AUTOMATON MODEL OF ONE STATISTICAL RULE

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Abstract: *In the paper, a construction (algorithm of behavior) of a finite automaton in a stationary random environment with three possible reactions (win, loss, indifference) is proposed. It is constructed on the basis of a known statistical rule from the theory of recurrent events: "either a series of successes of length m, or a series of failures of length l". With methods of the theory of random walks, a formula is obtained for the generating function of the probability of changing the action of the finite automaton under consideration. It is shown that the sequence of finite automata of the construction under consideration converges to the corresponding infinite (with countably many states) automaton of the same structure and its possible behavior is investigated.*

Keywords: *finite automaton, stationary random environment, behavior algorithm, generating function, expediency of behavior.*

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Introduction

The problem of the behavior of a finite automaton in a binary stationary random environment was formulated and developed by M. L. Tsetlin [1]. The environment in the simplest case reacts to the actions of the automaton in two ways: either "punishes" or "encourages" the automaton with certain probabilities. The automaton a priori information about the medium does not have an. Then different authors the proposed various constructions of asymptotically optimal sequences of automata in both binary and non-binary stationary random environment (see, for example, [2-7]). For automata belonging to such a sequence, the mathematical expectation of the win increases with an increase in the memory capacity of the automaton and tends to the maximum possible in a given stationary random environment.

In the interest of technical applications, the synthesis problems of automata optimal at various criteria in a binary and ternary stationary random environment were investigated in [8-9].

However, studies related to the study of the behavior of automata in both binary and non-binary stationary random environments have shown that the construction of an automaton that is best for some feature in any medium is unrealistic. Therefore, it is necessary to construct structures and develop analytical and numerical methods for finding statistical characteristics of the behavior of wide classes of automata that can be used to solve various practical problems.

In the present paper, on the basis of a known statistical rule from the theory of recurrent events: "either a series of successes of length m, or a series of failures of length l" we propose a construction (algorithm of behavior) of the finite automaton in a stationary random environment with three possible reactions (win, loss, indifference). With methods of the theory of random walks, received a formula is obtained for the generating function of the probability of changing the action of the finite automaton under consideration. It is shown that the sequence of finite automata of the construction under consideration converges to the corresponding infinite (with countably many states) automaton of the same structure and its possible behavior is investigated.

The functioning of the finite automaton $T_{2n,2}(l,m)$ in a ternary stationary random environment

Consider the known scheme of behavior of automata in a random environment [1] and assume that all possible reactions $S\epsilon\{s_1, s_2, \dots, s_g\}$ of the environment C are perceived by the automaton, unlike [1], as belonging to one of the three classes - class favorable reactions (win, s = + 1), to the class of adverse reactions (loss, s = -1) and to the class of neutral reactions (indifference, s = 0).

Definition. Will say that the automaton A_k functions in a ternary stationary random environment $C(a_1, r_1; a_2, r_2; \dots; a_k, r_k)$, if the actions of the automaton and the values of the input signal are connected as follows: if the automaton performs the action $f_\alpha(\alpha = \overline{1;k})$, then the medium $C(a_1, r_1; a_2, r_2; \dots; a_k, r_k)$ forms the signal value $s = +1$ at the input of the automaton with the probability $q_\alpha = \frac{1-r_\alpha+a_\alpha}{2}$, the value of the signal $s = -1$ with the probability $p_\alpha = \frac{1-r_\alpha-a_\alpha}{2}$ and the value of the signal $s = 0$ with the probability $r_\alpha = 1 - q_\alpha - p_\alpha$ $(\alpha = \overline{1;k})$.

Here the quantity $a_\alpha = q_\alpha - p_\alpha$ ($|a_\alpha| < 1 - r_\alpha$) has the meaning of the mathematical expectation of the payoff for the action f_α in the environment $C(a_1, r_1; a_2, r_2; \dots; a_k, r_k)$. In solving problems of analyzing the possible behavior of an automaton in some random environment, by changing the numbering of the actions of the automaton, one can achieve that $a_1 > a_2 \geq \dots \geq a_k$, i.e. in the environment $C(a_1, r_1; a_2, r_2; \dots; a_k, r_k)$ the action f_1 of the automaton A_k with the average win a_1 is optimal.

Let the finite automaton $T_{2n,2}(l,m)$, which has $2n$ $(n = l + m - 1)$ internal states $L^{(n)} = L_1^{(n)} \cup L_2^{(n)} = \{-(l+m-1), \dots, -2, -1, 1, 2, \dots, (l+m-1)\}$ and can perform two different actions f_1 and f_2, functions in a ternary stationary random environment $C(a_1, r_1; a_2, r_2)$.

We define the tactics of the behavior of the automaton $T_{2n,2}(l,m)$ in the environment $C(a_1,r_1;a_2,r_2)$ as follows: if the signal value $s=+1$ (win) arrives at the input of the automaton, then the automaton from any state x of the area $L_\alpha^{(n)}$ goes to the state $|x|=l$ of the same area; at a signal $s=-1$ (loss), the automaton passes from the state $|x|=i$, $(i=l,l+1,\dots,l+m-1)$ to the state $|x|=l-1$, and from state $|x|=i$, $(i=2,3,\dots,l-1)$ - in the state $|x|=i-1$; the state $x=1$ goes in the state $x=-l$, and the state $x=-1$ - to the state $x=l$. At the signal $s=0$ (indifference), the automaton passes from the state $|x|=i$ $(i=1,2,\dots,l)$ to the state $|x|=l+1$, and from the state $|x|=i$ $(i=l+1,\dots,l+m-2)$ - in the state $|x|=i+1$; the state $x=l+m-1$ goes to the state $x=-l$, and the state $x=-(l+m\text{-}1)$ - to the state $x=l$ (Fig.1).

Thus, the automaton $T_{2n,2}(l,m)$ has one input and two outputs: the input state in the area L_α, $\alpha=1,2$ is the state with the number $|x|=l$, and the output states are states with numbers $|x|=1$ and $|x|=l+m-1$.

It is easy to see that the automaton $T_{2n,2}(l,m)$ changes the action if its input receives a penalty of length l or indifference of length m and it is an automaton analogue in the ternary stationary random environment of the known statistical rule from the theories of recurrent events: "Either a success series of length l, or a series of failures of length m " [10]. We note that an automaton realization of this rule in a binary stationary random environment was considered in [11].

To study the possible behavior of an automaton in a stationary random environment $C(a_1,r_1;a_2,r_2;\dots;a_k,r_k)$, are initial the following statistical characteristics of the behavior [4]: the probabilities σ_α change (ever) the action f_α and the mathematical expectations of a random time τ_α before the change of action f_α by at starting from the state $x\in L_\alpha$, $\alpha=\overline{1;k}$.

We denote by $u_{x,d}^{(n)}$ the probability that the automaton $T_{2n,2}(l,m)$ at the instant d changes for the first time the action f_α, starting from any state with the number x of the domain L_α.

Taking into account the behavior of the automaton $T_{2n,2}(l,m)$ in the stationary random environment $C(a_1,r_1;a_2,r_2)$, with respect to the probabilities $u_{x,d}^{(n)}$, we obtain the following difference equation

$$u_{x,d}^{(n)}=pu_{x-1,d}^{(n)}+ru_{l+1,d}^{(n)}+qu_{l,d}^{(n)},\quad x=1,2,\dots l, \tag{1}$$

$$u_{x,d}^{(n)}=pu_{l-1,d}^{(n)}+ru_{x+1,d}^{(n)}+qu_{l,d}^{(n)},\qquad x=l+1,\dots,l+m-1, \tag{2}$$

$$d=0,1,2,\ \ .\ \ .\ \ .$$

and the boundary conditions arising from the probabilistic meaning of $u_{x,d}^{(n)}$

$$u_{0,0}^{(n)}=1,\ u_{l+m,0}^{(n)}=1,\ u_{x,0}^{(n)}=0\quad \forall x\neq 0,l+m. \tag{3}$$

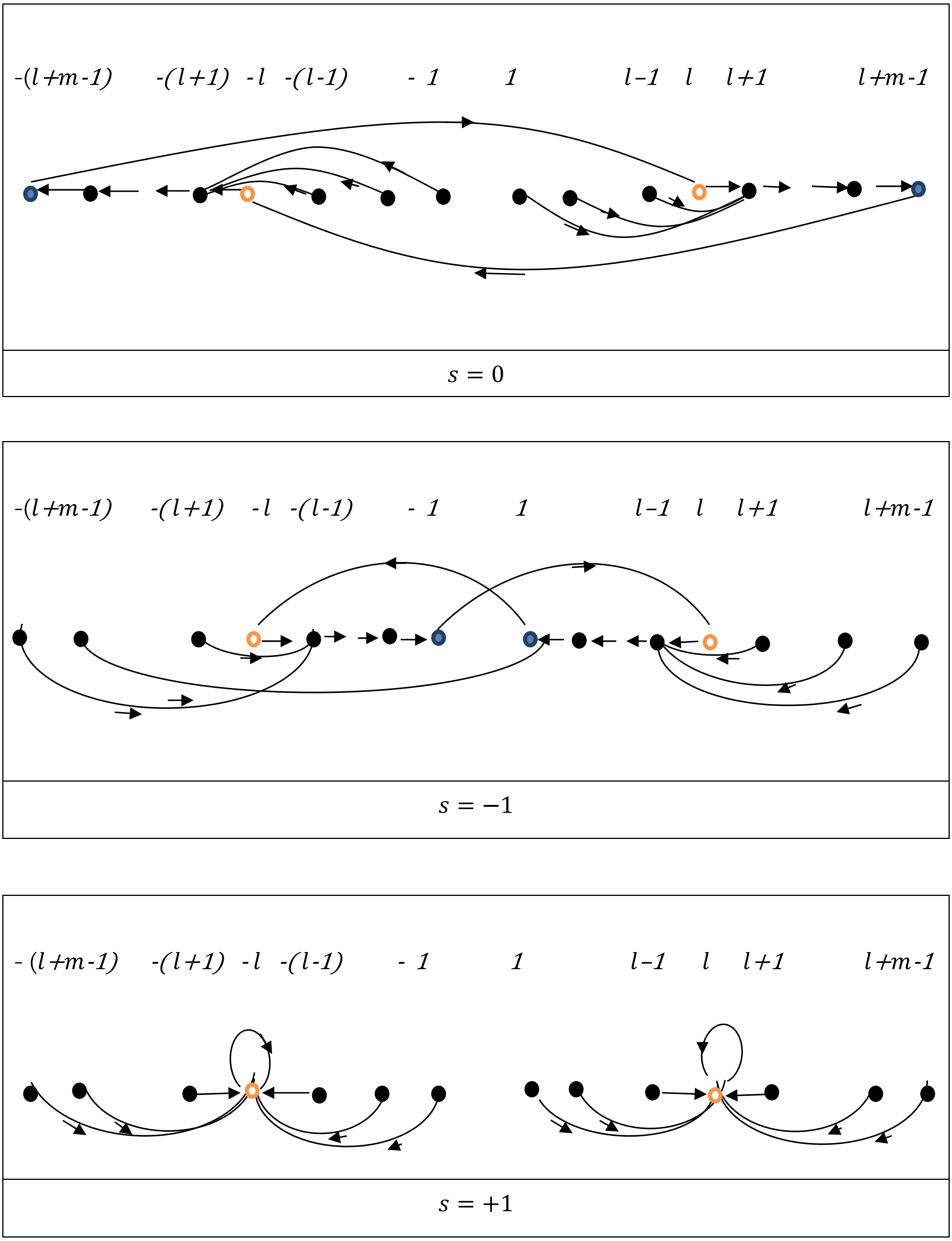

Fig.1. The graph of transitions between the states of the automaton $T_{2n,2}(l,m)$ in the area $L_{\alpha}, \alpha = 1,2$ at the signal $s = 0,\ \ s = -1,\ \ s = +1.$

Relative to the generating function of the probability of a action change

$$U_x^{(n)}(z) = \sum_{d=0}^{\infty} u_{x,d}^{(n)} z^d,$$

from (1) - (3) we obtain the boundary value problem

$$U_x^{(n)}(z) = pzU_{x-1}^{(n)}(z) + rzU_{l+1}(z) + qzU_l^{(n)}(z), \quad x = 1,2, \ldots, l, \tag{4}$$

$$U_{l+x}^{(n)}(z) = rzU_{l+x+1}(z) + pzU_{l-1}^{(n)}(z) + qzU_l^{(n)}(z), \quad x = 1, \ldots, m-1, \tag{5}$$

$$U_0^{(n)}(z) = 1\,, \ U_{l+m}^{(n)}(z) = 1. \tag{6}$$

From (4) - (6) we finally obtain that for the generating function $U_x^{(n)}(z)$ the action changes of the automaton $T_{2n,2}(l,m)$

$$U_l^{(n)}(z) = \frac{(1-pz)(pz)^l[1-(rz)^m] + (1-rz)(rz)^m[1-(pz)^l]}{1-z+(pz+qz)(rz)^m+(qz+rz)(pz)^l-(1+qz)(rz)^m(pz)^l}.$$

The functioning of the finite automaton $T_{2n,2}(l,m)$ in the environment $C(a_1, r_1; a_2, r_2)$ is described by a finite homogeneous ergodic Markov chain. For finite automata the probabilities $\sigma_\alpha^{(n)}$ of the change of the action f_α are equal to one, and the mean times $\tau_\alpha^{(n)}$ are finite in any non-degenerate ($|\, a_\alpha \,| \neq 1 - r_\alpha$) environment $C(a_1, r_1; a_2, r_2)$. Consequently, according to [4], the optimality of the behavior of a finite automaton is excluded and the quality of its behavior is determined by the degree of expediency of its functioning.

By the definition of [1-4], the automaton A_k has a expediency behavior in a stationary random environment $C(a_1, r_1; a_2, r_2; \ldots; a_k, r_k)$, if

$$M(A;C) > M_0, \tag{7}$$

where $M(A_k;C)$ is calculated by the formula

$$M(A_k;C) = \frac{\sum_{i=1}^{k} a_i \tau_i^{(n)}}{\sum_{i=1}^{k} \tau_i^{(n)}}, \tag{8}$$

and

$$M_0 = \frac{1}{k}\sum_{i=1}^{k} a_i$$

is the mathematical expectation of winning if the automaton with equal probability chooses its actions regardless of the reaction of the environment. Note that if $M(A_k;C) = M_0$, then the automaton is indifferent, and if $M(A_k;C) < M_0$ - is inexpedient.

Taking into account (7) and (8), for the expediency behavior of the automaton $T_{2n,2}(l,m)$ in the environment $C(a_1, r_1; a_2, r_2)$ the following condition must be fulfilled: $\tau_1^{(n)} > \tau_2^{(n)}$.

Accordingly, at $\tau_1^{(n)} = \tau_2^{(n)}$ the behavior of the automaton is indifferent, at $\tau_1^{(n)} < \tau_2^{(n)}$ - the behavior of the automaton is inexpedient.

For the automaton $T_{2n,2}(l,m)$ the statistical characteristics of the behavior - $\sigma_{l,\alpha}^{(n)}$ and $\tau_{l,\alpha}^{(n)}$ are calculated using the generating function:

$$\sigma_{l,\alpha}^{(n)} = U_l^{(n)}(z)\,|_{z=1} = 1,$$

$$\tau_{l,\alpha}^{(n)} = \frac{dU_l^{(n)}(z)}{dz}\,\Big|_{z=1} = \frac{(1-p_\alpha^l)(1-r_\alpha^m)}{(1-p_\alpha)p_\alpha^l+(1-r_\alpha)r_\alpha^m-(1+q_\alpha)p_\alpha^l r_\alpha^m} < \infty\,,\ \alpha = 1,2.$$

In the particular case, at $l = m = 1$, the behavior of the automaton $T_{2n,2}(1,1)$ in the environment $C(a_1, r_1; a_2, r_2)$:

- expedient, if $q_1 > q_2$;
- indifferent, if $q_1 = q_2$;
- inadvisable, if $q_1 < q_2$.

The functioning of the infinite automaton $T_2(l,m)$ in a ternary stationary random environment

Consider we now the functioning of the infinite (with a countable number of states) analog $T_2(l,m)$ of the automaton $T_{2n,2}(l,m)$ in the stationary random environment $C(a_1, r_1; a_2, r_2)$, the subsets of the states L_α $(\alpha = 1,2)$ of which are equally powerful.

Let $u_{x,d}$ be the probability that the automaton $T_2(l,m)$ at the instant of time d first changes the action of f_α, starting from any state with the number x of the area L_α.

Assume that l is fixed and $m \to \infty$ $(n = l + m - 1 \to \infty)$.

Then, taking into account the probabilistic meaning of the quantity $u_{x,d}$ and the construction of the infinite automaton $T_2(l,\infty)$, with respect to the generating function of the probability of changing the action

$$U_x(z) = \sum_{d=0}^{\infty} u_{x,d}\, z^d$$

we have the boundary value problem

$$\begin{gathered} U_x^{(n)}(z) = pzU_{x-1}^{(n)}(z) + (rz+qz)U_l^{(n)}(z), \quad x = 1,2,\dots,l, \\ U_0^{(n)}(z) = 1. \end{gathered} \tag{9}$$

The solution to this problem is:

$$U_l(z) = \frac{(1-pz)(pz)^l}{1-z+(qz+rz)(pz)^l}. \tag{10}$$

With the help of (10), the probability characteristics $\sigma_{l,\alpha}$ and τ_α are computed:

$$\sigma_{l,\alpha} = U_l(1) = 1, \qquad \tau_{l,\alpha} = \frac{dU_l(z)}{dz}\Big|_{z=1} = \frac{(1-p_\alpha^l)}{(1-p_\alpha)p_\alpha^l} < \infty, \ \alpha = 1,2.$$

Now let m be fixed and $l \to \infty$ $(n = l + m - 1 \to \infty)$. Then, renumbering the states of the automaton in the reverse order, it is easy to verify that the generating function of the probability of changing the action is a solution of the boundary value problem (9), if in it we replace l by m, p by r and r by p.

The solution obtained has the following form

$$U_m(z) = \frac{(1 - rz)(rz)^m}{1 - z + (qz + pz)(rz)^m}$$

and

$$\sigma_{l,\alpha} = U_m(1) = 1, \qquad \tau_{l,\alpha} = \frac{dU_m(z)}{dz}\Big|_{z=1} = \frac{(1-r_\alpha^m)}{(1-r_\alpha)r_\alpha^m} < \infty, \ \alpha = 1,2.$$

If $l \to \infty$ and $m \to \infty$, then the infinite automaton forever remain in the subset of states in which it was at the initial instant of time. In this case $U_l(z) = 0$ and $\sigma_{l,\alpha} = 0$, $\tau_{l,\alpha} = \infty$.

Thus

$$\lim_{m\to\infty} U_l^{(n)}(z) = U_l(z), \qquad \lim_{l\to\infty} U_l^{(n)}(z) = U_m(z), \qquad \lim_{\substack{l\to\infty \\ m\to\infty}} U_l^{(n)}(z) = U_l(z) = 0.$$

Thus, the sequence of finite automata $\{T_{2n,2}(l,m)\}_{l=1}^{\infty}$, $\{T_{2n,2}(l,m)\}_{m=1}^{\infty}$ and $\{T_{2n,2}(l,m)\}_{l,m=1}^{\infty}$ converges to the corresponding infinite automata $T_2(\infty, m)$, $T_2(l, \infty)$ and $T_2(\infty,\infty)$ for the same structure and, consequently, the asymptotic behavior of the finite automaton $T_{2n,2}(l,m)$ under consideration is defined, as in [4], by the behavior of the corresponding infinite automaton $T_2(l,m)$.

Conclusion

Analysis of the results obtained, taking into account the condition $a_1 > a_2$, makes it possible to draw the following conclusion.

1. The behavior of the infinite automaton $T_2(\infty,\infty)$ in the environment $C(a_1, r_1; a_2, r_2)$ is indifferent.
2. The behavior of the infinite automaton $T_2(\infty, m)$ in the environment $C(a_1, r_1; a_2, r_2)$ is:
   - expedient, if $r_1 < r_2$;
   - inexpedient, if $r_1 > r_2$;
   - indifferent, if $r_1 = r_2$;
3. The behavior of the infinite automaton $\mathrm{T}_2(\mathrm{l}, \infty)$ in the environment $\mathrm{C}(\mathrm{a}_1, \mathrm{r}_1; \mathrm{a}_2, \mathrm{r}_2)$ is:
   - expedient, if $\mathrm{p}_1 < \mathrm{p}_2$;
   - inexpedient, if $\mathrm{p}_1 > \mathrm{p}_2$;
   - indifferent, if $\mathrm{p}_1 = \mathrm{p}_2$.

Bibliography

[1] M.L.Tsetlin. On the behavior of finite automata in random environments. Automation and Telemechanics, N10, 1961.

[2] V.G. Sragovich. Theory of adaptive systems. M, Nauka,1976.

[3] V.I.Varshavsky. Collective behavior of automata. M, Science, 1973.

[4] V.S. Korolyuk, A.I. Pletnev, S.D. Eidelman. Automata. Walks. Games. Successes of Mathematical Sciences. T. 43, issue 1 (259), 1988.

[5] E.I. Paltsev. Behavior of finite automata at three types of reaction of a stationary random environment. Problems of information transfer. 1975, Vol. 11, N3.

[6] T. Khvedelidze, G.Tsertsvadze. Analysis of the behavior of the stochastic finite automaton in a ternary stationary random environment. http//gesj. Internet-academy. Org.ge N2(46), 2015.

[7] E.N. Vavilov, S.D. Eidelman, A.I. Ezrokhi. Asymptotic behavior of stochastic automata in stationary random environments. Cybernetics. 1977, N5.

[8] V.A. Andryushchenko, E.N. Vavilov, L.P. Lobanov. Synthesis of automata asymptotically optimals in stationary random environments. Cybernetics, 1972, Vol.1.

[9] T. Khvedelidze, G.Tsertsvadze. On a problem of synthesis of optimal constructions of automata. Message of the Georgian Academy of Sciences, 145, N1, 1992.

[10] V. Feller. Introduction to probability theory and its applications. T.1, Moscow, Mir, 1967.

[11] T. Khvedelidze, I. Aslanishvili. On one Construction of a Finite Automaton. Proceedings of the EUROPEAN COMPUTING CONFERENCE (ECC′ 09) Tbilisi, Georgia, iune26-28, 2009.

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