On Multiple Hypotheses LAO Testing With Liberty of Rejection of Decision for Two Independent Objects

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Abstract: *Multiple statistical hypotheses testing with possibility of rejecting of decision for discrete independent observations is investigated for models consisting of two independent objects. The matrix of optimal asymptotical interdependencies of possible pairs of the error probability exponents (reliabilities) is studied.*

Keywords: *Multiple statistical hypotheses, Optimal tests, Rejection option, Two objects*

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Introduction

For an asymptotically optimal test the probability of error decreases exponentially when the number of observations tends to infinity. Such tests for two hypotheses were studied by Hoeffding [Hoeffding, 1965], Csiszár and Longo [Csiszár and Longo, 1971], Tusnady [Tusnády, 1977], Longo and Sgarro [Longo and Sgarro, 1980]. Following Birgé [Birgé, 1981] we called the sequence of such tests logarithmically asymptotically optimal (LAO). Haroutunian [Haroutunian, 1988; Haroutunian, 1990] investigated the problem of multiple hypotheses LAO testing.

This paper is devoted to study of characteristics of logarithmically asymptotically optimal (LAO) hypotheses testing with possibility of rejection of decision for the model consisting of two independent objects. In publications [Ahlswede and Haroutunian, 2006]– [Haroutunian and Hakobyan, 2013] many hypotheses LAO testing for the model consist- ing of many independent objects was studied. The multiple hypotheses testing problem with possibility of rejection of decision for arbitrarily varying object with side information was examined by Haroutunian, Hakobyan and Yessayan [Haroutunian, Hakobyan and Yessayan, 2011a; Haroutunian, Hakobyan and Yessayan, 2011b]. Diverse formulations of hypotheses testing with no-match decision were considered by Hellman and Raviv [Hellman and Raviv, 1970] and Gutman [Gutman, 1989].

Our study is based on information theoretic methods. Applications of information theory in mathematical statistics, specifically in hypotheses testing, are exposed also in the monographs by Csiszár and Körner [Csiszár and Körner, 1981], Blahut [Blahut, ,1987], Cover and Thomas [Cover and Thomas, 2006], Csiszár and Shields [Csiszár and P. Shields, 2004].

Problem statement and formulation of results

Rejection of decision is allowed to one of the objects

Let $\mathcal{P}(\mathcal{X})$ be the space of all probability distributions (PDs) on finite set $\mathcal{X}$. Let X_1 and X_2 be independent RVs taking values in the same finite set $\mathcal{X}$ with one of M PDs $G_m \in \mathcal{P}(\mathcal{X})$, $m = \overline{1,M}$. These RVs are the characteristics of the corresponding independent objects. The random vector (X_1, X_2) assumes values $(x^1, x^2) \in \mathcal{X} \times \mathcal{X}$. Let $(\mathbf{x}^1, \mathbf{x}^2) \stackrel{\triangle}{=} \big((x_1^1, x_1^2), ..., (x_n^1, x_n^2), ..., (x_N^1, x_N^2)\big)$, $x_n^1, x_n^2 \in \mathcal{X}$, $n = \overline{1,N}$, be a vector of results of N independent observations of the vector (X_1, X_2) called sample.

The test has to determine unknown PDs of the objects from the set of hypotheses $H_m : G = G_m, m = \overline{1,M}$ on the base of sample $(\mathbf{x}^1, \mathbf{x}^2)$.

We call this procedure the compound test for two objects and denote it by Φ_N, it can be composed of two individual tests φ_N^1, φ_N^2 for each of two objects.

The test φ_N^1, can be defined by division of the space $\mathcal{X}^N$ into M disjoint subsets $\mathcal{A}_m^1$, $m = \overline{1, M}$, where $\mathcal{A}_m^1$ contains all vectors $\mathbf{x}^1$ for which the hypothesis H_m is adopted.

The test φ_N^2, can be defined by division of the space $\mathcal{X}^N$ into $M+1$ disjoint subsets: $\mathcal{A}_m^2$, $m = \overline{1, M}$, containing vectors $\mathbf{x}^2$ for which the hypothesis H_m is adopted and subset $\mathcal{A}_{M+1}^2$ containing all vectors $\mathbf{x}^2$ for which certain answer is refused. Hence Φ_N is division of the space $\mathcal{X}^N \times \mathcal{X}^N$ into $M \times (M+1)$ subsets $\mathcal{A}_{m_1,m_2} = \mathcal{A}_{m_1}^1 \times \mathcal{A}_{m_2}^2$, $m_1 = \overline{1, M}$, $m_2 = \overline{1, M+1}$. We denote the infinite sequence of compound tests by $\Phi = (\varphi^1, \varphi^2)$.

Let $\alpha_{l_1,l_2|m_1,m_2}(\Phi_N)$ be the probability of the erroneous acceptance of the hypotheses (H_{l_1}, H_{l_2}) by the test Φ_N provided that hypotheses (H_{m_1}, H_{m_2}) are true, where $(m_1, m_2) \neq (l_1, l_2)$, $m_1, m_2, l_1, l_2 = \overline{1, M}$:

$$\alpha_{l_1,l_2|m_1,m_2}(\Phi_N) = G_{m_1}^N(\mathcal{A}_{l_1}^1) \cdot G_{m_2}^N(\mathcal{A}_{l_2}^2). \tag{1}$$

When hypotheses H_{m_1}, H_{m_2} are true, but we decline the decision concerning to hypotheses regarding the second object the corresponding probability of error is:

$$\alpha_{l_1,M+1|m_1,m_2}(\Phi_N) = G_{m_1}^N(\mathcal{A}_{l_1}^1) \cdot G_{m_2}^N(\mathcal{A}_{M+1}^2),\ l_1 = \overline{1, M}, l_1 \neq m_1. \tag{2}$$

The probability not to accept a true pair of hypotheses (H_{m_1}, H_{m_2}), $m_1, m_2 = \overline{1, M}$ is the following:

$$\alpha_{m_1,m_2|m_1,m_2}(\Phi_N) = \sum_{(l_1,l_2)\neq(m_1,m_2),\, l_1=\overline{1,M},\, l_2=\overline{1,M+1}} \alpha_{l_1,l_2|m_1,m_2}(\Phi_N). \tag{3}$$

We study reliabilities (error probability exponents) $E_{l_1,l_2|m_1,m_2}(\Phi)$ corresponding to the sequence of tests Φ,

$$E_{l_1,l_2|m_1,m_2} = E_{l_1,l_2|m_1,m_2}(\Phi) \triangleq \lim_{N\to\infty}\left(-\frac{1}{N}\log \alpha_{l_1,l_2|m_1,m_2}(\Phi_N)\right),$$

$$m_1, m_2, l_1 = \overline{1, M}, l_2 = \overline{1, M+1}. \tag{4}$$

Definitions (2) and (3) imply that

$$E_{m_1,m_2|m_1,m_2}(\Phi) \triangleq \min_{(l_1,l_2)\neq(m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi),\ m_1, m_2, l_1 = \overline{1, M}, l_2 = \overline{1, M+1}. \tag{5}$$

The reliability martrix $\mathbf{E}(\Phi)$ has $M \times M$ lines and $M \times (M+1)$ columns, for the simple case, when $M = 2$ it is the following:

$$\mathbf{E}(\Phi) = \begin{pmatrix} E_{1,1|1,1} & E_{1,2|1,1} & E_{2,1|1,1} & E_{2,2|1,1} & E_{1,3|1,1} & E_{2,3|1,1} \\ E_{1,1|1,2} & E_{1,2|1,2} & E_{2,1|1,2} & E_{2,2|1,2} & E_{1,3|1,2} & E_{2,3|1,2} \\ E_{1,1|2,1} & E_{1,2|2,1} & E_{2,1|2,1} & E_{2,2|2,1} & E_{1,3|2,1} & E_{2,3|2,1} \\ E_{1,1|2,2} & E_{1,2|2,2} & E_{2,1|2,2} & E_{2,2|2,2} & E_{1,3|2,2} & E_{2,3|2,2} \end{pmatrix}.$$

We call the test sequence Φ^* LAO for the model with two objects if for given positive values of certain part of elements of the reliability matrix $\mathbf{E}(\Phi^*)$ the procedure Φ^* provides maximal values for all other elements of it.

We must consider also error probabilities and reliabilities of tests sequences for two seperate objects for $i = \overline{1, 2}$, $m_1, m_2, l_1 = \overline{1, M}, l_2 = \overline{1, M+1}$:

$$\alpha_{l_i|m_i}^i = \alpha_{l_i|m_i}(\varphi_N^i) \triangleq G_{m_i}^N(\mathcal{A}_{l_i}^i),$$

$$E_{l_i|m_i}^i = E_{l_i|m_i}(\varphi^i) \triangleq \lim_{N\to\infty}\left(-\frac{1}{N}\log \alpha_{l_i|m_i}(\varphi_N^i)\right).$$

Consider the following reliability matrices of tests sequences φ^1 and φ^2 for separate objects respectively:

$$\mathbf{E}(\varphi^1) = \begin{pmatrix} E_{1|1}^1 \ \ldots\ E_{l|1}^1 \ \ldots E_{M|1}^1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ E_{1|m}^1 \ldots E_{l|m}^1 \ldots E_{M|m}^1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ E_{1|M}^1 \ldots E_{l|M}^1 \ldots E_{M|M}^1 \end{pmatrix},$$

$$\mathbf{E}(\varphi^2) = \begin{pmatrix} E^2_{1|1} \ \dots \ E^2_{l|1} \ \dots \ E^2_{M|1} \ E^2_{M+1|1} \\ \dots\dots\dots\dots\dots\dots\dots \\ E^2_{1|m} \dots E^2_{l|m} \dots E^2_{M|m} \ E^2_{M+1|m} \\ \dots\dots\dots\dots\dots\dots\dots \\ E^2_{1|M} \dots E^2_{l|M} \dots E^2_{M|M} \ E^2_{M+1|M} \end{pmatrix}.$$

The following lemma establishes relations of reliabilities of two separate objects and of the paired complex object.

Lemma 1: *The following equalities hold for elements of* $\mathbf{E}(\Phi)$ for $m_1, m_2, l_1 = \overline{1, M}, l_2 = \overline{1, M+1}$

$$E_{l_1,l_2|m_1,m_2}(\Phi) = \sum_{i=1}^{2} E^i_{l_i|m_i}, \quad \text{if} \quad m_i \neq l_i, \quad i = \overline{1,2}, \tag{6}$$

$$E_{l_1,l_2|m_1,m_2}(\Phi) = \begin{cases} E^1_{l_1|m_1}, & \text{if} \quad m_1 \neq l_1, \ m_2 = l_2, \\ E^2_{l_2|m_2}, & \text{if} \quad m_1 = l_1, \ m_2 \neq l_2. \end{cases} \tag{7}$$

Proof: It follows from the independence of the objects that

$$\alpha_{l_1,l_2|m_1,m_2} = \prod_{i=1}^{2} \alpha^i_{l_i|m_i}, \quad \text{if} \quad m_1 \neq l_1, \ m_2 \neq l_2,, \tag{8}$$

$$\alpha_{l_1,l_2|m_1,m_2}(\Phi_N) = \left(1 - \alpha^i_{l_k|m_k}\right) \alpha^i_{l_i|m_i}, \text{ if } m_k = l_k, \ m_i \neq l_i, \ i, k = \overline{1,2}, \ \ i \neq k. \tag{9}$$

Here we consider not only error probabilities but also the probabilities of correct decisions. Since the error probabilities $\alpha^i_{l_i|m_i}$ are less than 1, we have

$$\lim_{N\to\infty} \left(-\frac{1}{N} \log \left(1 - \alpha^i_{l_i|m_i} \right) \right) = 0. \tag{10}$$

From definitions (3) and (4), equalities (8) and (9), applying (10) we obtain relations (6) and (7). Lemma is proved.

Remark: It follows from (7) that each M elements of the matrix $E(\Phi)$ when $m_1 = l_1, \ m_2 \neq l_2$ and $m_1 \neq l_1, \ m_2 = l_2$ are equal each other. Consequently the number of different values of elements of $E(\Phi)$ is far less than $M^3(M+1)$.

We define the divergence (Kullback-Leibler distance) $D(Q||G)$ for PDs Q and G from $\mathcal{P}(\mathcal{X})$ as usual:

$$D(Q||G) = \sum_x Q(x) \log \frac{Q(x)}{G(x)}.$$

Now we shall show how we can construct the LAO test from the set of compound tests when $2M-1$ positive elements of the reliability matrix $E_{M,m|m,m}$, $m = \overline{1, M-1}$ and $E_{m,M+1|m,m}$, $m = \overline{1, M}$, are preliminarily given. The following subset of tests:

$$\mathcal{D} = \{\Phi = (\varphi^1, \varphi^2) : E^i_{m|m} > 0, \ m = \overline{1, M}, \ i = \overline{1, 2}\}$$

is distinguished by the property that when $\Phi \in \mathcal{D}$ the elements $E_{M,m|m,m}$ and $E_{m,M+1|m,m}$, $m = \overline{1, M}$, of its reliability matrix are strictly positive.

Really, because $E^i_{m|m} > 0$, then $E^1_{M|m}$ and $E^2_{M+1|m}$ are also strictly positive. From (5) we obtain that for $m = \overline{1, M}$,

$$E_{M,m|m,m}(\Phi) = E^1_{M|m}, \tag{11}$$

$$E_{m,M+1|m,m}(\Phi) = E^2_{M+1|m}. \tag{12}$$

Let us define the following family of decision sets of PDs for given positive elements $E_{M,m|m,m}$ and $E_{m,M+1|m,m}$,

$$\mathcal{R}_m^{(1)} \triangleq \{Q : D(Q||G_m) \leq E_{M,m|m,m}\}, \; m = \overline{1, M-1} \tag{13}$$

$$\mathcal{R}_M^{(1)} \triangleq \{Q : D(Q||G_m) > E_{M,m|m,m}, \; m = \overline{1, M-1}\}, \tag{14}$$

$$\mathcal{R}_m^{(2)} \triangleq \{Q : D(Q||G_m) \leq E_{m,M+1|m,m}\}, \; m = \overline{1, M}, \tag{15}$$

$$\mathcal{R}_{M+1}^{(2)} \triangleq \{Q : D(Q||G_m) > E_{m,M+1|m,m}, \; m = \overline{1, M}\}. \tag{16}$$

Define also the elements of the reliability matrix of the LAO test for the model:

$$E^*_{M,m|m,m} \triangleq E_{M,m|m,m}, \quad m = \overline{1, M-1} \tag{17}$$

$$E^*_{m,M+1|m,m} \triangleq E_{m,M+1|m,m}, \quad m = \overline{1, M}. \tag{18}$$

$$E^*_{l_1,l_2|m_1,m_2} \triangleq \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),$$

$$l_1 = \overline{1, M-1}, \; l_2 = \overline{1, M}, m_1, m_2 = \overline{1, M}, \;\; m_k = l_k, \; m_i \neq l_i, \;\; i \neq k, i, k = \overline{1, 2}. \tag{19}$$

$$E^*_{l_1,l_2|m_1,m_2} \triangleq \sum_{i=1}^{2} \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}), \quad m_i \neq l_i, i = \overline{1, 2}. \tag{20}$$

$$E^*_{m_1,m_2|m_1,m_2} \triangleq \min_{(l_1,l_2) \neq (m_1,m_2)} E^*_{l_1,l_2|m_1,m_2}. \tag{21}$$

The following theorem is the main result of the present paper.

Theorem 1: *As all distributions G_m, $m = \overline{1, M}$, are different, (and equivalently $D(G_l||G_m) > 0$, $l \neq m$, $l, m = \overline{1, M}$), the following statements are valid:*

a) when given strictly positive elements $E_{M,m|m,m}$ and $E_{m,M+1|m,m}$, $m = \overline{1, M}$, meet the following conditions

$$E_{M,1|1,1} < \min_{l=\overline{2,M}} D(G_l||G_1), \tag{22}$$

$$E_{1,M+1|1,1} < \min_{l=\overline{2,M}} D(G_l||G_1), \tag{23}$$

$$E_{M,m|m,m} < \min \left[\min_{l=\overline{1,m-1}} E^*_{l,m|m,m} \;\; \min_{l=\overline{m+1,M}} D(G_l||G_m) \right], \; m = \overline{2, M-1}, \tag{24}$$

$$E_{m,M+1|m,m} < \min \left[\min_{l=\overline{1,m-1}} E^*_{m,l|m,m}, \;\; \min_{l=\overline{m+1,M}} D(G_l||G_m) \right], \; m = \overline{2, M}, \tag{25}$$

then there exists a LAO test sequence $\Phi^* \in \mathcal{D}$, the reliability matrix of which $\mathbf{E}(\Phi^*) = \left\{E_{l_1,l_2|m_1,m_2}(\Phi^*)\right\}$ is defined in (17) – (21) and all elements of it are positive,

b) when even one of the inequalities (22)-(25) is violated, then there exists at least one element of the matrix $\mathbf{E}(\Phi^*)$ equal to 0.

Proof: Leaning upon Lemma 1 we want to construct the required test Φ^* for the compound object by considering LAO tests $\varphi^{1*}, \varphi^{2*}$ for two separate objects.

We will apply Theorem 7.3 from [Haroutunian, Haroutunian and Harutyunyan, 2008] (it was early published in [Haroutunian, 1988] and [Haroutunian, 1990]) conserning to many hypotheses LAO testing and Theorem 2 from [Haroutunian, Hakobyan and Yessayan, 2011a] for multiple hypotheses testing case with rejection option.

Our aim is to prove that the test $\Phi^* = \left(\varphi^{1,*}, \varphi^{2,*}\right)$, where $\varphi^{i,*}$, $i = \overline{1,2}$ are LAO tests for objects X_i, belongs to the set $\mathcal{D}$ and is a LAO test.

Let the test Φ^* is such that $E_{M,m|m,m}(\Phi^*) = E_{M,m|m,m}$, $m = \overline{1, M-1}$ and $E_{m,M+1|m,m}(\Phi^*) = E_{m,M+1|m,m}$, $m = \overline{1, M}$.

We shall prove that conditions (22) – (25) imply that inequalities analogous to compatibility conditions of Theorem 7.3 in [Haroutunian, Haroutunian and Harutyunyan, 2008] and of Theorem 2 in [Haroutunian, Hakobyan and Yessayan, 2011a] hold simultaneously for the LAO tests $\varphi^{1,*}$, $\varphi^{2,*}$ for two separate objects.

Really, taking into account (11) and (12) we can see that conditions (22) – (25) may be replaced by the following inequalities:

$$E_{M|m}(\varphi^{1,*}) < \min \left[\min_{l=\overline{1,m-1}} \inf_{Q:D(Q||G_m) \le E_{M|l}(\varphi^{1,*})} D\left(Q||G_l\right), \min_{l=\overline{m+1,M}} D(G_l||G_m) \right],$$

$$m = \overline{1, M-1}. \tag{26}$$

$$E_{M+1|m}(\varphi^{2,*}) < \min \left[\min_{l=\overline{1,m-1}} \inf_{Q:D(Q||G_m) \le E_{M+1|l}(\varphi^{2,*})} D\left(Q||G_l\right), \min_{l=\overline{m+1,M}} D(G_l||G_m) \right],$$

$$m = \overline{1, M}. \tag{27}$$

In the paper [Haroutunian, Hakobyan and Yessayan, 2011a] it was noted, that elements of the last column of the reliability matrix of LAO test coinside with diagonal elements of the same line and instead of diagonal ones they can be considered as given parameters for construction of the LAO test.

For tests $\varphi^{1,*}$ and $\varphi^{2,*}$ we can use corresponding elements $E_{M|m}$, $m = \overline{1, M-1}$ and $E_{M+1|m}$, $m = \overline{1, M}$. According to this fact we obtain that (26) and (27) meet compatibility conditions of Theorems from [Haroutunian, Haroutunian and Harutyunyan, 2008] and [Haroutunian, Hakobyan and Yessayan, 2011a]. Hence $\varphi^{i,*}$, $i = \overline{1,2}$ are LAO tests for the first and the second objects respectively.

It remains to show that Φ^* is a compound LAO test.

Let us consider each test $\Phi \in \mathcal{D}$, $E_{m_i|m_i}(\varphi^i) > 0$, $i = \overline{1,2}$. Provided that $E_{m_i|m_i}(\varphi^i) = \min\limits_{m_i \neq l_i} E_{m_i|l_i}(\varphi^i)$, all the elements $E_{m_i|l_i}(\varphi^i)$ are also strictly positive. For a test $\Phi \in \mathcal{D}$ assertions of Lemma 1 are fulfilled and the elements of the reliability matrix $\mathbf{E}(\Phi)$ coincide with elements of matrix $\mathbf{E}(\varphi^i)$, $i = \overline{1,2}$, or sums of them (see (6) – (7)). Then from definition of LAO test it follows that $E_{l_i|m_i}(\varphi^i) \le E_{l_i|m_i}(\varphi^{i,*})$, then $E_{l_1,l_2|m_1,m_2}(\Phi) \le E_{l_1,l_2|m_1,m_2}(\Phi^*)$. Consequently Φ^* is a LAO test and $E_{l_1,l_2|m_1,m_2}(\Phi^*)$ satisfy (22) – (25).

b) When even one of the inequalities $(22) - (25)$ is violated, then at least one of inequalities of Theorem 2 from [Haroutunian, Hakobyan and Yessayan, 2011a] is violated and it follows that this one of elements $E_{m_i|l_i}(\varphi^{i,*})$ is equal to zero. Suppose $E_{3|2}(\varphi^{1,*}) = 0$, then the elements $E_{3,m|2,m}(\Phi^*) = E_{3|2}(\varphi^{1,*}) = 0$.

Theorem 1 is proved.

Rejection of decision is allowed for both objects

Now let us study the case, when the rejection of decision for the first object is possible too. So in this case the test φ^1_N, will be defined by division of the space $\mathcal{X}^N$ into $M+1$ disjoint subsets $\mathcal{A}^1_m$, $m = \overline{1, M}$, where $\mathcal{A}^1_m$, $m = \overline{1, M}$, contains all vectors $\mathbf{x}_1$ for which the hypothesis H_m is adopted and $\mathcal{A}^1_{M+1}$ contains all vectors for which we refuse to take certain answer. In this case when hypotheses (H_{m_1}, H_{m_2}) are true, but we decline decision the corresponding probabilities of error will be:

$$\alpha_{l_1,M+1|m_1,m_2}(\Phi_N) = G^N_{m_1}(\mathcal{A}^1_{l_1}) \cdot G^N_{m_2}(\mathcal{A}^2_{M+1}),\ l_1 = \overline{1, M}, l_1 \neq m_1,$$

$$\alpha_{M+1,l_2|m_1,m_2}(\Phi_N) = G^N_{m_1}(\mathcal{A}^1_{M+1}) \cdot G^N_{m_2}(\mathcal{A}^2_{l_2}),\ l_2 = \overline{1,M}, l_2 \neq m_2.$$

$$\alpha_{M+1,M+1|m_1,m_2}(\Phi_N) = G^N_{m_1}(\mathcal{A}^1_{M+1}) \cdot G^N_{m_2}(\mathcal{A}^2_{M+1}).$$

So for $M = 2$ the matrix of reliabilities for two objects has this form

$$\mathbf{E}(\Phi) = \begin{pmatrix} E_{1,1|1,1} & E_{1,2|1,1} & E_{2,1|1,1} & E_{2,2|1,1} & E_{3,1|1,1} & E_{3,2|1,1} & E_{1,3|1,1} & E_{2,3|1,1} & E_{3,3|1,1} \\ E_{1,1|1,2} & E_{1,2|1,2} & E_{2,1|1,2} & E_{2,2|1,2} & E_{3,1|1,2} & E_{3,2|1,2} & E_{1,3|1,2} & E_{2,3|1,2} & E_{3,3|1,2} \\ E_{1,1|2,1} & E_{1,2|2,1} & E_{2,1|2,1} & E_{2,2|2,1} & E_{3,1|2,1} & E_{3,2|2,1} & E_{1,3|2,1} & E_{2,3|2,1} & E_{3,3|2,1} \\ E_{1,1|2,2} & E_{1,2|2,2} & E_{2,1|2,2} & E_{2,2|2,2} & E_{3,1|2,2} & E_{3,2|2,2} & E_{1,3|2,2} & E_{2,3|2,2} & E_{3,3|2,2} \end{pmatrix}$$

and the matrix of $\mathbf{E}(\varphi^1)$ will be like matrix $\mathbf{E}(\varphi^2)$ and Lemma 1 will be formulated as follows.

Lemma 2: *The following equalities hold for elements of* $\mathbf{E}(\Phi)$ for $m_1, m_2 = \overline{1,M},\ l_1, l_2 = \overline{1,M+1}$

$$E_{l_1,l_2|m_1,m_2}(\Phi) = \sum_{i=1}^{2} E^i_{l_i|m_i}, \quad \text{if} \quad m_i \neq l_i, \quad i = \overline{1,2},$$

$$E_{l_1,l_2|m_1,m_2}(\Phi) = \begin{cases} E^1_{l_1|m_1}, & \text{if} \quad m_1 \neq l_1,\ m_2 = l_2, \\ E^2_{l_2|m_2}, & \text{if} \quad m_1 = l_1,\ m_2 \neq l_2. \end{cases}$$

Redefine the following family of decision sets of PDs for given positive elements $E_{M+1,m|m,m}$ and $E_{m,M+1|m,m}$, $m = \overline{1,M}$,

$$\mathcal{R}^{(1)}_m \triangleq \{Q : D(Q||G_m) \leq E_{M+1,m|m,m}\},\ m = \overline{1,M}$$

$$\mathcal{R}^{(1)}_{M+1} \triangleq \{Q : D(Q||G_m) > E_{M,m|m,m},\ m = \overline{1,M}\},$$

$$\mathcal{R}^{(2)}_m \triangleq \{Q : D(Q||G_m) \leq E_{m,M+1|m,m}\},\ m = \overline{1,M},$$

$$\mathcal{R}^{(2)}_{M+1} \triangleq \{Q : D(Q||G_m) > E_{m,M+1|m,m},\ m = \overline{1,M}\}.$$

Define also the values of elements of the reliability matrix of the LAO test for the model:

$$E^*_{M+1,m|m,m} \triangleq E_{M+1,m|m,m}, \quad m = \overline{1,M} \tag{28}$$

$$E^*_{m,M+1|m,m} \triangleq E_{m,M+1|m,m}, \quad m = \overline{1,M}. \tag{29}$$

$$E^*_{l_1,l_2|m_1,m_2} \triangleq \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),$$

$$l_1,\ l_2 = \overline{1,M}, m_1, m_2 = \overline{1,M}, m_k = l_k,\ m_i \neq l_i, \quad i \neq k, i, k = \overline{1,2}. \tag{30}$$

$$E^*_{l_1,l_2|m_1,m_2} \triangleq \sum_{i=1}^{2} \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),\ m_i \neq l_i, i = \overline{1,2}. \tag{31}$$

$$E^*_{m_1,m_2|m_1,m_2} \triangleq \min_{(l_1,l_2) \neq (m_1,m_2)} E^*_{l_1,l_2|m_1,m_2}. \tag{32}$$

We formulate the theorem for this case

Theorem 2 [Haroutunian, Hakobyan and Yessayan, 2011b]: *If all distributions G_m, $m = \overline{1,M}$, are different, (and equivalently $D(G_l||G_m) > 0$, $l \neq m$, $l, m = \overline{1,M}$), then the following statements are valid:*

a) when given strictly positive elements $E_{M+1,m|m,m}$ and $E_{m,M+1|m,m}$, $m = \overline{1,M}$, meet the following conditions

$$\max(E_{M+1,1|1,1}, E_{1,M+1|1,1}) < \min_{l=\overline{2,M}} D(G_l||G_1), \tag{33}$$

$$E_{M+1,m|m,m} < \min\left[\min_{l=\overline{1,m-1}} E^*_{l,m|m,m}, \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \ m = \overline{2,M}, \tag{34}$$

$$E_{m,M+1|m,m} < \min\left[\min_{l=\overline{1,m-1}} E^*_{m,l|m,m}, \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \ m = \overline{2,M}, \tag{35}$$

then there exists a LAO test sequence $\Phi^* \in \mathcal{D}$, the reliability matrix of which $\mathbf{E}(\Phi^*) = \left\{E_{l_1,l_2|m_1,m_2}(\Phi^*)\right\}$ is defined in (28) – (32) and all elements of it are positive,

b) when even one of the inequalities $(33) - (35)$ is violated, then there exists at least one element of the matrix $\mathbf{E}(\Phi^*)$ equal to 0.

The proof of Theorem 2 is similar to proof of Theorem 1 with use of Lemma 2.

Conclusion

The problem of error probability exponents (reliabilities) investigation for optimal testing of many hypotheses with possibility of rejection of decision is solved for model of two independent objects. Two situations are considered when the rejection of decision is allowed to one of the objects and when the rejection of decision is allowed to both objects. In future, modifications of the problem in several directions can be studied.

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