level ($i$-th and $j$-th decision alternatives), $i \in [1; N_{L_p}], \ j \in [1; N_{L_p}], \ r \in [1; N_{L_{p-1}}]$, when the multiplicative aggregation rule is used for calculation of global weights satisfies the inequality:

$$\delta_{i,j,r}^a > \delta_{i,j,r}^{a_{porog}}, \text{ if } i < j,$$

$$\delta_{i,j,r}^a < \delta_{i,j,r}^{a_{porog}}, \text{ if } i > j,$$

where the threshold value $\delta_{i,j,r}^{a_{porog}}$ of $\delta_{i,j,r}^a$ is calculated as follows:

$$\delta_{i,j,r}^{a_{porog}} = \left( 1 - \frac{\hat{w}_{j}^{L_pL_{p-1}}}{\hat{w}_{i}^{L_pL_{p-1}}} \right) \cdot 100 \% \quad (17)$$

under conditions:

1) $\hat{w}_{i}^{L_p} \geq \hat{w}_{j}^{L_p}$ when $i < j$;
2) \( \delta_{i,j,r}^{a_{p_{\text{org}}}} < 100\% \).

**Corollary:** \( i \)-th element of \( L_p \)-th level is stable in terms of \( r \)-th element of \( L_{p-1} \)-th level if \( \delta_{i,j,r}^{a_{p_{\text{org}}}} > 100\% \) holds for all \( i \in [1; N_{L_p}], \ j \in [1; N_{L_{p-1}}] \), where threshold value \( \delta_{i,j,r}^{a_{p_{\text{org}}}} \) is calculated using the formula (17).

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**Sensitivity analysis of a global ranking of decision alternatives in a problem of evaluation of renewable energy technologies for an eco-house in Ukraine**

Let us consider a multiple-criteria decision-making problem of evaluation of renewable energy technologies for an eco-house and solve it using the analytic hierarchy process (AHP). Several technologies of renewable energy for an eco-house are selected for investigation by a decision-maker:

- geothermal thermal pump (\( a_1 \));
- biofuel production (\( a_2 \));
- solar plant (\( a_3 \)).

To evaluate these technologies (alternatives) a decision-maker develops the following four criteria:

- accessibility (\( c_1 \));
- economic efficiency during the use of a technology (\( c_2 \));
- initial costs (\( c_3 \));
- costs during the use of a technology (\( c_4 \)).

Criteria weights, local weights of decision alternatives in terms of each criterion and global weights of the alternatives using the distributive and multiplicative aggregation rules of the AHP are shown in the Table 1.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Global weights</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Distributive aggregation rule</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>0.090</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0.455</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0.455</td>
</tr>
</tbody>
</table>
Sensitivity analysis was done separately for the global rankings of the decision alternatives using the distributive and multiplicative aggregation rules.

**A case of the distributive aggregation rule.** The global ranking of decision alternatives in this case is: 
\( a_1 > a_2 > a_3 \), and alternative \( a_1 \) is the optimal one. The criterion \( c_2 \) is the most important, its weight equals 0.509. Let us calculate a threshold value of relative change of this criterion weight that leads to changing of the global ranking, for example, between alternatives \( a_1 \) and \( a_2 \). This value is calculated as follows:

\[
\delta_{1,2,2}^{\text{porog}} = \frac{0.335 - 0.372}{0.279 - 0.649} \cdot \frac{1}{0.509} = 0.198.
\]

A positive value of \( \delta_{1,2,2}^{\text{porog}} \) means that the criterion \( c_2 \) weight has to be decreased to change the ranking between alternatives \( a_1 \) and \( a_2 \). The relative value of this decreasing equals 19.8%.

\( \delta_{1,2,2} > \delta_{1,2,2}^{\text{porog}} = 0.198 \), since \( w_{22} < w_{12} \). Thus, an interval of relative change of the criterion \( c_2 \) weight that leads to changing of the global ranking between \( a_1 \) and \( a_2 \) is \( \delta_{1,2,2} \in (0.198; 1.000) \).

For example, suppose that decision-maker preferences are changed and the criterion \( c_2 \) weight is decreased up to the value 0.407 (that is on 20%). The criteria weights after renormalization are \( w_1^c = 0.105 \), \( w_2^c = 0.453 \), \( w_3^c = 0.271 \) and \( w_4^c = 0.171 \). Then the global weights of the decision alternatives are: \( w_1^{\text{glob}} = 0.341 \), \( w_2^{\text{glob}} = 0.341 \), \( w_3^{\text{glob}} = 0.318 \), and the alternative \( a_2 \) becomes as important as the \( a_1 \).

Relative changes of all criteria weights that lead to changing of the global ranking between different pairs of alternatives are given in the Table 2.

According to the definition, a critical criterion for changing of optimal alternative defines as the minimum by absolute value in rows of the Table 2, that correspond to the optimal alternative \( a_1 \). This minimum value equals 19.8% and corresponds to the criterion \( c_2 \) and alternatives \( a_1 \) and \( a_2 \). Decreasing of the criterion \( c_2 \) weight on 19.8% leads to changing of the optimal alternative, and \( a_2 \) becomes optimal.

The criterion \( c_2 \) is the most sensitive in this problem, the next less sensitive criteria are \( c_3 \), \( c_4 \) and \( c_1 \) (Table 3).
Table 2: Threshold values $\delta_{i,j}^{\text{porog}}$ (case of the distributive aggregation rule)

<table>
<thead>
<tr>
<th>Pair of alternatives $(i, j)$</th>
<th>$\delta_{i,j}^{\text{porog}}$, %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_1$</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-108.7*</td>
</tr>
<tr>
<td>(1,3)</td>
<td>-230.8</td>
</tr>
<tr>
<td>(2,3)</td>
<td>-</td>
</tr>
</tbody>
</table>

* Negative value of $\delta_{i,j}^{\text{porog}}$ means that the criterion $C_j$ weight has to be increased to change ranking between alternatives $a_i$ and $a_j$.

Table 3: Degrees of criticality $\text{CritVal}$ and sensitivity $\text{SensVal}$ for the criteria (case of the distributive aggregation rule)

<table>
<thead>
<tr>
<th>Criterion</th>
<th>$\text{CritVal}$, %</th>
<th>$\text{SensVal}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>108.7</td>
<td>0.009</td>
</tr>
<tr>
<td>$c_2$</td>
<td>19.8</td>
<td>0.051</td>
</tr>
<tr>
<td>$c_3$</td>
<td>51.8</td>
<td>0.019</td>
</tr>
<tr>
<td>$c_4$</td>
<td>83.2</td>
<td>0.012</td>
</tr>
</tbody>
</table>

A case of the multiplicative aggregation rule. The global weights of the decision alternatives in this case equal $w_1^{\text{glob}} = 0.312$, $w_2^{\text{glob}} = 0.436$, $w_3^{\text{glob}} = 0.252$ (see Table 1). So the global ranking of the alternatives is $a_2 \succ a_1 \succ a_3$, and $a_2$ is the best (optimal) one.

Relative changes of criteria weights that lead to changing of the global ranking are shown in the Table 4. For example, the value $\delta_{1,3}^{\text{porog}}$ of relative change of the criterion $c_2$ weight that leads to changing of the ranking between alternatives $a_1$ and $a_3$ is calculated as follows:

$$
\delta_{1,3}^{\text{porog}} = \frac{\ln(0.312) - \ln(0.252)}{\ln(0.649) - \ln(0.072)} \frac{1}{0.509} = 0.193 \Rightarrow \delta_{1,3}^{\text{porog}} \in (19.3\%; 100\%).
$$
Indeed the relative decrease of the criterion $c_2$ weight, for example, on 20%, results in a new weight $(w_2^{c})' = 0.509 - 0.2 \cdot 0.509 = 0.407$. Then global weights of the alternatives are $w_1^{glob} = 0.277$, $w_2^{glob} = 0.443$, $w_3^{glob} = 0.280$, and alternative $a_3$ becomes more important than alternative $a_2$.

Table 4: Threshold values $\delta_{i,j,\text{parog}}$ (case of the multiplicative aggregation rule)

<table>
<thead>
<tr>
<th>Pair of alternatives $(i,j)$</th>
<th>$\delta_{i,j,\text{parog}}$, %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_1$</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-</td>
</tr>
<tr>
<td>(1,3)</td>
<td>-141.7</td>
</tr>
<tr>
<td>(2,3)</td>
<td>-</td>
</tr>
</tbody>
</table>

A critical criterion for changing of optimal alternative defines as the minimum by absolute values in rows of the Table 4, that correspond to the optimal alternative $a_2$. This minimum value equals 77.8% and corresponds to the criterion $c_2$ and alternatives $a_1$ and $a_2$. Increase of the criterion $c_2$ weight more than 77.8% results in changing of the optimal alternative, and $a_1$ becomes optimal. The criterion $c_2$ is also critical for changing the global ranking between any two considered alternatives: relative change of its weight that equal 19.3%, is enough for changing the global ranking between nonoptimal alternatives $a_1$ and $a_3$.

The criterion $c_2$ is the most sensitive in this problem, the next less sensitive criteria are $c_3$, $c_4$ and $c_1$ (Table 5).

Table 5: Degrees of criticality $\text{CritVal}$ and sensitivity $\text{SensVal}$ for the criteria (case of the multiplicative aggregation rule)

<table>
<thead>
<tr>
<th>Criterion</th>
<th>$\text{CritVal}$, %</th>
<th>$\text{SensVal}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>141.7</td>
<td>0.007</td>
</tr>
<tr>
<td>$c_2$</td>
<td>19.3</td>
<td>0.052</td>
</tr>
<tr>
<td>$c_3$</td>
<td>40.8</td>
<td>0.025</td>
</tr>
<tr>
<td>$c_4$</td>
<td>97.3</td>
<td>0.010</td>
</tr>
</tbody>
</table>
Conclusion

The paper deals with the methods of complex sensitivity analysis of solution given by the Analytic Hierarchy Process. These methods include evaluation of sensitivity of a local ranking of hierarchy elements as to changes in an expert pairwise comparison judgments and evaluation of sensitivity of a global ranking of decision alternatives as to changes of weights of hierarchy elements.

Formulas for calculation of stability intervals of expert pairwise comparison judgments as to change of a local ranking are obtained. Within these intervals change of the expert judgments does not lead to change of the best decision alternative or an overall ranking of alternatives. The obtained formulas for the stability intervals calculation may be used when the Row Geometric Mean Method is applied to find local weights. The method of sensitivity analysis of multiple-criteria problem solution using the AHP is also considered. This method results in stability intervals of a global ranking of decision alternatives in terms of changing of hierarchy elements weights.

The stability intervals allow to find so called critical elements of a decision-making problem. Critical expert pairwise comparison judgments can be found that are sensitive to changes of a local ranking of decision alternatives. Also critical hierarchy elements, i.e. decision criteria, decision goals etc. can be determined – elements that are characterized by the least changes of their weights necessary for changes of a global ranking of decision alternatives.

Bibliography


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