THE METHOD OF ESTIMATING THE CONSISTENCY OF PAIRED COMPARISONS

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***Abstract**: The equivalence of the indicators, which are used in the assessment of the consistency of paired comparisons, is investigated. It is found for which matrix of paired comparisons the usage of various indicators of consistency leads to different results regarding the permissible inconsistency, and as a consequence, to different results regarding the necessity of this matrix correction. To illustrate how critical in practical applications such contradictory results on a variety of indicators are, the examples, in which the calculation of the weights of alternatives decisions on the basis of primary and adjusted matrix of paired comparisons leads to a variety of alternatives ranking, are considered. On the basis of the conducted research the method of estimating the consistency of paired comparisons is proposed.*

***Keywords**: indicators of paired comparisons matrix consistency, weak consistency, the permissible inconsistency and method of consistency estimation*

***ACM Classification Keywords**: H.4.2. INFORMATION SYSTEM APPLICATION: type of system strategy*

Introduction

And as a result the matrix are constructed $D_{n\times n} = \{d_{ij} \mid i, j = 1,...,n\}$ with the properties $d_{ij} > 0$, $d_{ji} = 1/d_{ij}$.

Methods of paired comparisons are one of the components of the majority of modern methodologies of decision-making support, such as methodology of analysis of hierarchies of criteria and alternatives solutions [Pankratova & Nedashkovskaya, 2011; Saaty, 1980; Saaty & Vargas, 1987; Noghin, 2004] "line", "triangle", "square" [Totsenko, 2002], PROMETHEE [Macharis et al, 2004]. Methods of paired comparisons are used for the solution of weakly structured tasks of evaluation of decisions alternatives on the quality criterion with the involvement of experts estimates [David, 1978; Larichev &Moszkowicz, 1996; Orlov, 2004; Zhilyakov ,2006]. In particular, in the methods of analysis of hierarchies expert in pairs compares alternatives in a special fundamental scale of the relative importance.

Methods of paired comparisons are aimed at the recovery of the coefficients of relative importance (weights) $w \in R_+^n$, $\sum_{i=1}^{n} w_i = 1$ solutions alternatives on the quality criterion from the matrix of paired comparisons (MPC) $D_{n\times n}$. The calculation of these weights are most often based on the idea of minimizing the norm of MPC deviations $D_{n\times n}$ from some unknown matrix $C = (w_i / w_j)$, which in methods of paired comparisons is considered to be the best approximation of MPC $D_{n\times n}$. Matrix $C = (w_i / w_j)$ is called the consistent or theoretical. The traditional method is the one of the main vector of the weights calculation with the MPC $D_{n\times n}$ [Saaty, 1980]. Depending on the choice of the functions of the matrices norms the other methods of paired comparisons are also applied: least squares, weighted least squares, the logarithmic least squares (the method of geometric mean) and others (see the review performed in [Pankratova & Nedashkovskaya, 2011].

Take a closer look at the concept of consistency, which is one of the key methods of paired comparisons.

Inconsistency is the manifestation of the contradictions in the assessments of the experts and appears if it is necessary to compare more than three objects. MPC is called inconsistent, if $\exists i, j, k$, such that $d_{ij} \neq d_{ik} d_{kj}$. MPC is called serially or ordinally inconsistent, if $\exists$ is three of indices (i, j, k) , for which there is a cycle $a_i \succ a_j \succ a_k \succ a_i$, or in terms of elements of the MPC $(d_{ij} > 1) \wedge (d_{jk} > 1) \wedge (d_{ik} < 1)$ is performed. The reasons for inconsistencies are considered to be psychological limitations human expert [Totsenko, 2002; Xu&Da, 2003], the mistakes of experts in expressing assessments, the usage of fundamental scale of the relative importance [Zhilyakov, 2006].

For the assessment of the consistency of the MPC some indicators [Saaty, 1980; Totsenko, 2002; Aguaron & Moreno-Jimenez, 2003; Stein & Mizzi, 2007; Peláez & Lamata, 2003] and criteria [Saaty, 1980; Totsenko, 2002] are developed. They, using these indicators, allow evaluating the permissibility of the inconsistency of the MPC for its usage in the decision-making process. In this case, the actual task is to study the equivalence of the various indicators, namely if they lead to the same conclusions regarding the permissible inconsistency of the MPC in the sense of the criterion.

Expert assessments without the allowable inconsistencies are considered as controversial and, accordingly, may not be used decisions making.

The aim of this work is the assessment of the equivalence of different measures of consistency of paired comparisons and the development of appropriate method of evaluation of the consistency of the MPC, depending on the properties of the MPC.

1. Problem formulation

The definition. The matrix of paired comparisons (hereinafter MPC) is called positive, back symmetric MPC $D_{n \times n}$: $d_{ij} > 0$, $d_{ji} = 1 / d_{ij}$, $i, j = 1,...,n$.

MPC are obtained in a result of the expert's implementation of paired n of elements (for example, alternatives) for the quality criterion in the scale of relations or, in the case of quantitative criteria - taking the relations of the numerical values of the alternatives according to the criterion.

The definition. MPC $\widehat{D}_{n \times n}$, for which transitivity are implemented: $\widehat{d}_{ij} = \widehat{d}_{ik} \widehat{d}_{kj}$ for $\forall i, j, k = 1,...,n$, is called strongly consistent. If $\exists\, i, j, k$, such that $d_{ij} \neq d_{ik} d_{kj}$, so such MPC is called strongly inconsistent.

The definition. MPC $\breve{D}_{n \times n}$, for which order transitivity are implemented: $(\breve{d}_{ij} > 1) \wedge (\breve{d}_{jk} > 1) \Rightarrow (\breve{d}_{ik} > 1)$, is called weak or order consistent. If $\exists\, i, j, k$, such that $(d_{ij} > 1) \wedge (d_{jk} > 1) \wedge (d_{ik} < 1)$, so the MPC is called weakly inconsistent, and the element d_{ik} - the ejection..

Statement 1. If $D_{n \times n}$ – strongly consistent MPC, so $D_{n \times n}$ – weakly consistent.

Statement 2. If $D_{n \times n}$ – weakly inconsistent МПП, so $D_{n \times n}$ – strongly consistent.

Strongly consistent MPC is very rare in the real practical problems with the expert paired comparisons of the elements, especially if the amount of the compared elements елементів n exceeds 3. Mainly, strongly consistent MPC is used as a kind of ideal MPC, it is also called theoretical, with which the specified expert or empirical MPC is compared, in the methods of calculating the local weights of the elements of the tasks of decision-making support.

For estimation of the level of inconsistency of the MPC in practical problems the indicators CR [Saaty, 1980],

GCI [Aguaron&Moreno-Jimenez, 2003], HCR [Stein & Mizzi, 2007], CI^{tr} [Peláez & Lamata, 2003], k_y [Totsenko, 2002] and criteria [Saaty, 1980; Totsenko, 2002; Aguaron&Moreno-Jimenez, 2003; Stein & Mizzi, 2007; Peláez & Lamata, 2003], which, using these indicators, allow to evaluate the admissibility of the inconsistency of the MPC for its usage in the decision-making process, are used.

Consistency indicator is connected with the method of the calculation of weights. So, the indicator CR is used with a method of the main eigenvector EM of weights calculation, the indicator GCI - with the method of geometric mean RGMM of weights calculation, the indicator HCR - with the method of arithmetic normalization of AN weights calculation.

It is known: $D_{n\times n} = \{d_{ij} \mid i, j = 1,...,n\}$ – MPC of decisions alternatives on criterion.

It is required:

– *to determine whether well-known* indicators *of consistency* CR, GCI, HCR, CI^{tr} *and* k_y *are equivalent in the sense that they lead to the same conclusions regarding the permissible inconsistency of the MPC in the sense of the criterion;*
– *to identify, how the property of weak consistency of the MPC impacts on the inconsistency permissibility of this MPC in the decision-making process;*
– *to develop the correct method of assessment of the consistency of the MPC, to calculate the weights* $w \in R_+^n$, $\sum_{i=1}^{n} w_i = 1$ *and to find the ranking of the solutions alternatives.*

First, let consider the calculation of known indicators of consistency, which are studied in the work. Then move on to the coverage of the results of the assessment of these indicators equivalence.

Traditional for methods of paired comparisons index CR is introduced for the MPC, which is a disturbance of strongly consistent MPC. Such MPC will be permissibly inconsistent in the sense of criterion (see below criterion 1), does not need the correction and can be used for the calculation of weights and decision support.

In this work the equivalence of indicators CR, GCI, HCR, CI^{tr} and k_y are estimated for MPC, which are perturbations of strongly consistent MPC, as well as for the MPC with a high level of inconsistency, namely: MPC, which have at the same time the properties of weak consistency and a strong inconsistency; weak inconsistent MPC in the condition of one or more ejections.

Research of MPC with high level of inconsistency (hereinafter such MPC are called inadmissible inconsistent or the information noise, depending on the level of inconsistency) will be necessary to develop a method of improving the consistency of this MPC.

2. Indicators and criteria of MPC consistency

Let $d_{ij} = c_{ij}\varepsilon_{ij}$, де $c_{ij} = w_i / w_j$ – the element of MPC consistency, $\varepsilon_{ij} > 0$ – the perturbation value.

Consistency relations [2] of MPC $D_{n\times n}$ is $CR(D_{n\times n}) \overset{def}{=} CI(D_{n\times n}) / MRCI(n)$, where the index of consistency $CI(D_{n\times n}) \overset{def}{=} -\sum_{i=1}^{n-1} \lambda_i / (n-1)$ – the average value of not the main eigenvalues $D_{n\times n}$ with the sign «–», after the changes $CI(D_{n\times n}) = (\lambda_{\max} - n)/(n-1)$, $\lambda_{\max}$ – the main eigenvalue $D_{n\times n}$, $MRCI(n) > 0$ – the index of random consistency - the average value of the consistency indices for randomly

filled MPC dimension $n \geq 3$ (table value).
Vector of decisions alternatives weights is calculated according the method of main eigenvalue vector EM: vector of normalized weights v – the solution of the equation $Dv = \lambda_{\max} v$, normalized weights $w_i = v_i / \sum_{j=1}^{n} v_j$, $\sum_{i=1}^{n} w_i = 1$.

$GCI(D_{n\times n}) \overset{def}{=} S^2(D_{n\times n}) / d.f(n)$, where $S^2(D_{n\times n}) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} (\ln d_{ij} - \ln(v_i / v_j))^2$, $d.f(n) = n(n-1)/2 - (n-1)$ – the number of degrees of freedom, weights vector $v \in R_+^n$ – optimization problem solution $S^2(D_{n\times n}) \to \min$ with restrictions $\prod_{i=1}^{n} v_i = 1$, $v \in R_+^n$, is called geometric index of consistency [Aguaron & Moreno-Jimenez, 2003]. The analytical solution of the last problem $v_i = \left(\prod_{j=1}^{n} d_{ij}\right)^{1/n}$ is known as a method of geometric middle RGMM. Normalized weights: $w_i = v_i / \sum_{j=1}^{n} v_j$, $\sum_{i=1}^{n} w_i = 1$.

$HCR(D_{n\times n}) \overset{def}{=} HCI(D_{n\times n}) / MRHCI(n)$, where $HCI(D_{n\times n}) \overset{def}{=} (HM(s) - n)(n+1) / (n(n-1))$ – the harmonic index of consistency, $HM(s) = n(\sum_{j=1}^{n} (s_j)^{-1})^{-1}$ – average harmonious of values $s_j = \sum_{i=1}^{n} d_{ij}$, $MRHCI(n) > 0$ – index of random consistency – the average value of the harmonic indices of consistency for randomly filled MPC dimension $n \geq 3$ (table value), is called the harmonious relations of consistency [Stein & Mizzi, 2007].
The vector of decisions alternatives weights is calculated according the method of arithmetical normalization AN: vector of unnormalized weights $v_i = (\sum_{j=1}^{n} d_{ji})^{-1}$.

$CI^{tr}(D_{n\times n}) \overset{def}{=} \frac{1}{NT} \sum_{i=1}^{NT} CI^{tr}(\Gamma_i)$ – the average value of consistency indices $CI^{tr}(\Gamma_i)$ of all different transitivities of MPC $D_{n\times n}$, if $n > 3$, $CI^{tr}(D_{n\times n}) \overset{def}{=} \det(D_{n\times n})$, if $n = 3$ and $CI^{tr}(D_{n\times n}) \overset{def}{=} 0$ in other case, where $NT = n!/((n-3)!\,3!)$ – the number of different transitivities of MPC $D_{n\times n}$, if $n \geq 3$ is called index of transitivities consistency [Peláez & Lamata, 2003]. Transitivity Γ –the weak order on the set of the three alternatives $\{a_i, a_j, a_k\}$.

$$k_y(D_{n\times n}) = \min_{k=1,\dots,n} (k_y(R^k)) \text{, where}$$

$$k_y(R^k) = \left(1 - \left(\tfrac{1}{n}\sum_{j=1}^{m} r_j^k \,|\, j - mean^k \,| - \sum_{j=1}^{m} \tfrac{r_j^k}{n} \ln\left(\tfrac{r_j^k}{n}\right)\right) \Big/ \left(G \sum_{j=1}^{m} \left| j - \tfrac{m+1}{2} \right| + \ln(m)\right)\right) z$$

- spectral coefficient of weight spectrum $R^k = \{(r_j^k) \mid j = \overline{1,m}\}$ alternatives a_k, r_j^k - number of weights of alternative a_k, which relates to the scale division $s_j = j/m$, m - number of scale divisions, $mean^k$ - the average value of weights set of alternative a_k, $G = n/\ln(n) m \ln(m)$ - scaling coefficient, $z \in \{0,1\}$ - Boolean function, which defines the necessary and sufficient conditions for the equality to zero of spectral coefficient of consistency $k_y(R^k)$, is called spectral coefficient of consistency [Totsenko, 2002].

Statement 3. $CR(D_{n\times n}) \geq 0$, $GCI(D_{n\times n}) \geq 0$, $HCR(D_{n\times n}) \geq 0$, $CI^{tr}(D_{n\times n}) \geq 0$, $k_y(D_{n\times n}) \geq 0$.

The indicators CR i HCR are constructed by normalizing in accordance with measures of consistency CI i HCI values that characterize the consistency of random set MPC. Therefore, indicators CR i HCR allow assessing whether the information is in the MPC, or MPC can be considered as information noise or randomly set.

For the evaluation of allowable inconsistencies of MPC with the purpose of its use in the decision-making process two criteria are developed. Let formulate the well-known criterion of consistency 1 [Saaty, 1980], by adding to it the case of a lack of information in the MPC.

<u>Criterion of consistency1:</u> MPC $D_{n\times n}$:

- *strongly consistent (consistent)*, if and only if $CR(D_{n\times n})=0$, $GCI(D_{n\times n})=0$, $HCR(D_{n\times n})=0$, $CI^{tr}(D_{n\times n})=0$;
- *permissibly inconsistent (the correction of MPC is not needed)* if $CR(D_{n\times n})\le CR^{porog}$ or $GCI(D_{n\times n})\le GCI^{porog}$, або $HCR(D_{n\times n})\le HCR^{porog}$, або $CI^{tr}(D_{n\times n})\le CI^{tr\ porog}$ (depending on which indicator is used), where $CR^{porog}, GCI^{porog}, HCR^{porog}, CI^{tr\ porog}$ – the threshold values of the respective indicators;
- contains information, but it is impermissible inconsistent (you need to correct MPC), if the consistency rate exceeds its threshold value;
- *is information noise (MPC should be corrected)*, if normalized indicators $CR(D_{n\times n})\ge 1$ or $HCR(D_{n\times n})\ge 1$.

<u>Criterion of consistency 2</u> [Totsenko, 2002]: MPC $D_{n\times n}$:

- *strongly consistent (consistent)*, if and only if $k_y(D_{n\times n})=1$;
- *permissibly inconsistent (the correction of MPC is not needed)*, if $k_y(D_{n\times n})\ge T_u$;
- *contains the information, , but it is impermissible inconsistent (you need to correct MPC), if* $(k_y(D_{n\times n})\ge T_0)\wedge(k_y(D_{n\times n})<T_u)$;
- *is information noise (MPC should be corrected), if* $k_y(D_{n\times n})<T_0$,

where T_0, T_u – the threshold values, which are defined, respectively, from the spectrum, which contains the minimum amount of information and a range of permissible accuracy. For a scale of [0,1] with the divisions $s_j=\{0, 0.1, 0.2, ..., 1\}$, N=11 these thresholds are equal $T_0=0.40$, $T_u=0.79$.

Consider the examples of the assessment of the MPC consistency.

Example 1. MPC $\breve{D}1_{7\times 7}$ according the definition has the properties of a weak consistency and a strong inconsistency:

$\breve{D}1_{7\times 7}$ =

1	3	8	1	5	1/3	1
1/3	1	2	1/3	1/2	1/4	1/6
1/8	1/2	1	1/4	1/2	1/8	1/7
1	3	4	1	4	1/5	1/2
1/5	2	2	1/4	1	1/4	1/4
3	4	8	5	4	1	1/4
1	6	7	2	4	4	1

Let consider if MPC $\breve{D}1_{7\times 7}$ is permissibly inconsistent according to the above-mentioned criteria of consistency. The values of MPC consistency indicators $\breve{D}1_{7\times 7}$ are equal to $CR=0.09$, $GCI=0.31$, $HCR=0.17$, $CI^{tr}=1.19$, $k_y=0.61$. Comparing them with threshold values $CR^{porog}=0.1$, $GCI^{porog}=0.37$, $HCR^{porog}=0.1$, $CI^{tr\ porog}=1.329$, $T_0=0.40$ і $T_u=0.79$ for $n=7$, come to the conclusion, that the given MPC $\breve{D}1_{7\times7}$ is permissibly inconsistent (the correction is needed) according indicators CR, GCI та CI^{tr} and is not permissibly inconsistent (correction is needed) according indicators HCR and k_y.

Thus, the use of different indicators of consistency for the MPC may lead to different results in terms of criteria.

Example 2. MPC $D2_{7\times7}$ according the definition is weakly inconsistent:

$D2_{7\times7}$ =

1	1/8	6	1/6	4	1/2	1/3
8	1	48	4/3	32	4	**3/8**
1/6	1/48	1	1/36	2/3	1/12	1/18
6	3/4	36	1	24	3	2
1/4	1/32	3/2	1/24	1	1/8	1/12
2	1/4	12	1/3	8	1	2/3
3	**8/3**	18	1/2	12	3/2	1

Comparing values of inconsistency indicators $CR(D2_{7\times7}) = 0.06$, $GCI(D2_{7\times7}) = 0.18$, $HCR(D2_{7\times7}) = 0.12$, $CI^{tr}(D2_{7\times7}) = 0.75$, $k_y(D2_{7\times7}) = 0.69$ with its thresholds values (see example.1), come to the conclusion, that MPC $D2_{7\times7}$ is permissibly inconsistent (increasing of consistency is needed) according the indicators CR, GCI та CI^{tr} and is not permissibly inconsistent (consistency increasing is required) according the indicators HCR and k_y.

Let find the ejection in the given weakly inconsistent MPC $D2_{7\times7}$, using the definition. The condition $(d_{ij} > 1) \wedge (d_{jk} > 1) \wedge (d_{ik} < 1)$ of violation of the order traransitivity on the set of alternatives is performed for $(i, j, k) = (2, 4, 7)$, so the ejection is the element $d_{ik} = d_{27} = 3/8$. It is easy to note, that after the change of this element at the back symmetric $d_{27} := 8/3$, the problem of MPC $D2_{7\times7}$ becomes strongly consistent.

The MPC of such a kind, that is weakly inconsistent with one ejection, when other elements form a very consistent transitivities, may arise during the operator error in the assessment of decisions alternatives on quantitative criteria. Evaluating the quantitative criterion, as a rule, there is a statistical information, which leads to strongly consistent MPC. However, the error of the operator may lead to the ejection in the MPC. Thus, for the weakly inconsistent MPC the use of various indicators of consistency can also lead to different results regarding the permissible inconsistencies in the sense of criteria.

3. Assessment of the equivalence of the consistency indicators *CR, GCI, HCR, CI^{tr}* and k_y of MPC

In p.2 above the examples of two MPC with different properties it is shown that the known indicators of consistency lead to different results regarding the permissible inconsistency of the MPC in the sense of criteria 1 and 2. For the assessment of the equivalence of consistency indicators CR, GCI, HCR, CI^{tr} and k_y a statistical modeling of the dependency between these indicators on MPC with different properties is carried out. Samples of 500 MPC of different levels of consistency are randomly generated. As a reference, take indicator CR, which is considered as traditional for a method of paired comparisons [Saaty, 1980], and compare with it the other indicators. For each randomly given the MPC calculate the rank, the values of transitivities indicators, as the basic elements of consistency, when the number of different transitivities of MPC is equal $NT = n!/((n-3)!3!)$, and also the values of the indicators CR, GCI, HCR, CI^{tr} and k_y. The calculations were carried out with the accuracy of 10^{-4}, and the conclusions were made with an accuracy of 10^{-2}. The results are presented in table 1.

Table 1. The characteristics of the MPC consistency with different properties

№	MPC consistency according the definition	Are there ejections? The number of ejections	Ranges of values of consistency indicators (for $n = 7$)					Rank of MPC	The value of MPC transitivities qualifiers
			CR	GCI	HCR	CI^{tr}	k_y		
1	Strongly consistent	No ejections	0	0	0	0	1	1	All are equal to zero
2	Perturbated strongly consistent	No ejections	[0.011; 0.100]	[0.041; 0.359]	[0.009; 0.114]	[0.129; 1.347]	[0.491; 0.845]	5, 6, 7	Take small values
3	Has the properties of strong inconsistency and weak consistency (fundamental scale)	No ejections	[0.117; 0.732]	[0.407; 2.230]	[0.039; 0.793]	[1.567; 16.480]	[0.378; 0.715]	7	Take small values within an order of the magnitude
4	Weakly inconsistent	One ejection, other elements of MPC are strongly consistent	[0.003; 3.398]	[0.012; 3.678]	[0.000; 2.825]	[0.037; 937.000]	[0.405; 1.000]	3	Equal to zero, except of a few transitivities, which correspond to a single ejection
5	Weakly inconsistent	One ejection, Other elements of MPC – perturbation of strongly consistent	[0.007; 2.633]	[0.025; 3.227]	[0.006; 1.171]	[0.076; 486.200]	[0.289; 0.845]	5, 6, 7	Take small values, with the exception of a few transitivities, which differ from others for more than an order magnitude and related the ejection
6	Weakly inconsistent (fundamental scale)	A few ejections	[0.149; 1.737]	[0.474; 4.684]	[0.071; 1.937]	[2.047; 117.200]	[0.309; 0.684]	7	There are several transitivities with large values, which differ from the others by more than an order of magnitude and related ejections

1. If the MPC is strongly consistent, so all indicators $\boldsymbol{CR}, \boldsymbol{GCI}, \boldsymbol{HCR}$, $\boldsymbol{CI^{tr}}$ and $\boldsymbol{k_y}$ equivalent and demonstrate the consistency of the MPC in the sense of criteria 1 i 2.

2. Let consider the MPC, which are the perturbation of strongly consistent MPC, when the values of perturbation are small, so that the MPC is **weakly consistent**. For such MPC observe the permissible inconsistency in the sense of criterion 1, that is the correction of the MPC is not necessary, according all indicators CR, GCI, HCR and CI^{tr} (table1), except for 3.2% of the cases, when the indicator HCR showed the need of correction. The indicator k_y and criterion 2, on the contrary, in 95% of cases have shown the necessity of the MPC correction and only in 5% of the cases - permissible inconsistency.

Let estimate the regression of the values depending on the indicator CR according the least-squares method. As the criterion of significance of the regression parameters the coefficient of determination is used. For $n = 7$ obtain: $GCI = 0.009 + 3.426 \cdot CR$ with a coefficient of determination $R^2 = 0.99$, $HCR = 0.017 + 0.516 \cdot CR$ with a coefficient of determination $R^2 = 0.27$, $CI^{tr} = -0.064 + 13.673 \cdot CR$ with a coefficient of determination $R^2 = 0.99$, $k_y = 0.782 - 1.337 \cdot CR$ with a coefficient of determination $R^2 = 0.16$.

It is known that the coefficient of determination takes its values in the interval [0,1] and the more of its value means the great importance of the regression. Thus, for perturbated strongly consistent MPC, indicators $\boldsymbol{CR}, \boldsymbol{GCI}$ *and* $\boldsymbol{CI^{tr}}$ *are equivalent.* The Indicator k_y in 95% of cases erroneously shows the necessity of MPC correction.

There are several tranzitivnosey with large values that are different from the others by more than an order of magnitude and related emissions

3. If strongly inconsistent MPC has the properties of weak inconsistency, so, analyzing the ranges of indicators changes (table.1), it can be concluded that the normalized indicators CR i HCR , as well as k_y in 99.4% of the cases, indicate the presence of information in the MPC (because $CR < 1$, $HCR < 1$ та $k_y > T_0 = 0.39$). In 97% of cases according to all indicators simultaneously MPC is impermissible inconsistent in the sense of criteria 1 and 2. The exception is 2.8% of the cases of contradictory results according to CR i HCR : CR indicates the need of the MPC correction, and HCR indicates that the correction is not required.

Regressions of the indicators, depending on the indicator CR according to the method of least squares for $n = 7$:

$GCI = 0.111 + 2.852 \cdot CR$, the coefficient of determination $R^2 = 0.99$;

$HCR = -0.035 + 0.871 \cdot CR$, коефіцієнт детермінації $R^2 = 0.50$;

$CI^{tr} = -1.866 + 22.708 \cdot CR$, the coefficient of determination $R^2 = 0.95$;

$k_y = 0.629 - 0.260 \cdot CR$, the coefficient of determination $R^2 = 0.20$.

Thus, if strongly inconsistent MPC has the property of weak consistency $\boldsymbol{CR}, \boldsymbol{GCI}$ and $\boldsymbol{CI^{tr}}$ are equivalent.

4. Let consider weakly inconsistent MPC with different quantity of ejections and various properties of elements, which are not ejections. If **the ejection is single, and other elements form strongly consistent transitivities**, for these MPC the criterion of consistency 1 operates bad, since all cases of this criterion come into action, namely, the MPC can be whether consistent (according the indicator CR in 1.8% of the cases, accuracy 10^{-2}), permissible inconsistent, that is the MPC correction is not required (according the indicator CR in 16.9% of the cases), and impermissible inconsistent (according the indicator CR in 41% of the cases) and the information noise (according the indicator CR in42.6% of the cases).

The same conclusions are true for the criterion of consistency 2: MPC can be whether consistent (1.8% of the cases, the accuracy 10^{-2}), i.e. the correction of the MPC is not required (21.6% of cases), and it is impermissible inconsistent (80.6% of the cases).

There are conflicting results according CR , HCR and k_y in the sense of the criteria 1 i 2:

- in 20.3% of the cases CR indicates the need of the MPC correction, and HCR HCR indicates that the correction is not required, in 13.7% of the cases such contradictory results have a place for CR and k_y ;
- in 2.3% of the cases the MPC are identified as such? Which are not required the correction according CR and such, which are required the correction according HCR, in 7.1% of the cases such contradictory results have a place for CR and k_y ;
- k_y , in contrast to all of the other indicators, never found the lack of information in the MPC (in all cases it was performed $k_y > T_0 = 0.39$), while in 42.6% i 6.4% of the cases CR and HCR respectively identified a total absence of information in the MPC.

Let estimate the regressions of indicators depending on the indicator CR according to the method of least squares. For $n = 7$ are obtained: $GCI = 0.324 + 1.138 \cdot CR$, коефіцієнт детермінації $R^2 = 0.97$;

$HCR = 0.050 + 0.293 \cdot CR$, coefficient of determination $R^2 = 0.46$;

$CI^{tr} = -70.649 + 191.796 \cdot CR$, coefficient of determination $R^2 = 0.84$;

$k_y = 0.776 - 0.100 \cdot CR$, coefficient of determination $R^2 = 0.57$.

Thus, if the ejection is single and other elements form strongly consistent transitivities, $\boldsymbol{CR}, \boldsymbol{GCI}$ and $\boldsymbol{CI^{tr}}$ are equivalent. These indicators operate badly, because in 18.7% of the cases it was wrongly shown, that the MPC is not required the correction, and in 42.6% of the cases the total absence of the information in the MPC was wrongly indicated. The indicator k_y operates better, because in 80.6% of the cases it was indicated the necessity of the MPC correction and never found the lack of information in the MPC.

5. If the ejection is single, and the other elements of the MPC are the perturbation of strongly consistent elements, for such MPC the criterion of consistency 1, the same as for previous MPC, operates bad, since all cases of this criterion come into action, namely, the MPC can be whether consistent (according CR in 0.4% of the cases, тthe accuracy 10^{-2}), permissibly inconsistent, it means that the MPC correction is not required (according CR in 31.4% of the cases), and impermissibly inconsistent (according CR in 54.4% of the cases) and the information noise (according CR in 17% of the cases).

The criterion 2 operates bad, since in 96.4% of the cases k_y truly indicated the existence of information in MPC and the necessity of its correction, and only in 1.8% of the cases it indicated the MPC as information noise.

The regressions of indicators depending on the indicator CR according to the method of least squares $n = 7$ are as follows:

$GCI = 0.283 + 1.300 \cdot CR$, coefficient of determination $R^2 = 0.97$,

$HCR = 0.086 + 0.186 \cdot CR$, coefficient of determination $R^2 = 0.49$,

$CI^{tr} = -23.297 + 119.071 \cdot CR$, coefficient of determination $R^2 = 0.87$,

$k_y = 0.667 - 0.080 \cdot CR$, coefficient of determination $R^2 = 0.28$.

Thus, if the ejection is single and the other elements of the MPC are the perturbation of strongly consistent elements, $\boldsymbol{CR}, \boldsymbol{GCI}$ *and* $\boldsymbol{CI^{tr}}$ *are equivalent.* These indicators operate badly, since in 31.8% of the cases it was wrongly shown, that the MPC is not required the correction. k_y Operates good, it in 96.4% of the cases showed the existence of the information in the MPC and the necessity of its correction.

6. If the ejections are a few, in 51.9% of the cases, all indicators show the presence of information and the necessity of MPC correction, and just with CR – in 60.9% of the cases, with HCR – in 56.8% of the cases, and k_y – in 99% of the cases. The lack of information is found in all three indicators CR , HCR and k_y and at the same time only in 0.7% of the cases, in terms of two indicators CR i HCR – in 40.1% of the cases, and k_y – in 1% of the cases. Regressions of the indicators depending on CR according to the method of least squares for $n = 7$ are as follows:

$GCI = 0.212 + 2.458 \cdot CR$, coefficient of determination $R^2 = 0.95$;

$HCR = -0.169 + 1.163 \cdot CR$, coefficient of determination $R^2 = 0.69$;

$CI^{tr} = -25.300 + 64.927 \cdot CR$, coefficient of determination $R^2 = 0.92$;

$k_y = 0.547 - 0.045 \cdot CR$, coefficient of determination $R^2 = 0.04$.

Thus, if the MPC is weakly inconsistent with a few ejections, $\boldsymbol{CR}, \boldsymbol{GCI}$ *and* $\boldsymbol{CI^{tr}}$ *are equivalent.*

4. Examples of calculation of local weights of decisions alternatives on the basis of primary and corrected MPC

To illustrate how critical in practical problems the found in section 3 the contradictory results of the consistency assessment are, consider examples in which the calculation of the weights of decisions alternatives on the basis of primary and corrected MPC lead to a variety of alternatives ranking.

For MPC, which have the property of weak consistency, conflicting results CR and HCR are in a small number of cases (3.2% and 2.8% of cases for different levels of weak consistency of the MPC). So let consider these contradictory results as practically unimportant and won’t illustrate it here.

Let consider the weakly inconsistent MPC with different properties of the elements, which are not the ejections.

Example 3. Consider the MPC $D3_{7\times7}$, which by definition is weakly inconsistent, has the one ejection, and other elements are strongly consistent:

$D3_{7\times7}$ =

1	1/5	1/6	1/2	1/5	1/2	1/6
5	1	5/6	5/2	1	5/2	5/6
6	6/5	1	3	6/5	**1/3**	1
2	2/5	1/3	1	2/5	1	1/3
5	1	5/6	5/2	1	5/2	5/6
2	2/5	**3**	1	2/5	1	1/3
6	6/5	1	3	6/5	3	1

As noted above, the MPC of this type appear in a result of operator errors in the assessment of decisions alternatives according the quantitative criteria.

Comparing the values of the consistency indicators $CR(D3_{7\times7}) = 0.076$, $GCI(D3_{7\times7}) = 0.23$, $HCR(D3_{7\times7}) = 0.068$, $CI^{tr}(D3_{7\times7}) = 1.016$, $k_y(D3_{7\times7}) = 0.783$ with their threshold values (see table1), come to the conclusion that the MPC $D3_{7\times7}$ is permissibly inconsistent (the increasing of consistency is not required) according the indicators CR, GCI, HCR та CI^{tr} and is not permissibly inconsistenti (the increasing of consistency is required) the indicator k_y. Calculate the weights of solutions alternatives from the initial MPC $D3_{7\times7}$:

– according the method of EM: $\text{vec}^{\text{T}} = (0.037\ \ 0.183\ \ 0.172\ \ 0.073\ \ 0.183\ \ 0.133\ \ 0.219)$;
– according the method of RGMM: $\text{w}^{\text{T}} = (0.038\ \ 0.191\ \ 0.168\ \ 0.077\ \ 0.191\ \ 0.105\ \ 0.23)$;
– according the method of AN: $\text{w\_AN}^{\text{T}} = (0.04\ \ 0.198\ \ 0.149\ \ 0.079\ \ 0.198\ \ 0.099\ \ 0.238)$.

All methods lead to the same ranking: $a_7 \succ a_2 = a_5 \succ a_3 \succ a_6 \succ a_4 \succ a_1$.

The ejection in the given weakly inconsistent MPC $D3_{7\times7}$ according the definition is the element $d_{36} = 1/3$. After the ejection correction by changing the element at the back symmetric $d_{36} := 3$, given MPC $D3_{7\times7}$ becomes strongly consistent. So the methods of EM, RGMM, AN give equal weights: $\text{vec}^{\text{T}} = (0.037\ \ 0.185\ \ 0.222\ \ 0.074\ \ 0.185\ \ 0.074\ \ 0.222)$, which present another ranking of the alternatives: $a_7 = a_3 \succ a_2 = a_5 \succ a_6 = a_4 \succ a_1$.

Thus, *after the ejection correction the ranking, which differs from the ranking according the methods of EM, RGMM and AN to the MPC correction, is received.*

Example 4. Let consider weakly inconsistent MPC $D4_{7\times7}$ with single ejection, other elements of which are the perturbation of strongly consistent, in other words, they form permissibly inconsistent transistivities:

$D4_{7\times7}$ =

1	1/8	6	1/6	4	1/2	1/3
8	1	9	2	9	4	**1/3**
1/6	1/9	1	1/9	1	1/9	1/9
6	1/2	9	1	9	3	2
1/4	1/9	1	1/9	1	1/8	1/8
2	1/4	9	1/3	8	1	1/3
3	**3**	9	1/2	8	3	1

This MPC arises, for example, in the assessment of alternatives according the quality criterion in the fundamental scale, when the expert made a mistake while recording elements of the symmetric position of the MPC.

The value of the indicators are $CR(D4_{7\times7}) = 0.093$, $GCI(D4_{7\times7}) = 0.329$, $HCR(D4_{7\times7}) = 0.173$, $CI^{tr}(D4_{7\times7}) = 1.239$, $k_y(D4_{7\times7}) = 0.603$. Comparing them with the threshold values (see example1), come to the conclusion, that MPC $D4_{7\times7}$ is permissibly inconsistent according the indicators CR, GCI та

CI^{tr} and is not permissibly inconsistent (the correction is required) according the indicators HCR and k_y. Let calculate the weights of decisions alternatives from the MPC $D4_{7\times7}$:

- according the method of EM: $\text{vec}^{\mathrm{T}} = (0.062\ \ 0.276\ \ 0.019\ \ 0.253\ \ 0.021\ \ 0.104\ \ 0.265)$;
- according the method of RGMM: $\text{w}^{\mathrm{T}} = (0.065\ \ 0.269\ \ 0.02\ \ 0.262\ \ 0.022\ \ 0.113\ \ 0.248)$;
- according the method of AN: $\text{w\_AN}^{\mathrm{T}} = (0.058\ \ 0.231\ \ 0.027\ \ 0.278\ \ 0.029\ \ 0.1\ \ 0.277)$.

All methods lead to the different ranking of alternatives. For example, ranking according the traditional method of EM is equal to $a_2 \succ a_7 \succ a_4 \succ a_6 \succ a_1 \succ a_5 = a_3$ with the accuracy 10^{-2}.

The ejection in the given MPC is the element $d_{27} = 1/3$. After changing the element at the back symmetric $d_{27} := 3$, the MPC $D4_{7\times7}$ becomes permissibly inconsistent ($CR = 0.058$, $GCI = 0.213$, $HCR = 0.025$, $k_y = 0.660$) according the all indicators, except k_y. After the ejection correction the following weights are received:

- according the method of EM: $\text{vec}^{\mathrm{T}} = (0.063\ \ 0.362\ \ 0.02\ \ 0.249\ \ 0.022\ \ 0.11\ \ 0.173)$;
- according the method of RGMM: $\text{w}^{\mathrm{T}} = (0.063\ \ 0.357\ \ 0.02\ \ 0.254\ \ 0.022\ \ 0.109\ \ 0.175)$;
- according the method of AN: $\text{w\_AN}^{\mathrm{T}} = (0.05\ \ 0.422\ \ 0.023\ \ 0.243\ \ 0.026\ \ 0.087\ \ 0.149)$,

which set the same ranking of alternatives $a_2 \succ a_4 \succ a_7 \succ a_6 \succ a_1 \succ a_5 = a_3$, however, this ranking does not coincide with the ranking obtained from the initial MPC.

5. The method of estimating the consistency of MPC

The method of estimating the consistency of the MPC, which is proposed, is based on the results of the research of consistency indicators CR, GCI, HCR, CI^{tr} and k_y MPC, obtained in section 3. So, for MPC, which are perturbations of strongly consistent MPC in 97% of cases all indicators CR, GCI, HCR and CI^{tr} properly identified the permissible inconsistency of the MPC, and k_y for those MPC in 95% of cases erroneously shows the need of correction.

For the MPC, which has the properties of weak consistency and a strong inconsistency, in 97% of cases indicator together with all other variables correctly shows impermissible inconsistency, that is the necessity of the MPC correction.

For weakly inconsistent MPC with single ejection the CR, GCI, HCR and CI^{tr} operates bad, as erroneously show that MPC does not require the correction, if its other elements are: 1)strongly consistent or 2)perturbations of strongly consistent (18.7% and 31.8% of the cases respectively). These indicators are also mistakenly identifying the total lack of information in such MPC (42.6% and 17% of cases for 1) and (2) respectively).

In this case, the ranking of the solutions alternatives, obtained on the basis of the initial weakly inconsistent MPC, and the ranking on the basis of the corrected MPC after the change of the ejection often differs among themselves (see examples 3 and 4 above). In our opinion, the right of these two is the ranking, based on the corrected MPC. So for the weakly inconsistent MPC the indicators CR, GCI, HCR *and* CI^{tr} *cannot be used without the prior correction of ejections in this MPC.*

The indicator k_y for weakly inconsistent MPC operates better, than other elements, since in general correctly showed the presence of information in the MPC and the necessity of the correction (80% and 96.4% of the cases for 1) and (2) respectively) and practically has not revealed the lack of information in the MPC (0 1.8% of the cases for 1) and (2) respectively).

Let $D_{n\times n} = \{d_{ij} \mid i, j = 1,...,n\}$ – MPC of solutions alternatives on criteria.

The method of estimating the consistency of the MPC, that is offered, consists of the following stages:

1) Determine if $D_{n\times n}$ has the properties of weak consistency.

2) If $D_{n\times n}$ is weakly consistent, for the estimation of the permissibility of inconsistency any of the indicators CR, GCI or CI^{tr} should be used – they are equivalent – and the criterion of the consistency 1 should also be used.

3) If $D_{n\times n}$ has not the properties of weak consistency (weakly inconsistent), the ejections should be looked for in $D_{n\times n}$, and should be corrected till $D_{n\times n}$ becomes weakly consistent.

The results of simulation, shown in table1, indicate that for the getting of the ejections in the MPC, you can use the value of the determinants of its transitivities. *The proposed method of getting the ejection consists of stages*:

1). Creation of the set of MPC transitivities $\Gamma = \{\Gamma_u\} : \Gamma_u = \{d_{ij}, d_{jk}, d_{ik}\}$, $i, j, k = 1,...,n$, $i < j < k$,

$$u = 1,...,NT\ ,\ NT = \frac{n!}{(n-3)!\,3!}\ ,\ n \geq 3\ .$$

2). Calculation of the set of the transitivities determinants values: $Det = \{\det(\Gamma_u)\}$,

$$\det(\Gamma_u) = \frac{d_{ij}d_{jk}}{d_{ik}} + \frac{d_{ik}}{d_{ij}d_{jk}} - 2\ .$$

3) Selection of the maximum value of the set of determinants values:

$$\Gamma_{u^*} = \{d_{i^*j^*}, d_{j^*k^*}, d_{i^*k^*}\} = \arg\max_{u=1,...,NT} \det(\Gamma_u)\ .$$

4) The transitivity Γ_{u^*} should be corrected.

Example 5. To illustrate the method of searching the ejection, using the values of MPC transitivities determinants, consider the MPC $D4_{7\times7}$ from the example 4.

The set $Det = \{\det(\Gamma_u)\}$ for the MPC $D4_{7\times7}$ is equal:

u	i	j	k	$\det(\Gamma_u)$
1	1	2	3	3.521
2	1	2	4	0.167
3	1	2	5	1.837
4	1	2	6	0
5	1	2	7	6.125

u	i	j	k	$\det(\Gamma_u)$
18	2	3	6	2.25
19	2	3	7	1.333
20	2	4	5	0.5
21	2	4	6	0.167
22	***2***	***4***	***7***	***10.083***

6	1	3	4	2.25
7	1	3	5	0.167
8	1	3	6	0.083
9	1	3	7	0.5
10	1	4	5	1.042
11	1	4	6	0
12	1	4	7	0
13	1	5	6	0
14	1	5	7	0.167
15	1	6	7	0.5
16	2	3	4	0.5
17	2	3	5	0

23	2	5	6	1.837
24	2	5	7	1.671
25	2	6	7	2.25
26	3	4	5	0
27	3	4	6	1.333
28	3	4	7	0.5
29	3	5	6	0.014
30	3	5	7	0.014
31	3	6	7	1.333
32	4	5	6	1.042
33	4	5	7	0.34
34	4	6	7	0.5
35	5	6	7	1.333

The maximum value of determinant, equal to 10.083, is archived in transitivity $\Gamma_{u^*} = \{d_{24}, d_{47}, d_{27}\}$. Therefore, the ejection is the element of MPC d_{27}. It should be corrected on the basis of the product $d_{24} \cdot d_{47}$.

Conclusion

In the research the assessment of equivalence of known indicators CR, GCI, HCR, CI^{tr} and k_y is carried out for the MPC with a wide range of changes of the level of consistency. The equivalence of indicators CR, GCI and CI^{tr} is shown for all researched MPC: linear regression between these indicators showed the significance of the coefficient of determination that exceed 0.95. The results for the indicators CR and GCI are consisted with known results, received in [Aguaron&Moreno-Jimenez, 2003]. However, in this work, the research was carried out for a wider range of changes of the level of consistency of the MPC. A weak linear dependence between CR and HCR (R^2 is equal to 0.27, 0.46, 0.50, 0.69 depending on the level of MPC consistency) and even less linear dependence between CR and k_y (R^2 takes the values 0.16, 0.20, 0.28, 0.04) are shown.

Such a weak linear dependence between CR and k_y can be explained by the fact that k_y is based on a completely different ideas in comparison with indicators CR, GCI, HCR and CI^{tr}. The indicator k_y in many cases leads to inconsistent results compared with CR therefore should be used carefully.

It is established, that the property of weak consistency guarantees the presence of information in the MPC.

The correct method of assessment of the consistency of the MPC depending on its characteristics is developed. If the MPC is weakly consistent, so for the assessment of the permissibility of its inconsistency any of the indicators CR, GCI або CI^{tr} and criterion of consistency 1should be used. If the MPC has not the property of weak consistency, then you need to look for ejections in this matrix, correct these ejections until the MPC becomes weakly consistent and already after that to assess the permissibility of consistency of corrected MPC according any of the indicators CR, GCI or CI^{tr}. The use of any indicators of CR, GCI, HCR and CI^{tr} and the criterion of consistency 1 to MPC, which has not the properties of weak consistency without the prior correction of the ejections, in many cases (18.7% and 31.8% of cases depending on the level of consistency) leads to erroneous vectors of weights.

The results, obtained in this work, is planned to be used in future for the development of a method of improving the MPC consistency.

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