DISCRETE TOMOGRAPHY OVERVIEW: CONSTRAINTS, COMPLEXITY, APPROXIMATION

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Abstract: *In this paper we give an overview of discrete tomography problems, addressing constraints/properties; complexity and approximation issues. We present also some notes on existence/reconstruction of binary images from the given horizontal and diagonal projections.*

Keywords: *Discrete tomography, constraints, approximation.*

ITHEA Keywords: *G.2. Discrete Mathematics: G.2.3 Applications*

Introduction

Reconstruction of discrete sets from given projections, - is one of the main tasks of *Discrete Tomography*. Discrete sets can be presented as binary images. The line sum of a line through the image is the number of points on this line. The projection of the image in a certain direction consists of all the line sums of the parallel lines passing through the image in this direction. Any binary image with exactly the same projections as the original image represents a reconstruction of that image.

Reconstruction algorithms that are intended to solve particular inverse problems, have many applications in areas such as the image processing, medicine, computer tomography assisted engineering and design, etc. A large number of well-known medical problems require discrete reconstruction technique ([PrauseOnnasch, 1996], [SlumpGerbrands, 1982]). For example, in angiography, the values 0 and 1 can represent the absence or presence of a contrast agent in heart chambers.

Opposite to methods of Computerized Tomography, which use several hundreds of projections, in Discrete Tomography a few projections are available. The main problem arising here is that different binary images may appear with the same projections; and in case of small number of projections the problem in this form can have large number of solutions ([GardGrizmPran, 1999]). For exactly two directions, the horizontal and vertical ones, it is possible to reconstruct an image in polynomial time ([Ryser, 1957]). But in general, if only the horizontal and vertical projections are given, then the number of solutions can be exponentially large ([DLungo, 1994]).

On the other hand, for any set of more than two directions, the problem of reconstructing a binary image from its projections in those directions is NP-complete.([GardGrizmPran, 1999]).

One way to eliminate these problems is to suppose that there is some prior knowledge on the image to be reconstructed and this can reduce the search space of the possible solutions. It can be assumed that the image has some geometrical properties.

Using geometrical knowledge about the discrete sets, such as convexity and connectedness, is a well-studied area. The existence problem for convex matrices, as well as the existence problem for connected matrices are NP-complete ([BarcDLungoNivatPinz, 1996], [Woeginger, 2001]. In the meantime, the existence problem for horizontally and vertically convex and connected matrices can be solved in polynomial time ([DurrChrobak, 1999]).

Another property of discrete sets, which is new and specific for the domain of discrete tomography, is that the rows of the matrix to be reconstructed are distinct [Sahakyan, 2009], [Sahakyan, 2013], Sahakyan, 2014]. This constraint comes from applications, such as design of experiments; it is also related to known problems of other domains (discrete isoperimetry problem ([Aslanyan, 1979], [AslanyanDanoyan, 2013]), hypergraph degree sequence problem ([Sahakyan, 2015]), and others).

Another strategy can be: to find a possibly good but not necessarily the exact solution. Approximation algorithms with greedy approach are introduced in [Sahakyan, 2010], [SahakyanAslanyan, 2011].

In this paper we give some notes/overview on discrete tomography problems: addressing constraints/properties, complexity and approximation issues. We consider also the existence/reconstruction of binary image from the given horizontal and diagonal projections.

Orthogonal projections

Consider T, a finite set in the two-dimensional integer grid Z^2. A projection of T in any direction calculates the number of points of T on the lines parallel to the projection direction. Given a finite set of projections, it is required to reconstruct T or to construct any set matching these projections. T may be presented as a binary matrix, where ones correspond to the points of T.

In the simplest case of the orthogonal projections the existence and construction problem is solved by Gayle and Ryser in combinatorial terms in 1957. Let $A = \{a_{i,j}\}$ be a binary matrix with m rows and n columns. Let $R = (r_1, \cdots, r_m)$ and $S = (s_1 \cdots, s_n)$ denote the row and column sums of A, correspondingly, where $r_i = \sum_{j=1}^{n} a_{i,j}$, $i = 1, \cdots, m$ and $s_j = \sum_{i=1}^{m} a_{i,j}$, $j = 1, \cdots, n$. $U(R, S)$ denotes the set of all binary matrices with row sum R and column sum S.

Theorem 1. [Ryser, 1957].

Let $R = (r_1, r_2, \cdots, r_m)$ and $S = (s_1, s_2, \cdots, s_n)$ be vectors with non-negative integer components arranged in decreasing order. $S^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is the conjugate vector of R: $s_i^* = |\{r_i: r_i \geq j, \ j=1,\cdots,m\}$. Then the class $U(R,S)$ is not empty if and only if S is majorised by $S*$: $\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} s_i^*$, $k = 1, \cdots, n-1$, and $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i^*$.

For a given finite set of projections there may exist different sets with the same projections. Any property of the recovering object, if such property exists, can narrow the class of possible solutions.

Geometrical properties

Definition 1. A binary matrix is h-convex, if the ones in every row form an interval; and is v-convex if the ones in every column form an interval. A binary matrix is hv-convex if it is both h-convex and v-convex.

Definition 2. A binary matrix is connected, if the ones are connected with respect to the adjacency relation.
Connected by 4-adjacency (vertical and horizontal) matrix is called polyomino.

Complexity

If there is no additional restriction, then according to Theorem 1, the existence problem of a binary matrix with given orthogonal projections has polynomial complexity. The existence problem of a binary matrix is NP-complete for h-convex, v-convex matrices, and h-convex, v-convex polyominos ([BarcDLungoNivatPinz, 1996]. But in case of hv-convex polyiominos there exists a polynomial time algorithm ([DurrChrobak, 1999]). NP-completeness of a number of other cases is proven in [Woeginger, 2001].

Distinct rows

Let $A = \{a_{i,j}\}$ be a binary matrix with m rows and n columns, and let $S = (s_1 \cdots, s_n)$ denote the column sum vector of A.

$U(S)$ denotes the class of all binary matrices of size $m \times n$, with the column sum vector S. Let $\overline{U}(S)$ denote the subclass of $U(S)$ where all matrices consist of all distinct rows.

For a given integer vector S the problem of existence/reconstruction of a binary $m \times n$ matrix in the class $\overline{U}(S)$ is investigated in [Sahakyan, 2009] –[Sahakyan 2014]. The complete structural characterization of the set of column sum vectors of all binary $m \times n$ matrices with distinct rows is given in [Sahakyan, 2009]–[Sahakyan, 2014]. It is worth mentioning that these problems have counterparts in terms of hypergraphs and degree sequences, which are long standing open problems in the graph theory ([Berge, 1989]).

Approximation

A strategy to solve hard discrete tomography problems can be: to search for possibly good but not necessarily the exact solutions of the problem.

A relaxed version of the existence problem (in the class $\overline{U}(S)$) is addressed in [SahakyanAslanyan, 2017], where some constant number of repeated rows is allowed; the complexity of the relaxed problem is investigated, and several properties/results are obtained.

Another approach to obtain approximate solutions is applied in [Sahakyan, 2010]-[SahakyanAslanyan, 2011]. Greedy algorithm is proposed which constructs a matrix from the given column sum $S = (s_1, \cdots, s_n)$. The strategy is the following: to construct the matrix column-by column in such a way that in each step the number of different pairs of rows is maximized. A schematic picture of the greedy partitioning is given in Figure 1:

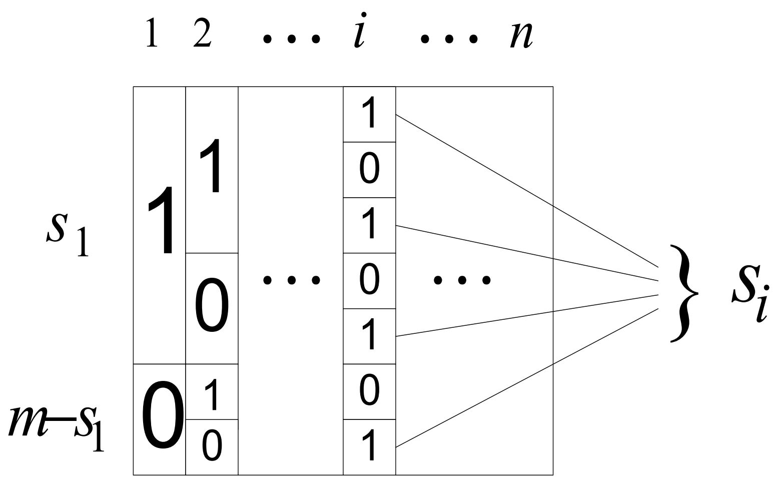

Figure 1

Diagonal projections

The problem of reconstructing binary images from given orthogonal and diagonal projections is studied in [GardGrizmPrang, 1999], [BarBrunDLunNivat, 2001]. In general the problem of reconstructing binary images, from given orthogonal and diagonal projections, is NP-complete [GardGrizmPrang, 1999].

The case of horizontal-vertical-diagonal connected and convex sets is studied in [BarBrunDLunNivat, 2001], and a polynomial-time algorithm is provided for reconstructing these sets.

Diagonal and anti-diagonal projections were studied in [VermaShriv, 2014], and an approach for reconstruction of binary images from diagonal and anti-diagonal projections is provided in [SrivastavaVermaPatel, 2012]; a comparison is done with the existed methods.

The uniqueness of solution for reconstruction problem with the diagonal and anti-diagonal projections is discussed in [SrivastavaVerma, 2013].

Horizontal and Diagonal projections

In this section we consider existence/reconstruction of binary matrices from the given horizontal and diagonal projections.

Consider a binary matrix $A = \{a_{i,j}\}$ with m rows and n columns. Let $R = (r_1, \cdots, r_m)$ denote the row sum, and $D = (d_1, \dots, d_{m+n-1})$ denote the diagonal sum vector of A, where $r_i = \sum_{j=1}^{n} a_{i,j}$, $i = 1, \cdots, m$, and $d_k = \sum_{i+j=k+1} a_{ij}$, $k = 1, \dots, m+n-1$.

For example, the image given in the Figure 2 has the following row and diagonal sums: $R = (3,4,5,4,5,5,6,2,1)$ and $D = (0,0,0,3,5,5,4,2,5,5,2,2,2,0,0,0)$.

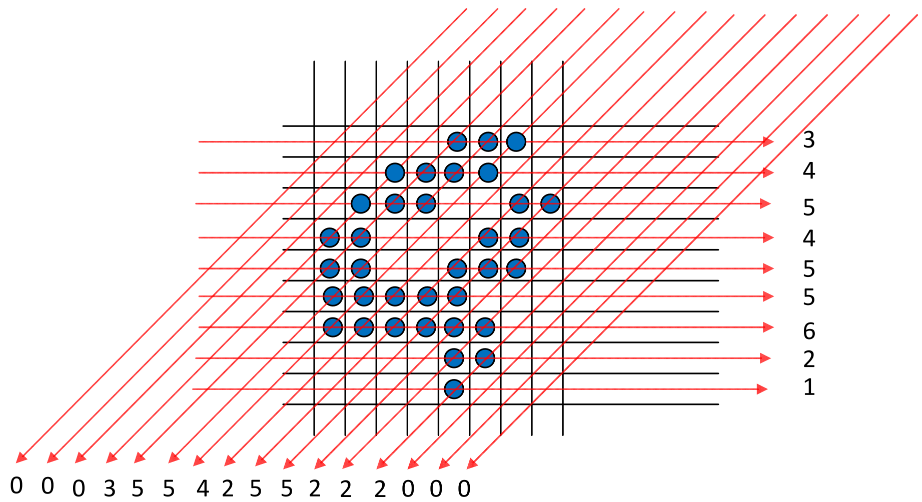

Figure 2

We say that a pair (R, D) is compatible if the following conditions hold:

$\sum_{k=1}^{m+n-1} d_k = \sum_{j=1}^{m} r_j$

$r_i \leq n$, and for $1 \leq i \leq m$;

$d_j \leq m_j$;

where m_j is the j-th component of the following $(m+n-1)$-length vector:

$$M = (1,2,3, \dots, \overbrace{\min(m,n), \min(m,n), \min(m,n)}^{m+n-1-2*(\min(m,n)-1)}, \dots, 3,2,1)$$

For a given vector $R = (r_1, r_2, \dots, r_m)$ we compose the maximal matrix of size $m \times n$ (denoted by $\bar{A}$), where each row has the following structure:

$\overbrace{1,1,\cdots,1}^{r_i}\overbrace{0,0,\cdots,0}^{n-r_i}.$

Let $R = (r_1, r_2, \dots, r_m)$ and $D = (d_1, \dots, d_{m+n-1})$ be a pair of compatible vectors.

We propose an algorithm for constructing a matrix A with the row sum R and diagonal sum D from the maximal matrix $\bar{A}$. Let R^i denote the collection of rows of $\bar{A}$ that intersect with the i-th diagonal line. For each i the algorithm shifts $d_{\rm i}$ ones from the rows of R^i and locates them in the i- th diagonal line of A.

To provide the performance of the algorithm we use/define several fragments in the maximal matrix and require majorization conditions/properties for each of them. These are necessary conditions for existing the matrix, and on the other hand they provide the construction of the matrix in case when such matrix exists.

Below is one of such fragments in the maximal matrix $\bar{A} = \{\bar{a}_{i,j}\}$.

Fragment 1.

For every i, $1 \le i \le min\,(m, n)$ we denote by $F1^i$ the left part of $\bar{A}$, bounded by the i-th diagonal line as shown in the Figure 3. $F1^i$ has i rows and i columns. $S^{F1\_i} = (s_1^{F1_i}, s_2^{F1_i}, \cdots, s_i^{F1\_i})$ is the column sum vector of $F1^i$, where $s_j^{F1_i} = \sum_{k=1}^{i-(j-1)} \bar{a}_{k,j}$ for each $j, 1 \le j \le i$.

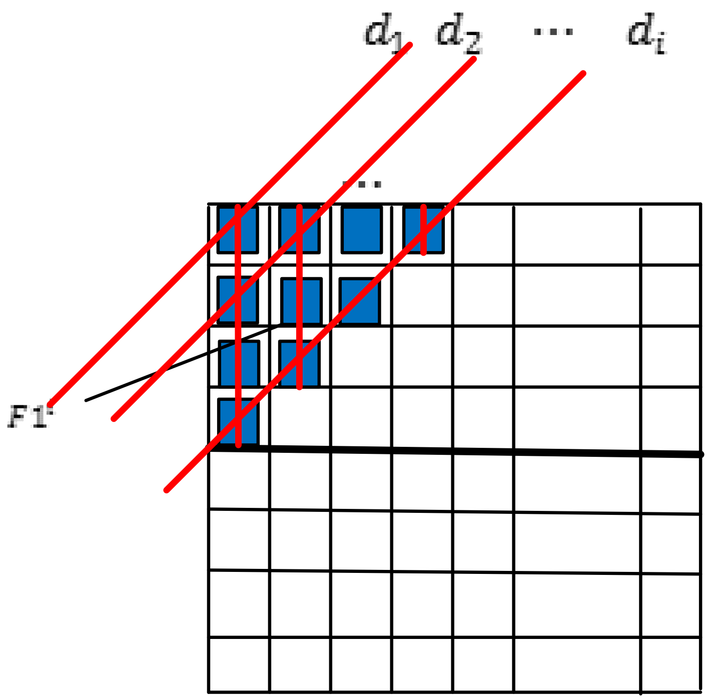

Figure 3.

Let D^{1_j} denote the initial part of D: $D^{1\_j} = \left(d_1, \dots, d_j\right)$.

Majorization 1.

For a given i, $1 \le i \le min\ (m, n)$ we say that S^{F1_i} majorizes D^{1_i}: $S^{F1\_i} \succcurlyeq^1 D^{1\_i}$ if for each $1 \le j \le i$ the following conditions hold:

$d_j \le s_1^{F1_i}$; $d_j + d_{j-1} \le s_1^{F1_i} + s_2^{F1_i}$; ... $d_j + d_{j-1} + \cdots + d_1 \le s_1^{F1_i} + s_2^{F1_i} + \cdots + s_i^{F1\_i}$.

Conclusion

In this paper we give an overview of discrete tomography problems, addressing constraints/properties; complexity and approximation issues. We present also some notes on existence/reconstruction of binary images from the given horizontal and diagonal projections.

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