Examples of NP-complete essential restrictions of the SUBSET SUM PROBLEM

Nikolay K. Kosovskii, Tatiana Kosovskaya, Michail Starchak

Abstract: *The problem SUBSET SUM Garey, Johnson, [1979] may be interpreted as the problem of solvability in $\{0, 1\}$ numbers checking of a linear Diophantine equation with positive coefficients and constant term. This problem is the one of the most simply formulated mathematical problems for which it is proved that it is NP-complete. Its essential restriction under which it remains to be NP-complete are offered in the paper.*

Keywords: *NP-completeness, SUBSET SUM, linear Diophantine equation*

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Introduction

Algorithms for the solving of a system of linear equations have broad implementation during computer simulation Krivyi [2016]. The use of a computer while solving mathematical and discrete problems involves to take into account the effectiveness of an algorithm to be programmed. At present an algorithm without proved polynomial-time complexity is regarded as not effective.

The notion of an NP-complete problem was introduced by Cook S.A. in 1971 Cook [1971]. At present for such a problem a polynomial-time algorithm is not found and there is a hypothesis that it does not exist. That's why the proof of NP-completeness for different mathematical problems are very actual (see fo example Kosovskii, Starchak, [2016]).

To prove NP-completeness of a problem the notion of polynomial reducibility is often used.

A very representative list of NP-complete problems is in Garey, Johnson, [1979]. The problem SUBSET SUM may be pointed as one of the most simply formulated mathematical problems.

SUBSET SUM Garey, Johnson, [1979]

Instance: Finite set A, a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$ and a positive number B.

Question: Is there exists a subset $A' \subseteq A$ such that the sum of sizes of the elements in A is exactly B?

This problem may be formulated as the problem of solvability in $\{0, 1\}$-numbers checking of a linear Diophantine equation with positive coefficients and constant term.

SUBSET SUM

Instance: A set of positive integers $\{s_1, \ldots, s_n\}$, a set of variables $\{x_1, \ldots, x_n\}$ and a positive integer B.

Question: Is there exists a $\{0, 1\}$-solution of the equation $s_1x_1 + \cdots + s_nx_n = B$?

The analysis of an NP-complete problem allows to extract such its subproblems that some of them remain to be NP-complete and the other turn into polynomial-time ones. The extracting of such subproblems is an important step to the defining the domain of input data which allows effective computer implementation.

Essential restriction of the problem SUBSET SUM, under which it remains to be NP-complete are offered in the paper. The essence of the restriction consists in the fact that the constant term of the equation is written in positional number system with the help of the only one non-zero figure.

To prove the result presented in the paper NP-completeness of the problem ONE-IN-THREE 3SAT Garey, Johnson, [1979] is used.

ONE-IN-THREE 3SAT

Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.

Question: Is there a truth assignment for U such that each clause in C has exactly one true literal?

Its restriction when each clause does not contain the symbol of negation is used in the paper. Note, that NP-completeness of such a restriction is announced in Garey, Johnson, [1979] but the reference to the proof is absent.

Main results

Let's prove the polynomial-time reduction of the problem ONE-IN-THREE 3SAT when each clause does not contain the symbol of negation to the main problem ONE-IN-THREE 3SAT.

Lemma 1. *The problem ONE-IN-THREE 3SAT is polynomially reducible to its subproblem ONE-IN-THREE 3SAT when each clause does not contain the symbol of negation.*

Proof. First of all show that $z = 1$ is equivalent to the consistency in $\{0, 1\}$-numbers of the system of equations

$$\begin{cases} y_1 + y_2 + z & = & 1 \\ y_2 + y_4 + z & = & 1 \\ y_3 + y_4 + z & = & 1 \\ y_1 + y_3 + z & = & 1 \end{cases}.$$

If $z = 1$ then the consistency is evident and $y_1 = y_2 = y_3 = y_4 = 0$.

If $z = 0$ then we have the system

$$\begin{cases} y_1 + y_2 & = & 1 \\ y_2 + y_4 & = & 1 \\ y_3 + y_4 & = & 1 \\ y_1 + y_3 & = & 1 \end{cases},$$

which is not consistent because the rank of its matrix is 3 and the rank of its augmented matrix is 4.

If the constant *true* is interpreted as the number 1 and the constant *false* is interpreted as the number 0, then the set of clauses $\{y_1 \vee y_2 \vee z,\ y_2 \vee y_4 \vee z,\ y_3 \vee y_4 \vee z,\ y_1 \vee y_3 \vee z\}$ in the problem ONE-IN-THREE 3SAT gives the condition $z = true, y_1 = y_2 = y_3 = y_4 = false$.

This statement allow to introduce in the problem ONE-IN-THREE 3SAT 4 new variables. One of which (the variable z) is identical *true* and the others y_1, y_2, y_3, y_4 are identical *false*. For every variable x in the problem ONE-IN-THREE 3SAT we can introduce a new variable $\overline{x}$ and the clause $x \vee \overline{x} \vee y_1$. The received in such a manner set of clauses does not contain the symbol of negation.

It is evident that the new set of variables and the new set of clauses may be obtained by a polynomial-time under the length of ONE-IN-THREE 3SAT input data algorithm. □

The proof of the following lemma uses NP-completeness of the following problem Schrijver [1986].

Lemma 2. *The problem of consistency in non-negative integers of the system of linear Diophantine equations of the form $a_1x_1 + ... + a_nx_n = 1$, with coefficients $a_i \in \{0, 1\}$ and exactly three non-zero coefficients in every equation, is NP-complete.*

Proof. The problem is from the class **NP** as the consequence of the NP-completeness of the general problem of consistency in non-negative integers of the system of linear Diophantine equations of the form $a_1x_1+...+a_nx_n = 1$ (corollary 18.1a in Schrijver [1986]).

The problem ONE-IN-THREE 3-SAT is polynomial-time reducible to the problem under consideration. The constants *true* and *false* are encoded with 1 and 0 respectively. As ONE-IN-THREE 3-SAT remains NP-complete even with no negated literals in disjunctions the representation of every disjunction of the form $x \vee y \vee z$ by $x + y + z = 1$ completes this polynomial reduction. □

Let SUBSET SUM-1f be the restriction of the problem SUBSET SUM when the constant term of the equation is written in positional number system with the help of the only one non-zero figure.

SUBSET SUM-1f

Instance: A set of positive integers $\{s_1, \ldots, s_n\}$ and a positive integer B written in a positional number system with the help of the only one non-zero figure.

Question: Is there exists a $\{0, 1\}$ solution of the equation $s_1 x_1 + \cdots + s_n x_n = B$?

Theorem 1. *The problem SUBSET SUM-1f is NP-complete.*

Proof. SUBSET SUM-1f belongs to the class **NP** as it is a subproblem of the NP-complete problem SUBSET SUM.

Let $u_1, \ldots, u_n, c_1, \ldots, c_m$ be input data of the problem ONE-IN-THREE 3SAT when each clause does not contain the symbol of negation.

If the constant *true* is interpreted as the number 1 and the constant *false* is interpreted as the number 0, then the truth with exactly one true literal of each clause c in the form $w_1 \vee w_2 \vee w_3$ (where w_1, w_2, w_3 are variables) in C may be interpreted as $w'_1 + w'_2 + w'_3 = 1$ (where w'_1, w'_2, w'_3 are variables with values from $\{0, 1\}$.

The jth equation multiply by 2^{j-1} ($j = 1, \ldots, m$) and sum the results. The equation in the form

$$c'_1 + 2c'_2 + \cdots + 2^{m-1} c'_m = 1 + 2 + \cdots + 2^{m-1}$$

is received. It is solvable in $\{0, 1\}$-numbers if and only if the problem ONE-IN-THREE 3SAT when each clause does not contain the symbol of negation is solvable.

This is a is polynomial-time reduction because the lengths of the binary notation of the numbers 2^{j-1} ($j = 1, \ldots, m$) are not greater then m, where m is the number of clauses in the input data.

The figure 1 may be changed by any non-zero figure f in the positional number system with the radix of number system p. In such a case the cofactor 2^{j-1} must be changed by $p^{j-1} f$ ($j = 1, \ldots, m$). □

Conclusion

NP-completeness of the restriction of the problem SUBSET SUM when the constant term of the equation is written in positional number system with the help of the only one non-zero figure is proved in the paper.

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Authors' Information

Nikolay K. Kosovskii *- Dr., Professor of Computer Science Chair of St. Petersburg State University, University av., 28, Stary Petergof, St. Petersburg, 198504, Russia; e-mail: kosov@NK1022.spb.edu*

Major Fields of Scientific Research: Mathematical Logic, Theory of Computational Complexity of Algorithms

Taniana Kosovskaya *- Dr., Professor of Computer Science Chair of St. Petersburg State University, University av., 28, Stary Petergof, St. Petersburg, 198504, Russia; e-mail: kosovtm@gmail.com*

Major Fields of Scientific Research:Logical Approach to Artificial Intelligence Problems, Theory of Computational Complexity of Algorithms

Mikhail Starchak *- PhD student of Computer Science Chair of St. Petersburg State University, University av., 28, Stary Petergof, St. Petersburg, 198504, Russia; e-mail: mikhstark@gmail.com*

Major Fields of Scientific Research: Theory of Computational Complexity of Algorithms