DECISION-MAKING IN GROUPS OF INTERVAL ALTERNATIVES

Gennady Shepelev

Abstract: *Problem of comparing alternatives with numerical quality indicators, which due to uncertainty have interval representations, is considered. It is demonstrated inevitability of presence in similar tasks of irremovable risk of making a wrong decision on preference of alternatives. It is emphasized necessity of the account in dealing with such problems both indicators of preference for analyzed alternative in comparison with other objects in their group and risk indicators of making wrong decision. Advantages and disadvantages of available methods of interval alternatives comparing are analyzed. Namely, methods of comparison are considered based on principles of dominance by probability, evaluation methods based on average indicators, new method of collective risk estimating, and methods of "mean – risk" approach. The arguments are given that methods of dominance by probability and methods based on average indicators as tools, which do not allow estimate risk of making a wrong decision explicitly, can play only an auxiliary role in decision-making. Advantages of the method of collective risk estimating consist in account of integral risk in concrete group of compared interval alternatives. Disadvantages are consequence of the fact that the method compares only relative preferences of alternative and permits take into account acceptability of separate alternative only after additional constructions. In the framework of "mean – risk" approach properties of such convenient for practice risk indicator as average semideviation were studied for some simple but important for applications distribution functions of preference chances. It is shown that in some cases instead of average semideviation indicator using of indicators of risk based on results of methods evaluating of deviation from target marks is more expedient. An advantage of methods of "mean – risk" approach is the possibility of calculating for each alternative pair of basic indicators needed for the evaluation interval alternatives, indicator of acceptability of alternatives and indicator of associated risk. The disadvantages include the fact that both of these indicators are calculated for an alternative as an independent object, which is not associated with others in compared group. Comparison of the alternatives on preference is based on values of these indicators but dependence of the risk on the context is not taken into account. Due to presence of both advantages and disadvantages of the methods is proposed to use a three-step approach to decision-making on the selection of interval alternatives, which is based on consistent using of different methods: A stage of the preliminary analysis of alternatives for the purpose of culling some of their number and reducing the dimension of the problem and size of collective risk by this is suggested to include in the procedure of decision making on the choice of the preferred objects. Method* *of average estimates is suggested to use at this stage. Using of the collective risk estimating techniques with isolating of a small number of alternatives as acceptable by the indicator of preference, as well as having the lowest risk of wrong estimates, is recommended at the following stage and methods of “mean – risk " approach at the final stage.*

***Keywords**: comparing interval alternatives, methods of probabilistic dominance, collective risk estimating, “mean-risk” methods, joint using of the methods.*

***ACM Classification Keywords**: H.1.2 Human information processing. G3 Distribution functions. I.2.3 Uncertainty, “fuzzy”, and probabilistic reasoning.*

Introduction

Let us call by interval alternatives such objects of comparison or evaluation whose quality indicators that are measured in quantitative scales have, due to uncertainty, interval representations. Such representation of numerical indicators arises, for example, in forecasting problems in attempts to predict the future values of point, by their nature, quantities. It is assumed that interval estimate includes all possible, up to the available knowledge, point implementations of forecast quantity. But in the future, when the uncertainty is removed, this quantity will receive certain the only numerical value.

The objective of comparing two or more alternatives is selection of one or more preferred (“best”) in a sense objects. The purpose of evaluation is comparison of quality indicator of interval alternative with some preassigned point benchmark that characterizes acceptable for assessing subject threshold value of the quality indicator. The criteria of comparison or evaluation, although have relations with the indicators of quality by their domains of definition, are not necessarily expressed in terms of the values of the lasts and may have dimensions, which are different from the dimensions of quality indicators. Further we will assume for definiteness that such situations take place when the greater values of the quality indicators are preferable than small values.

Problems of comparing and evaluation of interval alternatives due to their characteristics cannot be exhaustively solved by purely mathematical methods. Indeed in the case comparing of alternatives for general configurations, when the compared intervals have non-zero intersection, in principle one cannot with certainty conclude, which alternative in their entirety will be preferable. Any alternative may be “better” in the future, at the time of “removal” of uncertainty, when the interval estimates are replaced by exact (point) values of quality indicator. Therefore at the time of the comparison can be judged only on the chances that one alternative will be preferable to others. Therefore there is always a risk that in fact namely another alternative but not tested would be better. In problems estimating of the acceptability of separate alternative point benchmark, with which the interval quantity is compared, divides the interval object into two areas. All point implementations that suits to decision-maker (DM) are located in one area, those implementations, which are unacceptable to DM, lie in another area. Again one can only judge about chances of getting the point implementations in the desired area and the risk of them misses in it.

Thus formal methods of comparison and estimation serve here only as a tool of information-analytical support for the decision making process and cannot guarantee choice of truly the best object in the process of comparing or estimating. Thus such problems are problems of the decision-making theory as in the decision-making process have to involve preferences of DMs or experts and take account of their risk tolerance. So adequate to essence of the problem comparison should take into account at least two criteria, measure of preference of alternative in relation to others in their totality and risk measure. One can note that this requirement is not always fulfilled both in practice and in the recommendations on the use of comparison methods. Similarly indicator of the acceptability of alternative and indicator of risk that result of estimating will be in fact wrong must be taken into account during estimating. Human involvement in the decision-making process can determine not only the choice of the tools of describing the uncertainty but also, due to previous experience and knowledge of human being, the choice of the comparison and estimation methods that lead to the result indicators, which are familiar to the expert or DMs. Each of the available methods of comparison and evaluation allows to calculate its measures of alternative preference in relation to the other alternatives in their totality as well as indicators of the acceptability of alternatives during estimating and (but not always) their risk measures. For some configurations, i.e. relative location, of compared alternatives and types of uncertainty describing predictions of different methods are equivalent for others not. In this regard it should be stated that at present there is no approach transcending all others in quality of recommendations obtained on its basis. Each method has its advantages and disadvantages. Joint using of different methods on various successive stages of the decision-making process is probably the best way to combine the power of formal methods and knowledge of experts and DMs. Decision-making for comparing and estimating of interval alternatives remains both science and art, as it was before [Larichev, 1979].

The purpose of this paper is to compare different approaches and methods of evaluation and comparing of interval alternatives and identify their place in the decision-making process of the selection of preferred objects. We start with a matching of interval alternatives comparison methods.

Methods of interval alternatives comparison

We assume that all the alternatives are comparable in preference (i.e. system of alternatives is full). Disjunction containing comparable interval alternatives is not rigorous in this case that is such that only one object can be chosen as preferred. This choice depends now on the chances of preferences of the disjunction members. Similar relations of disjunction members are based on the degree of assurance in the truth of the hypotheses about preference an alternative from their totality, which is tested by DM. Such relations may be called by relations including the risk. Let us assume that from all possibilities of uncertainty description the tools of distribution functions, similar to apparatus that used in the theory of probability, is selected here for the quantification of preference chances for compared interval alternatives or subsets of values contained in them. This apparatus is the most familiar, in our experience, expert practitioners. It is important as expert analysis of practical problems is most productive when it is conducted in the usual for domain experts' language with using terminology understandable to them [Petrovsky, 1996].

Two main types of problems distinguish in the theory of alternatives comparing under uncertainty. These are of unique choice and problems of repeated choice. Each of these problems types requires for the analysis the specific comparison and risk criteria. So value of mathematical expectation of quality indicator as random variable is adequate criterion for problems of repetitive choice. The problems of forecasting deal mainly with situations of unique choice. This requires, in general, rejection of average estimates, or, if they are used, the mandatory accounting as no less important criterion estimations of risk calculated on the basis of certain indicators.

Let's note In addition to the above that the available methods of interval alternatives comparison compare them in different ways. In the methods of stochastic dominance pairwise comparison of alternatives carried out only on the basis of the behavior of the distribution functions defined on intervals-carriers, without taking into account the numerical characteristics of the firsts [Levy, 2006].

In the "mean – risk" approach [Fishburn, 1977] compared alternatives are considered as isolated, not "interactive" objects. Value of preference criterion is calculated for each alternative separately and then, regardless of this indicator, a risk indicator is computed again separately for each compared object. The problem of comparing is solved then as a two-criterion task. Mathematical expectation value is here the criterion of preference both for problems of unique choice and repeated choice. Such indicators as variance, left and right semivariances, left and right mean semideviations and the others act as risk criteria [Baker, 2015]. Let draw attention to the fact that the calculated values of the comparison criteria for these methods do not depend on the number of alternatives to be analyzed in their group.

In the method of collective risk estimating [Shepelev, 2015], when produced direct calculations of preference chances of alternatives in comparison with others in their totality, compared objects are viewed as interconnected community. Because apparatus of distribution functions was selected for quantification of preference chances and associated risks, the problem of comparing can be analyzed in the framework of probability logic approach [Nilsson, 1986]. In accordance with this approach in addition to the truth or falsity of logical statements intermediate logical values are possible. They are interpreted as chances of truth. The use of this approach to interval alternatives comparing allows calculating both the chances of alternative preferences and associated risks. Risk of choice of an alternative in their group as the preferred depends on the relative position of alternatives (configuration of alternatives) and on the number of compared objects. An original interaction of compared objects leads to a collective effect [Shepelev, 2015; Sternin, 2015], which consists in the fact that the properties of objects of interacting components of the system is significantly different from those of relatively independent objects. Therefore risk of making the wrong decision when choosing a preferred object increases with the growing number of compared alternatives. It seems that this approach is more consistent with the essence of problems of unique choice.

Let note that we deliberately leave aside the methods based on the construction of utility functions. These techniques allow us, on the one hand, to find point estimates, equivalent to the compared interval estimates ("deterministic equivalents") [Keeney, 1993], that is undoubtedly attractive for practitioners, but on the other hand requires detailed information about the preferences of the DM, her /his risk appetite and the use of complicated procedure to transform this information into specific utility functions. Allowing to calculate preference indicators of interval alternatives in their compared pair [Shepelyov, 2012], methods of comparing based on the construction of utility take into account the risk of making the wrong decision only indirectly during including the DM's preferences and propensity for risk in the specification of the parameters of the selected class of utility functions.

Let us consider now the above methods of comparing and analyze their possible place in the process of decision-making concerning choosing of a preferred alternative. Let start with the methods of stochastic dominance namely stochastic first-order dominance (or probabilistic dominance, dominance according to "chances" in line with the terminology adopted above) as having simple meaningful sense.

Methods of probability dominance

They say that the interval alternative I_1, where distribution F_1 of the random variable X_1 is given, dominates alternative I_2, where distribution F_2 of the random variable X_2 is given, if for a set of possible point implementations $I_1 \cup I_2$ for any point implementation x chances $F_1(X_1 < x)$ are not more than chances $F_2(X_2 < x)$, and at least for one point implementation they are smaller. In other words the graph of the distribution function F_1 for alternative I_1 lies always below the graph of the distribution function F_2, possibly coinciding with the first in some parts.

Keeping in mind our objectives we will analyze this method in more detail for the two distributions of chances, uniform and triangular ones. The first of them is consistent with the principle of maximum by Gibbs - Jaynes, the second is a visual model of common among practitioners unimodal distributions.

We will define interval objects $[L, R]$ by the left L and right R boundaries, $L < R$. From the four, up to permutations, different configurations of compared alternatives pairs, coinciding intervals; intervals without intersection; configurations of right shift and embedded intervals let focus on discussing the last two configurations, which have the greatest interest for the real-life problems.

In the case of right shift configurations, when $L_2 < L_1 < R_2 < R_1$, for uniform distributions alternative I_1 dominates alternative I_2 by probability. Indeed if by definition $\Delta I_i = R_i - L_i$, $i = 1, 2$ then

$$F_i(x) = \begin{cases} 0, x \le L_i \\ \dfrac{x - L_i}{\Delta I_i}, L_i < x < R_i \\ 1, x \ge R_i \end{cases} \qquad (1)$$

When $\Delta I_1 = \Delta I_2$ line segments, which represent graphs of distribution functions F_i, are parallel, in other cases they are intersected at a point $I_{in} = (L_2\Delta I_1 - L_1\Delta I_2)/(L_2\Delta I_1 - L_1\Delta I_2)$. One can verify that the inequality $L_1 < I_{in} < R_2$ is not met for the right shift configurations, and therefore in area $[L_2, R_1]$ $F_1 \le F_2$. Therefore the first alternative dominates the second by probability[1].

This conclusion is valid for any value of the uncertainty zone $[L_1, R_2]$ for point implementations of interval alternatives. Does it mean that DM should always select as the preferred first alternative due to the dominance of the second by probability? It seems that not because following this requirement means the neglect of the risk of making a wrong decision on the preference. DM can but should not make such a choice. We also call attention to the fact that after choosing one of the alternatives on base of the results of the analyzed approach DM still has not an estimate of the acceptability of the alternative. DM received only estimate that one alternative is preferable to another.

Will demonstrated above dominance by probability to have place for distributions other than the uniform? Answer is negative if distributions are strongly skewed in opposite directions (have "long tails", which are pointing in different directions). It is shown on Figure 1, with using the relations similar to (1), for the functions of triangular distributions with modes $M_1 = 2.1$; $M_2 = 6.9$ for intervals $I_1 = [2, 11]$, $I_2 = [1, 7]$.

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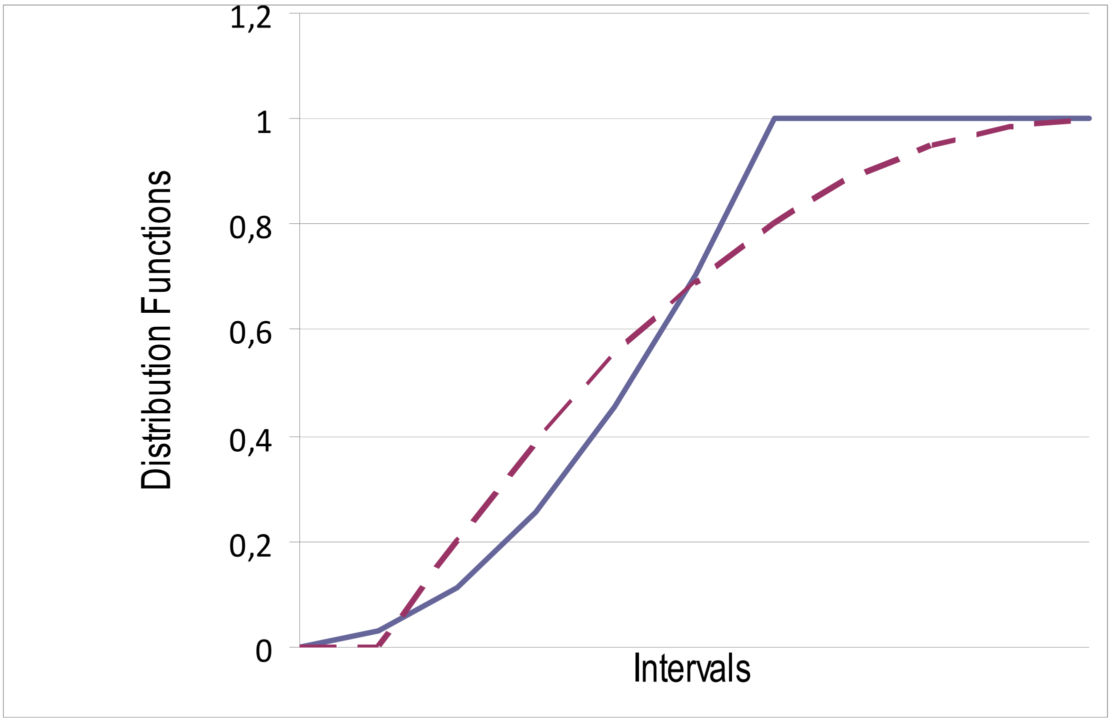
[1] Note that in this case I_1 is more preferable than I_2 also according to frequently used criterion of mathematical expectation.



Legend: - - - for I_1; — for I_2.

Fig. 1. An example of absence of dominance by probability for the right shift case

Turn now to the case of embedded intervals. Let I_2 embedded in I_1. The intersection point I_{in} of distribution functions lies for uniform distribution in the interval $[L_2, R_2]$. And what is more I_2 dominates I_1 by the probability in the area lying to the left of I_{in} and on the contrary to the right of this point.

One can see that $F(I_{in}) = F_1(I_{in}) = F_2(I_{in}) = (L_2 - L_1)/(\Delta I_1 - \Delta I_2)$. Length of the region where I_2 dominates I_1 is more than length of the region where I_1 dominates I_2 if $F(I_{in}) < ½$, that is if $R_1 + L_1 > R_2 + L_2$.

Thus for this configuration has place a certain harmonization of results based on comparing criterion of dominance by probability and mathematical expectation criterion. However this harmonization is expressed less clearly than for the right shift configurations.

In closing the discussion of this approach note that use of the dominance by probability principle to eliminate certain alternatives from their set for decreasing their number to reduce collective risk is problematic. In the framework of this approach there is only pairwise comparison of alternatives, the comparison results do not include risk estimates explicitly, there are no estimates of the acceptability of separate alternative.

Methods of collective risk estimating

In the framework of method of collective risk estimating compared alternatives are considered as an interconnected totality. Comparison is carried out here in general, “as a whole”. Risk of choosing an alternative as preferred in their set depends here not only on configuration of compared alternatives, but also on their amount in the set [Shepelev, 2015]. The presence of the group of mutually influencing alternatives increases the risk of making the wrong decision during choosing a preferred alternative. This is due to a “collective effect” just as it happens, for example, in condensed matter physics when properties of condensed matter systems composed of interacting components significantly differ from properties of more or less independent parts [Halperin, 2010].

Dimensionless chances of truth of tested by expert hypothesis on preference of an analyzed alternative relative to others are comparison criteria within this approach. Chances of truth of the opposite hypothesis, which complement the first chances up to unity, serve as a measure of risk. In this approach point implementations of different alternatives from analyzed set are considered as independent and priori all the alternatives have equal rights with respect to the choice.

The start step in the realization of this approach is pairwise comparing alternatives, when the number of objects to be compared and its impact on risk do not take into account [Shepelyov, 2013; Shepelev, 2014]. Criterion of comparison of interval alternatives with arbitrary distributions of chances, which was called “assurance factor”, was proposed on this way. It is equal to the difference between chances of the truth of tested hypothesis on preference of an alternative in their set and the chances of the truth of the opposite hypothesis. Numerical (for arbitrary distributions of chance) and analytical (for uniform and triangular distributions) methods calculating the assurance factor as well as decision-making procedures based on this criterion were proposed. The assurance factor and chances of preference are equivalent as comparison criteria. The first of them is more convenient in some cases. For example, for some simple distributions of chances one may establish a relation between such criteria as the difference of the averages for two compared alternatives and assurance factor [Shepelev, 2014] and their dissimilarity in other cases.

Let give a brief overview of the results obtained in the framework of the method of collective risk estimating in paper [Shepelev, 2015].

Suppose that there are K alternatives I_i, i = 1, 2,…, K with the same interval quality indicators. Let dimensionless quantity $C(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},…, I_K))$ is the chances, in other words degree of assurance, in the truth of a testable hypothesis of preference, that the alternative I_i is more preferable than all at once alternatives (I_1, I_2,.., I_{i-1}, I_{i+1},…, I_K) from initially given their set (I_i is “better” of others “on the whole”). The term “all at once” means here that

$$I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K) \equiv (I_i \succ I_1) \wedge (I_i \succ I_2) \wedge (I_i \succ I_3) \wedge \ldots \wedge (I_i \succ I_{i+1}) \wedge \ldots \wedge (I_i \succ I_K),$$

where $\equiv$ and $\wedge$ are symbols of equivalence and conjunction respectively.

Risk that I_i would not preferred in reality will be measured by means of dimensionless quantity $R_s(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K))$ complementing previous chances to unity so that

$$Rs(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K)) = 1 - C(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K)).$$

As can be seen $R_s(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K))$ is degree of assurance in the truth of a hypothesis, which is opposite to the testable hypothesis of preference.

Equivalently $R_s(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K)) = C(\neg(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K)))$, where $\neg$ is symbol of negation. It is shown in [Shepelev, 2015] that the following relations hold for chances

$$C(I_1 \succ (I_2, I_3,\ldots, I_K)) + C(I_2 \succ (I_1, I_3,\ldots, I_K)) + C(I_3 \succ (I_1, I_2, I_4,\ldots, I_K)) + \ldots + C(I_K \succ (I_1, I_2,\ldots, I_{K-1})) = 1 \qquad (2)$$

and for risks

$$Rs(I_1 \succ (I_2, I_3,\ldots, I_K)) + Rs(I_2 \succ (I_1, I_3,\ldots, I_K)) + Rs(I_3 \succ (I_1, I_2, I_4,\ldots, I_K)) + \ldots + Rs(I_K \succ (I_1, I_2,\ldots, I_{K-1})) = K - 1.$$

Let us prove the relation (2). The statement $\neg(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K))$ means that at least one alternative from their compared set would be preferable than I_i. Let illustrate the meaning of introduced in this way the measure of risk for case of three alternatives. Here we have the following possible preferences and chains of disjunctions:

$$((I_1 \succ I_2 \succ I_3) \vee (I_1 \succ I_3 \succ I_2)) \vee ((I_2 \succ I_1 \succ I_3) \vee (I_2 \succ I_3 \succ I_1)) \vee ((I_3 \succ I_1 \succ I_2) \vee ((I_3 \succ I_1 \succ I_2))$$
$$\equiv (I_1 \succ (I_2, I_3)) \vee (I_2 \succ (I_1, I_3)) \vee (I_3 \succ (I_1, I_2)) \text{ or } Rs(I_1 \succ (I_2, I_3)) = C(I_2 \succ (I_1, I_3)) + C(I_3 \succ (I_1, I_2)).$$

Hence for K alternatives we have (2).

One can see now that $C(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},.., I_K))$ is monotonically non-increasing function of K, that is the chances that a certain alternative would be preferable in comparison with all the others do not increase with increasing number of the alternatives. Indeed, if the number of compared alternatives is increased the number of non-negative terms in the unit sum of corresponding chances in (2) is also increased. Therefore

$$C(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_K)) \leq C(I_i \succ (I_1, I_2,.., I_{i-1}, I_{i+1},\ldots, I_{K-1})),$$

Then corresponding risk will be monotonically non-decreasing function of number of compared alternatives:

$$Rs(I_i \succ (I_1, I_2,\ldots, I_{i-1}, I_{i+1},\ldots, I_{K-1})) \leq Rs(I_i \succ (I_1, I_2,\ldots, I_{i-1}, I_{i+1},\ldots, I_K)).$$

By another words the more of number of the alternatives the more risk of wrong decision-making.

The effect of the comparison of interval alternatives “as a whole” is manifested primarily in reducing value of preference chances for each alternative with respect to its chances under pair-wise comparison. This leads to a quantitative increasing risk value of selection as preferred alternative such one, which may not actually be per se later. As already noted the nature of this effect lies in the fact that in the presence of non-zero intersection for already two compared alternatives there is a non-zero risk of making the wrong decision. This risk is enhanced with increasing amounts of compared alternatives especially if some of these chances are not too different from each other. However, it should be borne in mind that the perception of risk is individual and can vary from one DM to another. Therefore the risk value resulting from the use of the proposed method is nothing more than a calculated risk, which can serve only as an estimate for the DMs.

What can be done to reduce the calculated risk? During deciding on preferred alternative choice or in the process of ordering alternatives by preference it’s useful to conduct a preliminary analysis of their initial set. Firstly, after selecting an alternative that preference is tested, one should select in the set of alternatives those, which do not have the intersection with analyzed alternative. If the left boundary of such intervals no less than the right boundary of the tested one the latter is certainly worse. If the right boundary of such intervals not greater than the left boundary of the tested one they can be excluded because they are certainly worse than the last interval. Secondly, one may try to unify some similar or complementary alternatives. By reducing the number of intervals in their initial set one may increase the calculated preference chances of analyzed alternative and decrease risks. At last, after calculating the preference chances of tested alternative during pairwise comparisons it is advisable to exclude those alternatives whose preference chances with respect to tested alternative is less than 0.5 and, respectively, the risk is more than 0.5. Other possibilities to reduce the number of comparable alternatives and to decrease by that the collective risk discussed below in the discussion of using “mean-risk” methods.

Are there any other amendments to the results of the pairwise comparison of alternatives due to “collective” effect? Particularly important is the following question: is there difference of the alternatives order in their set defined by the “collective” preference chances and the order for pairwise comparison? The answer to this question is negative: the order established in the process of pairwise comparison is the same as the order for comparison “as a whole”.

Thus for this approach the “best” alternative will be the alternative with the highest chances at pairwise comparisons in the set of compared alternatives. However adequate the risk estimation of making wrong decision we obtain by comparing this alternative simultaneously with all the others, “as a whole”.

Advantages of this approach consist in account of integral risk in a concrete group of compared interval alternatives that permit to DM to get an idea of the true size of surprises lie in wait for him. Other methods of comparing do not permit do so. Disadvantages are a consequence of the fact that the method compares only relative preferences of alternative and does not take into account acceptability of separate alternative.

“Mean — risk” methods of comparing

“Mean-risk” methods are free from these disadvantages and complement method of collective risk estimating. Let us now consider “mean-risk” methods in a version, where the preference criterion is value of mathematical expectation and risk criterion, in accordance with the recommendations of the papers [Ogryczak, 1999; Grechuk, 2012], is the mean absolute semideviation S_I. It is natural to suppose that one alternative is more preferred than another in their compared pair if this alternative is better according to the one of the two criteria and no worse according to the second criterion.

As a measure of alternative preference (as well as of an acceptability) is here mathematical expectation E of a random variable X defined on I, then the measures of the average deviation of the random variable on the left S_{Il} and on the right S_{Ir} from E (“semideviations”) are good indicators of risk. Conveniently they have the same dimension as that of E and besides, as it can be shown, S_{Il} = S_{Ir}.

By the definition

$$S_{Il} = \int_L^E (E-x) f(x)dx,\ S_{Ir} = \int_E^R (x-E) f(x)dx$$

because

$$\int_L^R (E-x) f(x)dx = \int_L^E (E-x) f(x)dx - \int_E^R (x-E) f(x)dx = 0,$$

so S_{Il} = S_{Ir} = S_I.

Let consider the behavior of S_I for uniform and triangular distributions in more detail. For uniform distribution have: $E_U = (L + R)/2$, and $S_{IU} = (R - L)/8$.

In the case of triangular distribution with mode M its density consists of two branches, the left f_l lying on the graph of the density on the left of M, $f_l = 2(x - L)/[(R - L)(M - L)]$ and the right, $f_r = 2(R - x)/[(R - L)(R - M)]$. Mathematical expectation equals $E_T = (R + M + L)/3$. There are two possibilities for location of the mode: $M \le E_T$ and $M > E_T$. One can show that if $M > E_T$ then $M > (L + R)/2 = E_U$, if $M \le E_T$, then $M \le E_U$.

For $M \le E_T$ have:

$$S_I = S_{Ir} = \int_{E_T}^{R} (x - E) f_r(x) = \frac{(R - E_T)^3}{3(R - L)(R - M)} = \frac{(2R - L - M)^3}{81(R - L)(R - M)}.$$

And for $M > E_T$:

$$S_I = S_{Il} = \int_{L}^{E_T} (E - x) f_l(x) = \frac{(E_T - L)^3}{3(R - L)(M - L)} = \frac{(R + M - 2L)^3}{81(R - L)(M - L)}.$$

One can see that as a function of mode M semideviation $S_I(M)$ is convex downward, monotonically decreasing on the interval $[L, (L + R)/2]$ and monotonically increasing on the interval $[(L + R)/2, R]$ function. The function is symmetrical about a vertical axis $M = (L + R)/2 = E_U$, its minimum S_{Imin} is attained at the point $M = E_U$, $S_{Imin} = (R - - L)/12$ and maximum S_{Imax} at points $M = L$ and $M = R$, $S_{Imax} = S_I(L) = S_I(R) = 8(R - L)/81$.

It is useful for experts to take into account the following during selection of chances distribution functions to describe their knowledge about the interval indicators of quality. The transition from uniform distributions to the others, for example, triangular, means availability of more knowledge about the object. This reduces the risk indicator value and therefore $S_{Imax} < S_{IU}$ (for the same carrier interval). Choice of the mode, which equals to the mean of the corresponding uniform distribution E_U, in triangular distributions results in the lowest risk indicator value and deviations from the values of M on both sides from E_U leads to a symmetric growth of risk indicator. However, these deviations from the viewpoint of the alternatives comparison are not equivalent. Deviations of M from E_U on the right lead to increasing of expectation value E_T, i.e. to increasing of preference indicator in “mean – risk” approach, and on the left to decreasing of E_T.

The indicator S_{Il} characterizes possible, because of the uncertainty, the average unfavorable deviation of alternative quality indicator from the value of the mathematical expectation. Course the average deviation is significantly less than the maximum possible deviation. So S_{IU} is four times less than the highest possible negative deviation. However the choice of the mathematical expectation as a preference indicator may not reflect the preferences of DM and her/his risk appetite. So for uniform distribution of the chances for such a choice point exactly half of the possible implementations of the quality indicator are less than the value of mathematical expectation.

It is advisable therefore the introduction of a risk measure $S_{II}(a)$, which is similar to the average semideviation but is associated with any valid target value *a* of an alternative quality indicator:

$$S_{II}(a) = \int_{L}^{a} (a - x) f(x) dx .$$

Consider some of the properties of this indicator for the simplest case of uniform distributions of chances.

Let α ($0 < \alpha < 1$) is the desired for DM level of chances that the point implementations of the quality indicator will be less than *a*. It is clear that lower values of α are more desirable but such values may be connected with undesirable values of target indicator.

For uniform distributions $a = (1 - \alpha)L + \alpha R$. DM can now either appoint the value of α and determine *a* or do the opposite. For $S_{II}(a)$ can obtain the following relationship: $S_{II}(a) = (a - L)^2/[2(R - L)] = S_{II}(\alpha) = \alpha^2(R - L)/2$. Coefficient $\alpha^2/2$ can be called a risk factor K_r. By definition $K_r = \alpha^2/2$. It is easy to see that the smaller value of the target indicator the smaller the value of risk factor.

In real problems $R > 0$, $L < 0$ because for $R < 0$ alternative should be excluded from the number of compared and for $L > 0$ alternative most often is a priori suitable for the realization. Under such conditions $S_{II}(\alpha) > 0$ but the negative values of the expectation *E* are possible. Let require that only alternatives with $E > 0$ be permitted to the comparison; i.e. such that $R > - L$. We also require that for suitable to comparison alternatives values of the risk indicator were below the target indicators: $S_{II}(\alpha) < a$. This gives for suitable alternatives the condition: $R > - L(2 - - 2\alpha + \alpha^2)/[\alpha(2 - \alpha)]$. The coefficient $K_s = (2 - 2\alpha + \alpha^2)/[\alpha(2 - \alpha)]$ may be called the suitability factor.

Naturally to require also that the selected by DM value of the target mark was positive. This requirement leads to the following condition for *R*: $R > - [(1 - \alpha)/\alpha]L$. Coefficient $K_p = (1 - \alpha)/\alpha$ can be called a factor of positivity (for *a*). Values of chances α, for which the positivity condition is not met, should not be tested by DM.

One can be shown that $K_s > K_p$, since, besides, $K_s > 1$, the condition of the suitability of alternatives to comparison $R > - K_sL$ provides simultaneous fulfillment of the requirements $E > 0$, $a > 0$ and $S_{II}(\alpha) < a$. Thus alternatives with negative left boundaries for which $R < - K_sL$ may be excluded.

Let calculate values of the risk, suitability and positivity factors for chances α that less than 0.5, namely such chances most likely will be tested by DMs. Results of the calculations are shown in Table 1.

Table 1. Some risk, suitability and positive factors values

Chances (α)	0.1	0.2	0.3	0.4	0.5
Risk Factor	0.005	0.02	0.045	0.08	0.125
Suitability Factor	9.53	4.56	2.92	2.13	1.67
Positivity Factor	9	4	2.3	1.5	1

Let us dwell on the essence of entities appearing in Table 1.

The parameter α indicates the amount of chances that point realization of quality indicator will be less than the target mark, that is shows the magnitude of the risk on which the DM agree when sets a target mark. Clearly that the smaller the risk, which is accepted by DM, the smaller her/his target mark and vice versa. At the same time it is clear that this target mark can do not satisfy of DM but then she/he must accept a few larger risks.

Of course, in real cases the target mark must be positive. Positivity factor K_p shows at how many times, under the selected risk α, right boundary R of interval estimation must exceed its left boundary L (to be not less than this value) for the positivity of target mark. Naturally if DM may want to restrict himself by low risk then this factor may be too large and the analyzed interval can do not satisfy this requirement. DM must then either abandon the analysis or adopt more risk.

Suitability factor can be used in the analysis of interval estimates if DM desires to find those objects the risk factor values for which are less than the target mark.

One can see that this condition imposes stringent requirements on the right border of the interval estimation in comparison to the left so that the value of suitability factor exceeds the value of positivity factor. This difference is increased with growth of the risk, which DM accepts.

At last the value of the risk factor multiplied by the length of the analyzed interval object indicates the scope of the range where in the worst case may in average be point implementations of the quality indicator of the alternative. The lower the risk accepted by the decision maker the less is the scope of this range.

In concluding this section we distinguish the advantages and disadvantages of methods of "mean – risk" approach. An advantage of the methods of "mean – risk" approach is the possibility of calculating for each alternative both basic indicators needed for the evaluation interval alternatives, indicator of acceptability of alternatives and indicator of associated risk. The disadvantages include the fact that both of these indicators are calculated for an alternative as an independent object, which is not associated with the others in compared group. A comparison of the alternatives on preference is based on values of these indicators but dependence of the risk on the context is not taken into account. It should also be kept in mind that used here indicators are averaged estimates, which are not adequate, generally speaking, the situation of a unique choice.

At the same time in paper [Shepelev, 2014] was shown that results of interval objects selection on preference for the criterion of difference of mathematical expectations and the criterion of assurance factor are the same for a number of the simplest chances distributions. Recall that the assurance factor equals the difference in the chances of truth of tested hypothesis on preference of the analyzed alternative and in chances of truth of the opposite hypothesis.

Conclusion

Problems of comparing alternatives with numerical quality indicators, which due to uncertainty have interval representations, are fairly common in practice, in particular, in economics and engineering [Vilensky, 2015]. They play an important role and require special methods for their analysis. The inevitability of presence in similar tasks of irremovable risk of making a wrong decision on the preference of the alternatives requires human involvement in the decision-making processes. These processes are quite difficult; therefore DMs and experts need means of analytical support for their activity. They may permit to DMs and experts check how their knowledge and largely intuitive choice is consistent with the formal results and adjust their decisions. Besides use of different formal methods in the process of alternatives comparing and decision-making increases the volume and variety of accessible information that is useful to DMs and experts.

However formal methods of comparison and estimation of interval alternatives as a component of computer system for information-analytical support of the decision-making process cannot guarantee choice of truly the best object in the process of comparing or estimating. The results using of such methods can serve for DM only as a guideline, kind of a hint in the decision-making. At present there is no approach transcending all others in quality of recommendations obtained on its basis. Each method has its advantages and disadvantages.

Human involvement in the decision-making process can determine not only the choice of the tools of describing the uncertainty but also, due to previous experience and knowledge of human being, the choice of methods for comparison and estimation of alternatives that lead to the result indicators, which are familiar to the expert or DMs. Each of the available methods of comparison and evaluation allows calculating its measures of alternative preference in relation to the other alternatives in their set as well as indicators of the acceptability of alternatives during estimating and their risk measures.

Presence of the collective effect in groups of compared alternatives is manifested primarily in reducing value of preference chances for each alternative with respect to its chances under pair-wise comparison. This leads to a quantitative increasing risk value of selection as preferred alternative such one, which may not actually be per se later. The nature of this effect lies in the fact that in the presence of non-zero intersection for already two compared alternatives there is a non-zero risk of making the wrong decision. This risk is enhanced with increasing amounts of compared alternatives especially if some of these chances are not too different from each other. However, it should be borne in mind that the perception of risk is individual and can vary from one DM to another. Therefore the risk value resulting from the use of formal methods is nothing more than a calculated risk, which can serve only as an estimate for the DMs.

What can be done to reduce the calculated risk? Try to reduce the number of comparable alternatives as well as to take into account that joint using of different methods on various successive stages of the decision-making process is probably the best way to combine the power of formal methods and knowledge of experts and DMs.

Therefore decision-making process for the selection of interval alternatives is suggested in the paper to divide into three stages. Firstly, during deciding on preferred alternative choice or in the process of ordering alternatives by preference it’s useful to conduct a preliminary analysis of their initial set. After selecting an alternative that preference is tested it should select in the set of alternatives those, which do not have the intersection with analyzed alternative. If the left boundary of such intervals no less than the right boundary of the tested one the latter is certainly worse. If the right boundary of such intervals not greater than the left boundary of the tested one they can be excluded because they are certainly worse than the last interval. Then one may try to unify some similar or complementary alternatives. By reducing the number of intervals in their initial set one may increase the calculated preference chances of analyzed alternative and decrease collective risks.

Using of the method of collective risk estimating on the first stage of the decision-making process permits to appraise integral risks for each alternative in the group and find the "best" alternatives as the alternatives with the highest chances at pairwise comparisons in the set of compared alternatives.

At last, on the final stage, this narrowing set of alternatives should be evaluated according to the criteria of preference and risk, which are based on methods of "average – risk" approach. In the case of incomparability of alternatives to these criteria, when tested alternative is better by one criterion and worse by another, one may go to multi-criteria models or to models with one generalized optimality criterion. Both the initial criteria are harmonized in the generalized criterion by means of positive coefficient of replacement, which reflects the attitude of DM to take risks [Podinovsky, 2015].

One can expect that such a multi-step approach to decision-making on the choice of the preferred interval alternatives may contribute to increasing of adequacy of decision-making.

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Authors' Information

Gennady Shepelev – *Laboratory Head of Institute for Systems Analysis of Federal Research Center "Computer Science and Control" of RAS; e-mail: gis@isa.ru*

Major Fields of Scientific Research: *mathematical modeling, probabilistic methods, interval analysis, generalized interval estimations, comparing interval alternatives*