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DUAL PROBLEM OF FUZZY PORTFOLIO OPTIMIZATION

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Abstract: The dual problem of fuzzy portfolio optimization is considered and investigated. A mathematical model of this problem was constructed, explored and the sufficient conditions for its convexity were obtained. The sufficient optimality conditions of its solution are presented.

Keywords: fuzzy portfolio, optimization, dual problem, optimality conditions

ACM Classification Keywords: H.4.2. Information system Applications: Types of Systems Decision Support

Introduction

Last years the problem of portfolio optimization attracts great attention due to the development of financial markets in Ukraine. The determining of optimal portfolio enables the investors and financial funds to distribute available financial assets, to obtain high dividends and cut the risk of erroneous solutions. A novel approach to portfolio optimization which is an alternative to classical portfolio model of Markovitz considers this problem under uncertainty and based on application of fuzzy sets theory.

The fuzzy portfolio problem was considered and investigated in [Недосекин, 2003, Зайченко, 2007]. In these works the problem statement was the following: to optimize the total portfolio profitableness under constraints on possible risk. The algorithm of its solution was suggested and explored in [Зайченко, 2007]. In [Зайченко, 2008] the forecasting of stock prices was suggested to utilize in fuzzy portfolio optimization which increased the efficiency of the solution.

The goal of the present work is the consideration of the dual portfolio problem, investigation of the constructed mathematical model and determining the optimality conditions of its solution.

The Dual Fuzzy Portfolio Problem

The initial portfolio optimization problem which is naturally to be called as direct has the following form [Недосекин, 2003, Зайченко, 2007] :

Optimize the expected profitableness of a portfolio

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \to \max \qquad (1)$$

under constraints on risk

$$\beta(x) \le \beta_{задан} \ 0 < \beta < 1, \tag{2}$$

$$\sum_{i=1}^{N} x_i = 1, \tag{3}$$

$$x_i \ge 0. \tag{4}$$

Let's consider the case when the criterion value r^* meets the conditions

$$\sum_{i=1}^{N} x_i r_{i1} \le r^* \le \sum_{i=1}^{N} x_i \tilde{r}_i = \tilde{r}. \tag{5}$$

Then

$$\beta(x) = \frac{1}{\sum_{i=1}^{N} x_i r_{i2} - \sum_{i=1}^{N} x_i r_{i1}} \left[\left(r^* - \sum_{i=1}^{N} x_i r_{i1} \right) + \left(\sum_{i=1}^{N} x_i \tilde{r}_i - r^* \right) \ln \left(\frac{\sum_{i=1}^{N} x_i \tilde{r}_i - r^*}{\sum_{i=1}^{N} x_i \tilde{r}_i - \sum_{i=1}^{N} x_i r_{i1}} \right) \right]. \tag{6}$$

Now consider the dual portfolio optimization problem dual related to the problem (1)-(4):

to minimize

$$\beta(x) \tag{7}$$

under conditions

$$\tilde{r} = \sum_{i=1}^{N} x_i \tilde{r}_i \ge r_{çŕä} = r^*, \tag{8}$$

$$\sum_{i=1}^{N} x_i = 1, x_i \ge 0. \tag{9}$$

Let's prove that the risk function $\beta(x)$ is convex where $\beta(x)=\left(A(x)+B(x)\ln\frac{B(x)}{C(x)}\right)-D(x)$.

It's necessary to prove that function $D(x)=\frac{1}{\sum_{i=1}^{N}x_i r_{i2}-\sum_{i=1}^{N}x_i r_{i1}}$ is convex and function

$A(x)+B(x)\ln\frac{B(x)}{C(x)}$ is also convex. And besides, both functions are decreasing and nonnegative by x_i,

where $A(x)=r^*-\sum_{i=1}^{N}x_i r_{i1}$; $B(x)=\sum_{i=1}^{N}x_i\tilde{r}_i-r^*$; $C(x)=\sum_{i=1}^{N}x_i\tilde{r}_i-\sum_{i=1}^{N}x_i r_{i1}$.

Really $A(x)$ is linear and therefore is not strictly convex and both $B(x)$ and $C(x)$ are linear.

Besides, $\tilde{r}_i \ge r_{i1}$, $i=\overline{1,N}$, $\sum_{i=1}^{N}x_i\tilde{r}_i-r^*>0$ by assumption

Consider the function $D(x)$:

$$D(x)=\frac{1}{\sum_{i=1}^{N}x_i r_{i2}-\sum_{i=1}^{N}x_i r_{i1}},$$

$$\frac{\partial D(x)}{\partial x_i}=-\frac{r_{i2}-r_{i1}}{(\sum_{i=1}^{N}x_i r_{i2}-\sum_{i=1}^{N}x_i r_{i1})^2},$$

$$\frac{\partial^2 D(x)}{\partial x_i^2}=\frac{2(r_{i2}-r_{i1})^2}{(\sum_{i=1}^{N}x_i r_{i2}-\sum_{i=1}^{N}x_i r_{i1})^3}>0.$$

And as $r_{i2}\ge r_{i1}$ then $\frac{\partial^2 D(x)}{\partial x_i^2}>0$, for all $i=\overline{1,N}$ and $\frac{\partial^2 D}{\partial x_i x_j}=\frac{2(r_{i2}-r_{i1})(r_{j2}-r_{j1})}{(\sum_{i=1}^{N}x_i r_{i2}-\sum_{i=1}^{N}x_i r_{i1})^3}$.

Besides, all the diagonal minors of the second order

$$\begin{vmatrix} \frac{\partial^2 D}{\partial x_i^2} & \frac{\partial^2 D}{\partial x_i x_j} \\ \frac{\partial^2 D}{\partial x_j x_i} & \frac{\partial^2 D}{\partial x_j^2} \end{vmatrix} = 0 .$$

Therefore function $D(x)$ is convex.

Calculate

$$\frac{\partial}{\partial x_i}\left[B(x)\ln\frac{B(x)}{C(x)}\right] = B'(x)\ln\frac{B(x)}{C(x)} + B(x)\frac{B(x)}{C(x)}\frac{B'(x)C(x)-C'(x)B(x)}{C^2(x)} =$$

$$= B'(x)\ln\frac{B(x)}{C(x)} + B'(x) - \frac{C'(x)B(x)}{C(x)} \tag{10}$$

But

$$C'(x) = \frac{\partial B(x)}{\partial x_i}(\tilde{r}_i - r_{i1}) > 0 ,$$

$$B'(x) = \frac{\partial B(x)}{\partial x_i} = \tilde{r}_i ,$$

and substituting in (10) obtain

$$\frac{\partial}{\partial x_i}\left[B(x)\ln\frac{B(x)}{C(x)}\right] = \tilde{r}_i \ln\frac{B(x)}{C(x)} + \tilde{r}_i - (\tilde{r}_i - r_{i1})\frac{B(x)}{C(x)}. \tag{11}$$

Find he second partial derivatives

$$\frac{\partial^2}{\partial x_i^2}\left[B(x)\ln\frac{B(x)}{C(x)}\right] = \tilde{r}_i \frac{C(x)}{B(x)}\frac{B'(x)C(x)-C'(x)B(x)}{C^2(x)} - (\tilde{r}_i - r_{i1})\frac{B'(x)C(x)-C'(x)B(x)}{C^2(x)} =$$

$$= \tilde{r}_i\left(\frac{B'(x)}{B(x)} - \frac{C'(x)}{C(x)}\right) - (\tilde{r}_i - r_{i1})\left[\frac{B'(x)}{C(x)} - \frac{C'(x)B(x)}{C^2(x)}\right] = \tag{12}$$

$$= \tilde{r}_i \left[\frac{\tilde{r}_i}{\sum_{i=1}^{N} x_i \tilde{r}_i - r^*} - \frac{\tilde{r}_i - r_{i1}}{\sum_{i=1}^{N} x_i \tilde{r}_i - \sum_{i=1}^{N} x_i r_{i1}} \right] -$$

$$- (\tilde{r}_i - r_{i1}) \left[\frac{\tilde{r}_i}{\sum_{i=1}^{N} x_i \tilde{r}_i - \sum_{i=1}^{N} x_i r_{i1}} - \frac{(\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)(\tilde{r}_i - r_{i1})}{(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2} \right] = \qquad (13)$$

$$= \frac{\tilde{r}^2}{\sum_{i=1}^{N} x_i \tilde{r}_i - r^*} - \frac{2\tilde{r}(\tilde{r}_i - r_{i1})}{\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1})} + \frac{(\tilde{r}_i - r_{i1})^2 (\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)}{(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2} = \qquad (14)$$

After transferring to common denominator we obtain

$$= \frac{\tilde{r}_i^2 (\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2 - 2\tilde{r}_i (\tilde{r}_i - r_{i1})(\sum_{i=1}^{N} x_i \tilde{r}_i - r^*) \sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1})}{(\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2} +$$

$$+ \frac{(\tilde{r}_i - r_{i1})^2 (\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)^2}{(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2 (\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)} =$$

$$= \frac{\left[\tilde{r}_i \left(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}) - (\tilde{r}_i - r_{i1})(\sum_{i=1}^{N} x_i \tilde{r}_i - r^*) \right) \right]^2}{(\sum_{i=1}^{N} x_i \tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i (\tilde{r}_i - r_{i1}^*))^2} \geq 0 \qquad (15)$$

As $\tilde{r}_i > (\tilde{r}_i - r_{i1})$ and $\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1}) > \sum_{i=1}^{N} x_i\tilde{r}_i - r^*$, the expression (15) is strictly greater than 0. Thus all partial derivatives of the second order $\Delta_{ii} = \frac{\partial^2}{\partial x_i^2}\left[B(x)\ln\frac{B(x)}{C(x)}\right] > 0$ and respectively $\frac{\partial^2}{\partial x_i^2}\left[A(x) + B(x)\ln\frac{B(x)}{C(x)}\right] > 0.$

Now it's necessary to show that all the diagonal minors of the following form

$$\begin{bmatrix} \Delta_{ii} & \Delta_{ij} \\ \Delta_{ji} & \Delta_{jj} \end{bmatrix} = \Delta_{ii}\Delta_{jj} - \Delta_{ij}\Delta_{ji} = \Delta_{ii}\Delta_{jj} - \Delta_{ji}{}^2 \geq 0. \tag{16}$$

These are sufficient convexity conditions of function $B(x)\ln\frac{B(x)}{C(x)}$, and therefore of the initial function $A(x) + B(x)\ln\frac{B(x)}{C(x)}$.

Find the mixed partial derivatives $\frac{\partial^2}{\partial x_i \partial x_i}\left[B(x)\ln\frac{B(x)}{C(x)}\right]$:

$$\frac{\partial^2}{\partial x_i \partial x_i}\left[B(x)\ln\frac{B(x)}{C(x)}\right] = \frac{\partial}{\partial x_j}\left(B'(x)\ln\frac{B(x)}{C(x)} + B'(x) - \frac{C'(x)B(x)}{C(x)}\right) =$$

$$= \tilde{r}_i \frac{C(x)}{B(x)} \frac{B'_j(x)C(x) - C'_j(x)B(x)}{C^2(x)} - (\tilde{r}_i - r_{i1}) \frac{B'_j(x)C(x) - C'_j(x)B(x)}{C^2(x)} =$$

$$= \tilde{r}_i\left(\frac{B'_j(x)}{B(x)} - \frac{C'_j(x)}{C(x)}\right) - (\tilde{r}_i - r_{i1})\left[\frac{B'_j(x)}{B(x)} - \frac{C'_j(x)B(x)}{C^2(x)}\right] \tag{17}$$

where $B'_j(x) = \frac{\partial}{\partial x_j}B(x) = \tilde{r}_j$; $C'_j(x) = \frac{\partial}{\partial x_j}C(x) = \tilde{r}_j - r_{j1}$ and substituting these assignments in (17) obtain

$$\tilde{r}_i\left(\frac{\tilde{r}_j}{\sum_{i=1}^{N} x_i\tilde{r}_i - r^*} - \frac{\tilde{r}_j - r_{j1}}{\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})}\right) - (\tilde{r}_i - r_{i1})\left[\frac{\tilde{r}_j}{\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})} - \frac{(\tilde{r}_j - r_{j1})\sum_{i=1}^{N} x_i\tilde{r}_i - r^*}{\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2}\right] =$$

$$= \frac{\tilde{r}_i\tilde{r}_j(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2 - \tilde{r}_i(\tilde{r}_j - r_{j1})(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1}) - \tilde{r}_j(\tilde{r}_i - r_{i1})}{(\sum_{i=1}^{N} x_i(\tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2)} -$$

$$\frac{\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*) + (\tilde{r}_i - r_{i1})(\tilde{r}_j - r_{j1})\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2}{(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2)} =$$

$$\frac{\tilde{r}_i\tilde{r}_j(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2 - (2\tilde{r}_i\tilde{r}_j - \tilde{r}_i r_{j1} - \tilde{r}_j r_{i1})(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)}{(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2)} +$$

$$+\frac{(\tilde{r}_i - r_{i1})(\tilde{r}_j - \tilde{r}_{j1})(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)^2}{(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)(\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2)}. \qquad (18)$$

For convenience and cut of transformations denote the denominator as

$$(\sum_{i=1}^{N} x_i\tilde{r}_i - r^*)\sum_{i=1}^{N} x_i(\tilde{r}_i - r_{i1})^2 = E(x). \qquad (19)$$

Substituting the expressions for Δ_{ii} и Δ_{ij} from (15) and (18) into (16) and obtain

$$\Delta_{ii}\Delta_{jj}-\Delta_{ji}{}^{2}=\frac{\left[\tilde r_i\sum_{i=1}^{N}x_i(\tilde r_i-r_{i1})-(\tilde r_i-r_{i1})(\sum_{i=1}^{N}x_i\tilde r_i-r^{*})\right]^2}{E^2(x)}\times$$

$$\times\left[\frac{\tilde r_j(\sum_{i=1}^{N}x_i(\tilde r_i-r_{i1})-(\tilde r_j-r_{j1})(\sum_{i=1}^{N}x_i\tilde r_i-r^{*}))}{E^2(x)}\right]^2-$$

$$-\frac{\{\tilde r_i\tilde r_j(\sum_{i=1}^{N}x_i(\tilde r_i-r_{i1}))^2-(2\tilde r_i\tilde r_j-\tilde r_i r_{j1}-\tilde r_j r_{i1})(\sum_{i=1}^{N}x_i(\tilde r_i-r_{i1}))(\sum_{i=1}^{N}x_i\tilde r_i-r^{*})}{E^2(x)}+$$

$$+\frac{(\tilde r_i-r_{i1})(\tilde r_j-r_{j1})(\sum_{i=1}^{N}x_i\tilde r_i-r^{*})\}^2}{E^2(x)}. \qquad (20)$$

For further simplicity assign

$$\sum_{i=1}^{N}x_i(\tilde r_i-r_{i1})=\tilde r-r_{\min}\,;\sum_{i=1}^{N}x_i\tilde r_i-r^{*}=\tilde r-r^{*}. \qquad (21)$$

Substituting them into (20) obtain

$$\Delta_{ii}\Delta_{jj}-\Delta_{ji}{}^{2}=\frac{\left[\tilde r_i(\tilde r-r_{\min})-(\tilde r_i-r_{i1})(\tilde r-r^{*})\right]^2}{E^2(x)}\times$$

$$\times\left[\frac{\tilde r_j(\tilde r-r_{\min})-(\tilde r_j-r_{j1})(\tilde r-r^{*})}{E^2(x)}\right]^2-$$

$$-\frac{\{\tilde{r}_i\tilde{r}_j(\tilde{r}-r_{\min})^2-(2\tilde{r}_i\tilde{r}_j-\tilde{r}_i r_{j1}-\tilde{r}_j r_{i1})(\tilde{r}-r_{\min})(\tilde{r}-r^*)}{E^2(x)}+ \tag{22}$$

$$+\frac{(\tilde{r}_i-r_{i1})(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)\}^2}{E^2(x)}.$$

Further assign

$$\tilde{r}_i(\tilde{r}-r_{\min})-(\tilde{r}_i-r_{i1})(\tilde{r}-r^*)=F\,;\;\; \Delta_{ji}=\frac{H}{E},$$

$$\tilde{r}_j(\tilde{r}-r_{\min})-(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)=G.$$

Substituting into (23), obtain

$$\Delta_{ii}\Delta_{jj}-\Delta_{ji}^2=\frac{F^2G^2-H^2}{E^2}=\frac{(FG-H)(FG+H)}{E^2}>0. \tag{23}$$

Non-negativeness condition (24) is following:

$$FG-H>0\,.$$

Thereof (hence)

$$FG-H=\left[\tilde{r}_i(\tilde{r}-r_{\min})-(\tilde{r}_i-r_{i1})(\tilde{r}-r^*)\right].\left[\tilde{r}_j(\tilde{r}-r_{\min})-(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)\right]-$$

$$-\tilde{r}_i\tilde{r}_j(\tilde{r}-r_{\min})^2-(2\tilde{r}_i\tilde{r}_j-\tilde{r}_i r_{j1}-\tilde{r}_j r_{i1})(\tilde{r}-r_{\min})(\tilde{r}-r^*)-(\tilde{r}_i-r_{i1})(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)^2=$$

$$=\tilde{r}_i\tilde{r}_j(\tilde{r}-r_{\min})^2-\tilde{r}_j(\tilde{r}_i-r_{i1})(\tilde{r}-r^*)(\tilde{r}-r_{\min})-\tilde{r}_i(\tilde{r}-r_{\min})(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)+$$

$$+(\tilde{r}_i-r_{i1})(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)-\tilde{r}_i\tilde{r}_j(\tilde{r}-r_{\min})^2+(2\tilde{r}_i\tilde{r}_j-\tilde{r}_i r_{j1}-\tilde{r}_j r_{i1})(\tilde{r}-r_{\min})(\tilde{r}-r^*)-$$

$$-(\tilde{r}_i-r_{i1})(\tilde{r}_j-r_{j1})(\tilde{r}-r^*)^2=$$

$$= (\tilde{r} - r_{\min})(\tilde{r} - r^*)\left[-\tilde{r}_j(\tilde{r}_i - r_{i1}) - \tilde{r}_i(\tilde{r}_j - r_{j1}) - 2\tilde{r}_i\tilde{r}_j - \tilde{r}_i r_{j1} - \tilde{r}_j r_{i1}\right] = 0 \qquad (24)$$

Thus we have obtained the following conditions

$\Delta_{ii} = \frac{\partial^2}{\partial x_t^2}\left[B(x)\ln\frac{B(x)}{C(x)}\right] > 0$; for all $i = \overline{1, N}$.

Diagonal minors of the form $\mu_{i1} = \begin{bmatrix} \Delta_{ii} & \Delta_{ij} \\ \Delta_{ji} & \Delta_{jj} \end{bmatrix} \geq 0.$

These are the sufficient conditions of convexity of function $B(x)\ln\frac{B(x)}{C(x)}$, and therefore the convexity of function $A(x) + B(x)\ln\frac{B(x)}{C(x)}$.

Now it's only left to show that the product of convex functions $A(x) + B(x)\ln\frac{B(x)}{C(x)}$ и $D(x)$ will be convex as well at the interval $x_i \in [0,1],\ i = \overline{1, N}$ taking into account that $D(x) = \frac{1}{\sum_{i=1}^{N} x_i(r_{i2} - r_{i1})}$

where $r_{i2} \geq r_{i1}$, $x_i \in [0,1]$, $\sum_{i=1}^{N} x_i = 1$.

Notice that $A(x) + B(x)\ln\frac{B(x)}{C(x)}$ и $D(x)$ as proved previously are positive and monotonically decreasing.

For convenience denote $A(x) + B(x)\ln\frac{B(x)}{C(x)} = \varphi(x)$.

Let's prove that $\varphi'(x) < 0$.

$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial}{\partial x_i}\left(A(x) + B(x)\ln\frac{B(x)}{C(x)}\right) = A'(x) + B'(x) + B(x)\frac{C(x)}{B(x)}\frac{B'(x)C(x) - C'(x)B(x)}{C^2(x)} =$$

$$= A'(x) + B'(x)\ln\frac{B(x)}{C(x)} + B'(x) - \frac{B'(x)C(x)}{C(x)}. \qquad (25)$$

Substituting the values $A'(x)$ and $B'(x)$ obtain

$$\frac{\partial\varphi}{\partial x_i} = -r_{i1} + \tilde{r}_i \ln\frac{B(x)}{C(x)} + \tilde{r}_i - (\tilde{r}_i - r_{i1})\frac{B(x)}{C(x)} =$$

$$= \tilde{r}_i\left(1 + \ln\frac{B(x)}{C(x)}) - r_{i1} - (\tilde{r}_i - r_{i1})\frac{B(x)}{C(x)}\right). \qquad (26)$$

As $\frac{B(x)}{C(x)} < 1$, $-r_{i1} + \tilde{r}_{i1}\frac{B(x)}{C(x)} < 0.$

Therefore, after simplifying (27), we obtain

$$\frac{\partial\varphi}{\partial x_i} = \tilde{r}_i(1 + \ln\frac{B(x)}{C(x)} - \frac{B(x)}{C(x)}), \qquad (27)$$

$$1 + \ln\frac{B(x)}{C(x)} - \frac{B(x)}{C(x)} = 1 + \ln\frac{\tilde{r} - r^*}{\tilde{r} - r_{\min}} - \frac{\tilde{r} - r^*}{\tilde{r} - r_{\min}}, \qquad (28)$$

According previous assumptions $r^* > r_{\min} = \sum_{i=1}^{N} x_i r_{i1}$ and $\tilde{r} > r^*$. Lets show that (28) is greater than 0.

Denote $\tilde{r} - r^* = a$. Then $\tilde{r} - r_{\min} = \tilde{r} - r^* + (r^* - r_{\min}) = a + y$, $y = r^* - r_{\min} > 0$.

Then

$$1 + \ln\frac{\tilde{r} - r^*}{\tilde{r} - r_{\min}} - \frac{\tilde{r} - r^*}{\tilde{r} - r_{\min}} = 1 + \ln\frac{a}{a+y} - \frac{a}{a+y}. \qquad (29)$$

Let's show that

$$\Delta = 1 + \ln\frac{a}{a+y} - \frac{a}{a+y} < 0 \text{, for all } y>0.$$

Evidently, $\Delta = 1 + \ln\frac{a}{a+y} - \frac{a}{a+y} = 0$ by y=0 and function is motonnically decreasing, as

$$\Delta'(y) = -\frac{1}{a+y} + \frac{a}{(a+y)^2} = -\frac{y}{(a+y)^2} < 0 \text{ for all } y>0.$$

Therefore $\Delta(y) < 0$ under y>0. Previously we have proved that $D(x) = \frac{1}{\sum_{i=1}^{N} x_i(r_{i2} - r_{i1})}$ is convex.

Consider

$$\frac{\partial}{\partial x_i}\big(\varphi(x)D(x)\big) = \varphi'(x)D(x) + D'(x)\varphi(x) < 0. \qquad (30)$$

Find second partial derivatives

$$\frac{\partial^2}{\partial x_i^2}\left[\varphi(x)D(x)\right] = \varphi''(x)D(x) + \varphi'(x)D'(x) + D''(x)\varphi(x) + D'(x)\varphi'(x) =$$

$$= \varphi''(x)D(x) + \varphi(x)D''(x) + 2D'(x)\varphi'(x). \qquad (31)$$

It was proved earlier that $\varphi'(x) < 0$; $D'(x) < 0$; $D''(x) > 0$; $\varphi''(x) > 0$ therefore

$$\frac{\partial^2}{\partial x_i^2}(\varphi(x)D(x)) > 0.$$

Therefore, risk function $\beta(x) = \varphi(x)D(x)$ is convex. End of proof.

Optimality Conditions for Dual Fuzzy Portfolio Problem

As it was earlier shown the dual portfolio problem (7)-(8) is convex programming problem under the corresponding conditions. Taking into account that constraints (8) are linear compose Lagrangian function

$$L(x,\lambda,\mu)=\beta(x)+\lambda(r^{*}-\sum_{i=1}^{N}x_i\tilde{r}_i)+\mu(\sum_{i=1}^{N}x_i-1). \tag{32}$$

The optimality conditions by Kuhn-Tucker are such:

$$\frac{\partial L}{\partial x_i}=\frac{\partial\beta(x)}{\partial x_i}-\lambda r_i+\mu\geq 0\,;i=\overline{1,N}, \tag{33}$$

$$\frac{\partial L}{\partial\lambda}=-\sum_{i=1}^{N}x_i\tilde{r}_i+r^{*}\leq 0,$$

$$\frac{\partial L}{\partial\mu}=\sum_{i=1}^{N}x_i-1=0,$$

and conditions of complementary non-fixedness

$$\frac{\partial L}{\partial x_i}x_i=0\,,\ i=\overline{1,N},$$

$$\frac{\partial L}{\partial\lambda}\lambda=\lambda\left(-\sum_{i=1}^{N}x_i\tilde{r}_i+r^{*}\right)=0, \tag{34}$$

$$x_i\geq 0\,,\ x\geq 0,$$

where $\lambda\geq 0$ и μ are Lagrange multipliers.

This problem may be solved using standard methods of convex programming, for instance by the method of feasible directions or penalty functions method.

Conclusion

The dual fuzzy portfolio problem is considered. The sufficient conditions for this problem to be convex were obtained and investigated. The optimality conditions for the solution were obtained.

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Bibliography

[Недосекин, 2003] Недосекин А.О. Система оптимизации фондового портфеля от Сименс Бизнес Сервисез / Банковские технологии. – 2003. – № 5. – Также на сайте: http: //www. finansy. ru/publ/fin/004. Htm

[Зайченко, Малихех, 2008] Юрий Зайченко Юрий, Малихех Есфандиярфард. Оптимизация инвестиционного портфеля в условиях неопределенности на основе прогнозирования курсов акций // Proceedings of XIV-th International Conference “KDS-2008” (Knowledge, Dialogue, Solution). June, 2008, Varna, Bulgaria.-Sophia. - Pp. 212-228.

[Зайченко, 2007] Зайченко Юрий, Малихех Есфандиярфард. Анализ и сравнение результатов оптимизации инвестиционного портфеля при применении модели Марковитца и нечетко-множественного метода // Proceedings of XIII-th International Conference “KDS-2007” (Knolwledge, Dialogue, Solution), Vol.1 , pp.278-286.

[Зайченко, 2008] Зайченко Ю.П., Есфандиярфард М. Оптимизация инвестиционного портфеля в условиях неопределенности.// Системнi дослiдження та iнвестицiйнi технологiї. - 2008. - №2. - С.59-76.

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